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ANALYSIS OF OSCILLATIONS OF ONE-DIMENSIONAL SPATIALLY PERIODIC STRUCTURES. AN UNCONVENTIONAL APPROACH AND SOME NEW EFFECTS

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Abstract. *The present study is concerned with the analysis of oscillations of one-dimensional spatially periodic structures. There are many approaches for wave examination in such structures, in particular, methods, based on the utilization of the Floquet theory [1]. However, in the framework of this theory it is problematic to incorporate the (external) boundary conditions. The averaging procedure for processes in periodic systems based on the multiple scales method [2] combined with the averaging method [3] was proposed in the monograph [4].*

In the present paper a new approach for the analysis of oscillations of one-dimensional spatially periodic structures, which is based on the method of direct separation of motions (MDSM) [5,6], is proposed. The approach is introduced in order to detect new effects in such structures' behavior, which may be employed to produce systems with advanced properties.

As an example of the approach application the study of oscillations of a string with variable cross-section is conducted. As the result, analytical expressions for the eigenmodes and the eigenfrequencies of the system are obtained. It is shown that modulation of the string cross-section leads to a change of the eigenfrequencies as compared with their non-modulated values, and to the emergence of a spectrum of additional high eigenfrequencies, which correspond to large wave lengths. A simple physical explanation of the latter effect, which is noted, apparently for the first time, is proposed. It is shown that character of string oscillations may be controlled by modulation of its cross-section, in particular, for given initial conditions the effect of high-frequency components suppression may be achieved.

1 INTRODUCTION

The present study is concerned with the analysis of oscillations of one-dimensional spatially periodic structures. There are many approaches for wave examination in such structures, in particular, methods, based on the utilization of the Floquet theory [1]. The so-called frequency stop bands, i.e. frequencies ranges, in which a wave does not propagate through the considered system, can be determined by their means [7-9]. However, in the framework of this theory it is problematic to incorporate the (external) boundary conditions. The averaging procedure for processes in periodic structures based on the multiple scales method [2] combined with the averaging method [3] was proposed in the monograph [4]. There are several approaches [10, 11] based on this procedure, which are applicable to study other classes of dynamical systems.

In the present paper a new approach for the analysis of oscillations of one-dimensional spatially periodic structures, which is based on the method of direct separation of motions (MDSM) [5, 6], is proposed. The approach is introduced in order to detect new effects in such structures' behavior, which may be employed to produce systems with advanced properties. As an example of the approach application the study of oscillations of a string with variable cross-section is conducted. The aim is to identify the eigenfrequencies and the eigenmodes of this system. Effects revealed for the considered simple model may facilitate solving the practically important problem of design of a periodic structure with tailored characteristics. In particular, a system with advanced performance of sound and vibration suppression may be produced.

It is noted that application of the multiple scales method (MSM) and other classical asymptotic methods for solving the considered equations is rather cumbersome, because an unknown parameter is present in them. The fundamental difference between the MDSM and these methods is specified. In particular, it is shown that the MDSM may be employed for solving equations without small parameter; thereby, the applicability range of this method turns out to be broader than the one of the classical asymptotic methods.

We note that the proposed approach is applicable for solving problems, which are opposite to those studied by the asymptotic method of G. Wentzel, H. Kramers, and L. Brillouin (the WKB method) [12]. Because here long waves in rapidly varying structures are considered that is opposite to short waves in slowly varying structures analyzed by the WKB.

2 OSCILLATIONS OF A STRING WITH VARIABLE CROSS-SECTION. INITIAL EQUATIONS

Consider oscillations of the simplest one-dimensional periodic system – string with variable cross-section, which are described by the equation

$$\rho S \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(T \frac{\partial u}{\partial x} \right) = 0 \tag{1}$$

Here ρ is the density of the string material, S is the cross-sectional area, T is the tension force, $u(x,t)$ is the lateral deflection of the string. Spatial modulations of the cross-sectional area are determined by the relation $S = S_0(1+\alpha \sin kx)$, here $0 \le \alpha < 1$, $k >> \frac{\pi}{l}$, where l is the length of the string (Figure 1). The boundary conditions are set to be homogeneous

$$u\big|_{x=0} = u\big|_{x=l} = 0 \tag{2}$$

The classical separation of variables technique is used to solve equation (1)

$$u(x,t) = A(x)B(t) \tag{3}$$

Consequently for the new variable $A(x)$ the following equation is obtained

$$A'' = -\frac{\Omega^2 \rho S_0}{T}(1+\alpha \sin kx)A \tag{4}$$

here Ω is a frequency of string oscillations, which value and order are not given a priori and thence unknown, prime designates the spatial derivative.

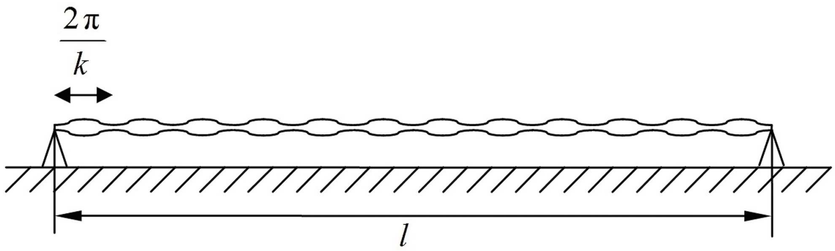


Figure 1: String with variable cross-section.

Generally, application of the averaging method, MSM and other asymptotic methods requires the presence of a small parameter in the equation considered. In addition, orders of all parameters, involved in the equation, must be known to employ these methods. In our case the unknown parameter Ω is present in equation (4). Application of the asymptotic methods for solving such equations is rather cumbersome and may lead to erroneous results. For example Ω may have such value that it will be impossible to assign a small parameter in equation (4).

The condition of the MDSM applicability is the fulfillment of the main assumption of vibrational mechanics, which was formulated in the monograph [5]. In substance, this method also implies the presence of a small parameter. But in contrast to the asymptotic methods, this small parameter must be present in the resulting equation, and not in the initial one. This distinction broadens the applicability range of the MDSM, in particular, enables to apply it for solving the considered equation (4).

So, the applicability range of the MDSM turns out to be broader than the one of the classical asymptotic methods. The validity of this statement is illustrated, particularly, in paper [13], where a classical problem about the stability of a pendulum with vibrating suspension axis is considered at unconventional values of parameters.

3 OSCILLATIONS OF A STRING WITH VARIABLE CROSS-SECTION. SOLUTION BY THE MDSM

The vibrational mechanics approach and the MDSM are employed for solving equation (4). The procedure of the method application remains the same as in conventional cases, but equation with respect to spatial coordinate x, and not to time t, is considered. Solution of equation (4) is sought in the form

$$A = A_1(x) + \psi(x, kx) \tag{5}$$

where A_1 is "slowly varying", and ψ is "rapidly varying", 2π-periodic in dimensionless ("micro") coordinate $x_1 = kx$ variable, with period x_1 average being equal to zero:

$$\langle \psi(x, x_1) \rangle = 0 \tag{6}$$

Here $\langle ... \rangle$ designates averaging in the period 2π on coordinate x_1. The slowly varying component A_1 describes processes taking place at macro scale, which is determined by the characteristic spatial dimension of the problem. In our case this scale is defined by the length of the string l. The rapidly varying component ψ captures processes taking place at micro scale, which is determined by linear dimensions of the inhomogeneity in the considered structure, i.e. by $\frac{2\pi}{k}$. The aim of the method is to compose equations, which describe the spatially averaged processes in periodic system, i.e. equations for the slowly varying variable A_1.

The following equation is obtained for variable A_1 by averaging (4) by x_1

$$\frac{\mathrm{d}^2 A_1}{\mathrm{d}x^2} + \frac{\Omega^2 \rho S_0}{T} A_1 = -\frac{\Omega^2 \rho S_0}{T} \alpha \langle \psi \sin x_1 \rangle \tag{7}$$

Equation for the "rapidly varying" variable ψ may be obtained by subtracting equation (7) from equation (4)

$$\psi'' + \frac{\Omega^2 \rho S_0}{T} \psi = -\frac{\Omega^2 \rho S_0}{T} \alpha \left((A_1 + \psi) \sin x_1 - \langle \psi \sin x_1 \rangle \right) \tag{8}$$

We note that in the absence of modulations these equations will have a natural solution $\psi = 0$ and $A = A_1$.

To achieve more accurate results the MDSM is applied in its modified form [14,15]. We abandon one of the conventional simplifications of the method, which in the considered case has the form: while solving the equation for variable ψ it is possible to consider variable A_1 as constant ("frozen"). Detailed discussion of the reasons for using the modified method is given in section 4.

So, we take into account the dependence of variable A_1 on x while solving equation (8). The solution is sought in the form of series

$$\psi = B_{11} \sin x_1 + B_{12} \cos x_1 + B_{21} \sin 2x_1 + B_{22} \cos 2x_1 + B_{31} \sin 3x_1 + B_{32} \cos 3x_1 + ... \tag{9}$$

Taking into account only the first harmonic in the solution, that is sufficient at small α, we obtain the following expression for ψ

$$\psi = \alpha A_1 \frac{\Omega^2 \mu}{1 - \Omega^2 \mu} \sin x_1 + \alpha \frac{2}{k} A_1' \frac{\Omega^2 \mu}{\left(1 - \Omega^2 \mu\right)^2} \cos x_1 \tag{10}$$

here $\mu = \frac{\rho S_0}{Tk^2}$. Using this expression, equation (7) for the "slowly varying" variable A_1 is transformed into

$$\frac{\mathrm{d}^2 A_1}{\mathrm{d}x^2} + \frac{\Omega^2 \rho S_0}{T} \left(1 + \frac{\alpha^2}{2} \frac{\Omega^2 \mu}{1 - \Omega^2 \mu} \right) A_1 = 0 \tag{11}$$

Now consider the boundary conditions (2). Taking into account decomposition (5) these conditions may be written in the form

$$A_1(0) + \psi(0,0) = A_1(l) + \psi(l, kl) = 0 \tag{12}$$

Employing expression (10) for ψ, from (12) we obtain
at $x=0$

$$A_1(0)+\alpha\frac{2}{k}\frac{\Omega^2\mu}{\left(1-\Omega^2\mu\right)^2}A_1'(0)=0 \tag{13}$$

at $x=l$

$$A_1(l)+\alpha\frac{\Omega^2\mu}{1-\Omega^2\mu}A_1(l)\sin kl+\alpha\frac{2}{k}\frac{\Omega^2\mu}{\left(1-\Omega^2\mu\right)^2}A_1'(l)\cos kl=0 \tag{14}$$

So, the equation and the boundary conditions for the slowly varying variable A_1 are formulated. As the result, the following equation for the eigenfrequencies Ω is derived

$$\begin{gathered}\left(\sin\lambda l-\alpha\frac{2}{k}\frac{\Omega^2\mu}{\left(1-\Omega^2\mu\right)^2}\lambda\cos\lambda l\right)\left(1+\alpha\frac{\Omega^2\mu}{1-\Omega^2\mu}\sin kl\right)+\\+\alpha\frac{2}{k}\frac{\Omega^2\mu}{\left(1-\Omega^2\mu\right)^2}\lambda\left(\cos\lambda l+\alpha\frac{2}{k}\frac{\Omega^2\mu}{\left(1-\Omega^2\mu\right)^2}\lambda\sin\lambda l\right)\cos kl=0\end{gathered} \tag{15}$$

Here $\lambda=\sqrt{\frac{\Omega^2\rho S_0}{T}\left(1+\frac{\alpha^2}{2}\frac{\Omega^2\mu}{1-\Omega^2\mu}\right)}$ is corresponding wave number. The “macro-scale” eigenmode of the inhomogeneous string is determined by the expression

$$A_1=\sin\lambda x-\alpha\frac{2}{k}\frac{\Omega^2\mu}{\left(1-\Omega^2\mu\right)^2}\lambda\cos\lambda x \tag{16}$$

4 VERIFICATION OF THE MDSM APPLICABILITY

Verification of the MDSM applicability for solving equation (4) is provided. As it was noted above, in conventional cases criterion of the MDSM applicability is the fulfillment of the main assumption of vibrational mechanics [5]. In the present paper equations with respect to spatial coordinate x and not to time t are considered. However, the procedure of the method application remains the same. So, condition of the MDSM applicability in the considered case is similar to the conventional one. It takes the following form

$$\frac{\psi'}{\psi_0}>>\frac{A_1'}{A_{10}} \tag{17}$$

Here ψ_0 is characteristic amplitude of the rapidly varying variable ψ oscillations, A_{10} is characteristic amplitude of the slowly varying variable A_1 oscillations. Employing expressions (10) and (16) for ψ and A_1, we obtain $\frac{A_1'}{A_{10}}\sim\lambda$, $\frac{\psi'}{\psi_0}\sim k$. So, condition (17) may be reduced to

$$\frac{\lambda}{k} << 1 \tag{18}$$

In substance, this condition may be interpreted as the requirement for variable A_1 to vary "indeed slowly" in comparison with variable ψ. So, only long waves in rapidly varying structures may be studied by means of the MDSM.

As is seen from condition (18), application of the MDSM implies the presence of a small parameter. However, in contrast to the asymptotic methods, this small parameter must be present in the resulting equation, and not in the initial one. In the considered case this parameter is determined by the expression

$$\varepsilon = \frac{\lambda}{k} = \sqrt{\Omega^2 \mu \left(1 + \frac{\alpha^2}{2} \frac{\Omega^2 \mu}{1 - \Omega^2 \mu}\right)} \tag{19}$$

Relation (19) may be considered as an equation for the product $\Omega^2 \mu = \frac{1}{k^2} \frac{\Omega^2 \rho S_0}{T}$, which is present in the initial equation (4). Obtained information about the order of $\Omega^2 \mu$ may facilitate solving equation (4) by means of the asymptotic methods. In our case from relation (19) we obtain $\Omega^2 \mu \sim 1$ and $\Omega^2 \mu \sim \varepsilon^2$. Substituting the first one in (4), equation without small parameter is composed. Such equation can not be properly solved by means of the classical asymptotic methods. If we substitute the second value $\Omega^2 \mu \sim \varepsilon^2$ in (4) and solve this equation by the MSM, then the following equation for the main component A_{1m} will be obtained

$$\frac{\mathrm{d}^2 A_{1m}}{\mathrm{d}x^2} + \frac{\Omega^2 \rho S_0}{T} A_{1m} = 0 \tag{20}$$

As is seen, this equation does not capture the change of the system's effective parameters due to modulation. I.e. application of the MSM in the considered case leads to the erroneous results. Thereby, even the introduction of the small parameter (19), which was determined from the condition of the MDSM applicability, does not allow solving equation (4) by means of the asymptotic methods.

In section 3 the MDSM was applied without one of the conventional simplifications, which in the considered case has the form: while solving the equation for variable ψ it is possible to consider variable A_1 as constant ("frozen"). The abandonment of this simplification complicates the solution of equation (8) for the rapidly varying variable ψ, but in the considered case it turns out to be necessary. To illustrate it we solve equation (8) considering variable A_1 as frozen, or parameter x as constant. Taking into account only the first harmonic in the solution, we obtain

$$\psi = \alpha A_1 \frac{\Omega^2 \mu}{1 - \Omega^2 \mu} \sin x_1 \tag{21}$$

Equation for the slowly varying variable A_1 will, as before, have form (11). But the boundary conditions will change: $A_1(0) = A_1(l) = 0$. As the result, equation for the eigenfrequencies will take the form

$$\lambda = \sqrt{\frac{\Omega^2 \rho S_0}{T}\left(1+\frac{\alpha^2}{2}\frac{\Omega^2\mu}{1-\Omega^2\mu}\right)} = \frac{\text{n}\pi}{l} \tag{22}$$

Solutions of this equation (values of frequencies Ω and wave numbers λ) differ from the correct values Ω and λ, obtained from equation (15).

5 THE EIGENFREQUENCIES AND THE EIGENMODES OF THE INHOMOGENEOUS STRING

To solve equation (15) for the eigenfrequencies the finite-product method [16] is employed. The application of this method implies the representation of $(\sin\lambda l)/\lambda l$ and $\cos\lambda l$ as the following products

$$\frac{\sin\lambda l}{\lambda l} = \prod_{\text{m}'=1}^{\text{m}}\left(1-\left(\frac{\lambda l}{\pi}\right)^2\frac{1}{\text{m}'^2}\right) \tag{23}$$

$$\cos\lambda l = \prod_{\text{m}'=1}^{\text{m}}\left(1-\left(\frac{\lambda l}{\pi}\right)^2\frac{1}{(\text{m}'-\frac{1}{2})^2}\right) \tag{24}$$

Number of retained factors in products (23)-(24) affects the number and the values of the resulting roots of equation (15). We are interested in the frequencies, for which corresponding wave numbers are relatively small $\lambda << k$ (see section 4). To determine such frequencies it is sufficient to retain only a few factors in products (23)-(24). Limiting ourselves to just two factors, we obtain five roots – values of frequency Ω, one of which corresponds to $\lambda = 0$. It should be noted that in the case of a homogeneous string, if we take into account two factors in each product, corresponding equation for the eigenfrequencies $\sin\lambda l = 0$ will have not five, but two roots $\lambda_{1(0)} = \pi/l$ 1/cm, $\lambda_{2(0)} = 2\pi/l$ 1/cm. As an illustration, wave numbers λ and corresponding eigenfrequencies Ω of the inhomogeneous and homogeneous strings are presented in Figure 2 for the following values of the parameters $\rho S_0/T = 1\text{s}^2/\text{cm}^2, l = 5\,\text{cm}, \alpha = 0.5, k = 9\pi/l$ 1/cm.

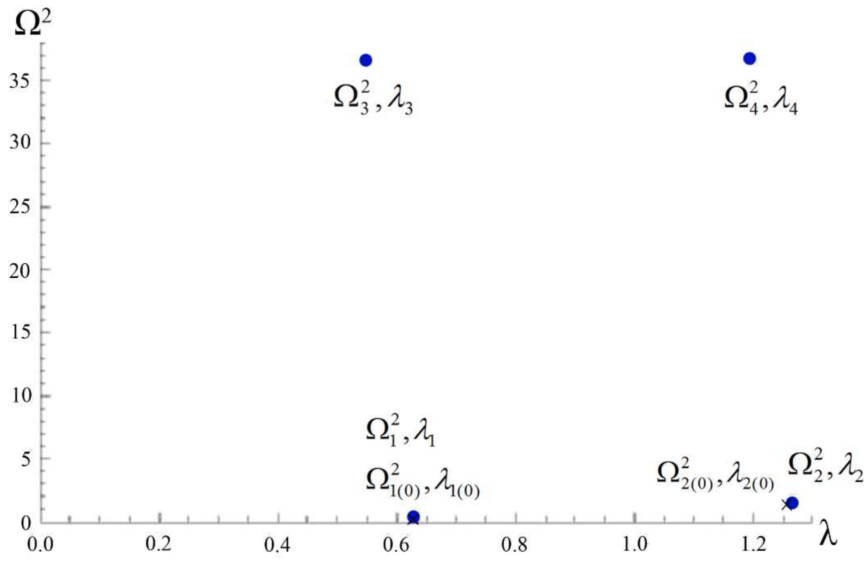


Figure 2: Wave numbers and corresponding eigenfrequencies of the homogeneous (crosses) and inhomogeneous (circles) strings.

As is seen from Figure 2, modulation of the string cross-section leads to a small change of (the basic) eigenfrequencies Ω_1, Ω_2 as compared with their non-modulated values $\Omega_{1(0)}$ and $\Omega_{2(0)}$. Corresponding wave numbers also change due to modulation. Another interesting effect is the emergence of additional eigenfrequencies Ω_3, Ω_4, which values are significantly higher than $\Omega_{1(0)}$ and $\Omega_{2(0)}$. We note that wave numbers, which correspond to these frequencies, are close to $\lambda_{1(0)}$ and $\lambda_{2(0)}$.

The same effects will occur if we retain more factors in products (23)-(24). Thus, if we take into account three factors, equation (15) will have not five, but seven roots, one of which corresponds to $\lambda = 0$. In the case of a homogeneous string we would obtain three roots $\lambda_{1(0)} = \pi/l$ 1/cm, $\lambda_{2(0)} = 2\pi/l$ 1/cm, $\lambda_{3(0)} = 3\pi/l$ 1/cm. Taking into account four factors, we get nine roots of equation (15) etc.

Values of the wave numbers and corresponding eigenfrequencies depend significantly on the character of modulation. To illustrate it wave numbers and corresponding eigenfrequencies are shown in Figure 3 at $\rho S_0/T = 1\text{s}^2/\text{cm}^2, l = 5\text{cm}, \alpha = 0.5$ for different values of parameter k: $k = 8\pi/l$ 1/cm, $k = 17\pi/2l$ 1/cm, $k = 9\pi/l$ 1/cm. As before, we retain two factors in products (23)-(24). It is interesting that at $kl = 2m\pi$ corresponding wave numbers turn out to be equal to each other $\lambda_1 = \lambda_3$ и $\lambda_2 = \lambda_4$.

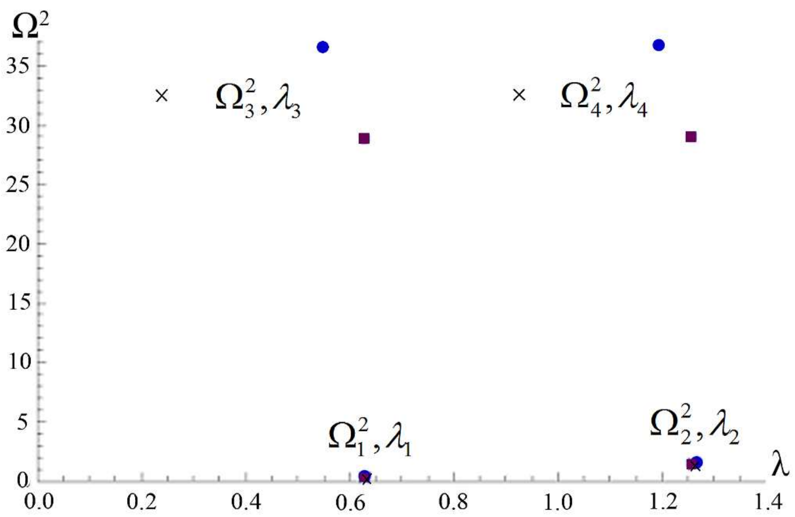


Figure 3: Wave numbers and eigenfrequencies of the inhomogeneous string at different modulations: $k = 8\pi/l$ 1/cm (squares), $k = 17\pi/2l$ 1/cm (crosses), $k = 9\pi/l$ 1/cm (circles).

So, we may conclude that modulation of the string cross-section leads to two effects:1. The emergence of additional high eigenfrequencies, which correspond to small wave numbers or large wave lengths 2. Shift of (the basic) eigenfrequencies Ω_1, Ω_2 etc. as compared with their non-modulated values $\Omega_{1(0)}$, $\Omega_{2(0)}$ etc. This shift may be as in the direction of increasing (e.g. at $k = 9\pi/l$ 1/cm), as in the direction of decreasing (e.g. at $k = 8\pi/l$ 1/cm).

Taking into account relations (10) for ψ and (16) for A_1, the following expression for the eigenmode of the inhomogeneous string, which corresponds to eigenfrequency Ω, is derived

$$A(x)=\left(\sin\lambda x-\alpha\frac{2}{k}\frac{\Omega^2\mu}{\left(1-\Omega^2\mu\right)^2}\lambda\cos\lambda x\right)\left(1+\alpha\frac{\Omega^2\mu}{1-\Omega^2\mu}\sin kx\right)+$$
$$+\alpha\frac{2}{k}\frac{\Omega^2\mu}{\left(1-\Omega^2\mu\right)^2}\lambda\left(\cos\lambda x+\alpha\frac{2}{k}\frac{\Omega^2\mu}{\left(1-\Omega^2\mu\right)^2}\lambda\sin\lambda x\right)\cos kx \quad (25)$$

This expression may be considered as correct at $\alpha<0.5$.

6 ON THE EFFECT OF THE EMERGENCE OF A SPECTRUM OF ADDITIONAL HIGH EIGENFREQUENCIES, WHICH CORRESPOND TO LARGE WAVE LENGTHS

While solving equation (8) for the rapidly varying variable ψ only the first harmonic was taken into account. If we take into account other harmonics, resulting equation for the slowly varying variable A_1 will change. Thus, retaining two harmonics, we will obtain

$$\frac{\mathrm{d}^2A_1}{\mathrm{d}x^2}+\frac{\Omega^2\rho S_0}{T}\left(1+\alpha^2\frac{2\Omega^2\mu(\Omega^2\mu-4)}{(\alpha^2-4)\Omega^4\mu^2+20\Omega^2\mu-16}\right)A_1=0 \quad (26)$$

Corresponding boundary conditions for variable A_1 will also change. As the result, equation for the eigenfrequencies will take the form, which is different from (15). Solving this equation by the finite-product method and retaining two factors in products (23)-(24), we obtain not five, but seven roots - Ω values, one of which corresponds to $\lambda=0$. The first four non-zero roots will differ only slightly from those determined above, for the remaining two the following relations will hold true $\Omega_5^2>>\Omega_3^2$, $\lambda_5\approx\lambda_3$ and $\Omega_6^2>>\Omega_4^2$, $\lambda_6\approx\lambda_4$. Taking into account three harmonics in series (9), we obtain nine eigenfrequencies etc. As an illustration, wave numbers λ and corresponding eigenfrequencies Ω, obtained retaining three harmonics, are shown in Figure 4 for the following values of parameters $\rho S_0/T=1\,\mathrm{s}^2/\mathrm{cm}^2, l=5\,\mathrm{cm}, \alpha=0.5, k=8\pi/l$ 1/cm.

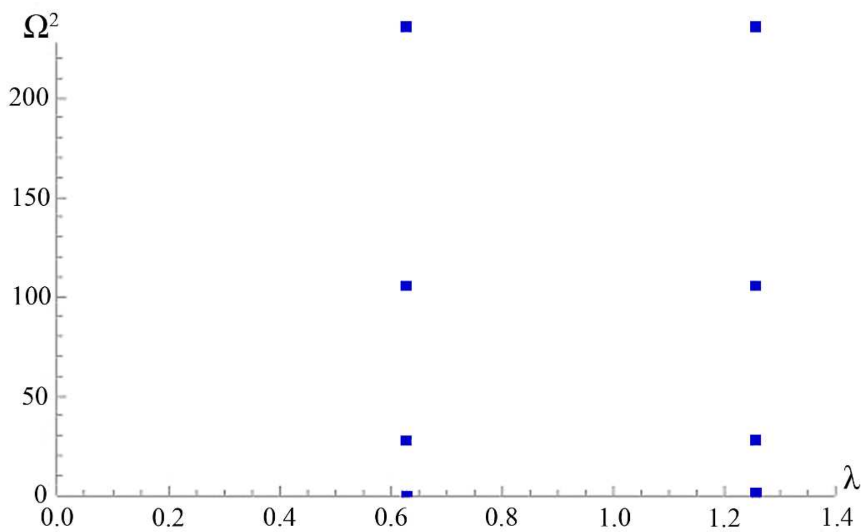


Figure 4: Wave numbers and corresponding eigenfrequencies, obtained retaining three harmonics in series (9), for $k=8\pi/l$ 1/cm.

So, we may conclude that modulation of the string cross-section leads the emergence of a whole spectrum of additional high eigenfrequencies, which correspond to large wave lengths.

A simple physical explanation of this effect is proposed in the paper. Consider oscillations of a homogeneous string fixed at both ends. As is well-known, its first eigenmode is $\sin \lambda_{1(0)} x$ ($\lambda_{1(0)} = \pi/l$ 1/cm), while oscillations on high frequencies occur at micro-level (Figure 5), i.e. large wave lengths correspond to low eigenfrequencies, and small wave lengths to high eigenfrequencies.

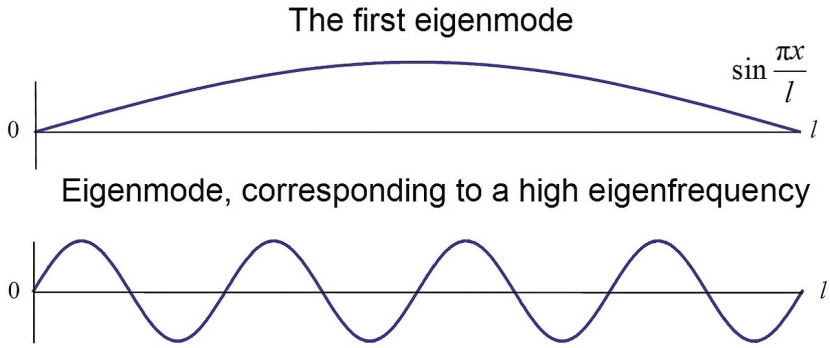


Figure 5: Eigenmodes of a homogeneous string.

Now consider an inhomogeneous string. Due to modulation its eigenmode, corresponding to the first eigenfrequency, is not $\sin \lambda_{1(0)} x$, but is $\sin \lambda_{1(0)} x$ plus imposed additional (small) oscillations on the micro-level. So, this mode is a superposition of the main macro-component $\sin \lambda_{1(0)} x$ and the additional (small) micro-component $(\sin \lambda_{1(0)} x) \sin kx$. As in the case of the low eigenfrequency, oscillations on high eigenfrequencies change due to inhomogeneity of the system and involve both micro and macro (additional) components (Figure 6). The macro-component is small in comparison with the micro-component; low value of the wave number corresponds to it. So, oscillations of the inhomogeneous string on high eigenfrequencies contain long-wave component. And that means spectrum of additional high eigenfrequencies, which correspond to small wave numbers or large wave lengths, is present for the string due to modulation.

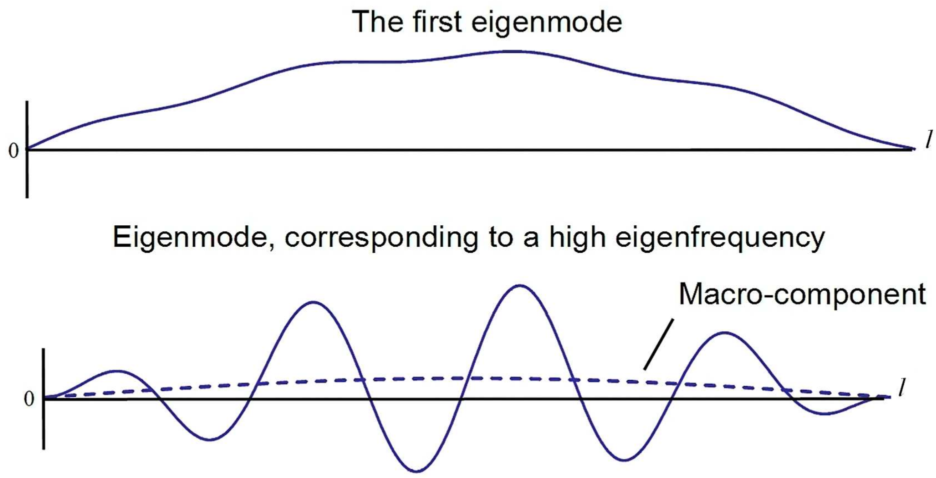
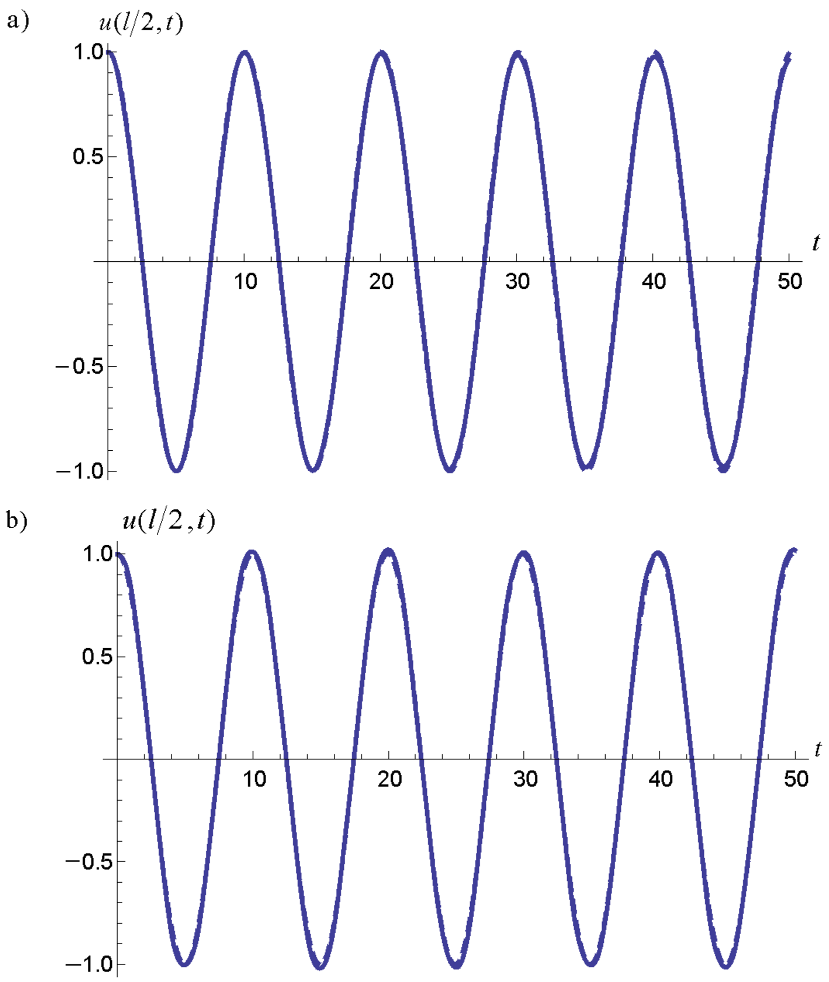


Figure 6: Eigenmodes of an inhomogeneous string.

At $\alpha \to 0$ (the basic) eigenfrequencies Ω_1, Ω_2 etc. tend to their non-modulated values $\Omega_{1(0)}$, $\Omega_{2(0)}$ etc. The long-wave component of string oscillations on high eigenfrequencies vanishes in this limit. This fact is in good agreement with the proposed physical explanation.

7 COMPARISON WITH THE RESULTS OF NUMERICAL EXPERIMENTS

Results obtained in the paper were verified by numerical experiments. Oscillations of an inhomogeneous string at the simplest initial conditions $u|_{t=0} = \sin \lambda_{1(0)} x$, $\dot{u}|_{t=0} = 0$ were considered. In accordance with our analytical predictions, at such initial conditions the string should oscillate "about the same" as the homogeneous one, because Ω_1 is close to $\Omega_{1(0)}$, and eigenmode (25), which corresponds to this frequency, differs only slightly from $\sin \lambda_{1(0)} x$ since $\lambda_1 \approx \lambda_{1(0)} = \dfrac{\pi}{l}$ 1/cm and $\Omega_1^2 \mu << 1$. The dependences of the deflection of the middle of the string $u(l/2, t)$ on time at $\rho S_0 / T = 1\text{s}^2/\text{cm}^2, l = 5\text{cm}$, $\alpha = 0.5$ are shown in Figure 7 for different values of parameter k: a) $k = 4\pi/l$ 1/cm b) $k = 5\pi/l$ 1/cm; solid line is the numerical solution, dashed line is the analytical solution. To illustrate the effect of modulation on basic eigenfrequencies these dependences, obtained numerically, are presented in Figure 7c) for $k = 4\pi/l$ 1/cm and $k = 5\pi/l$ 1/cm.



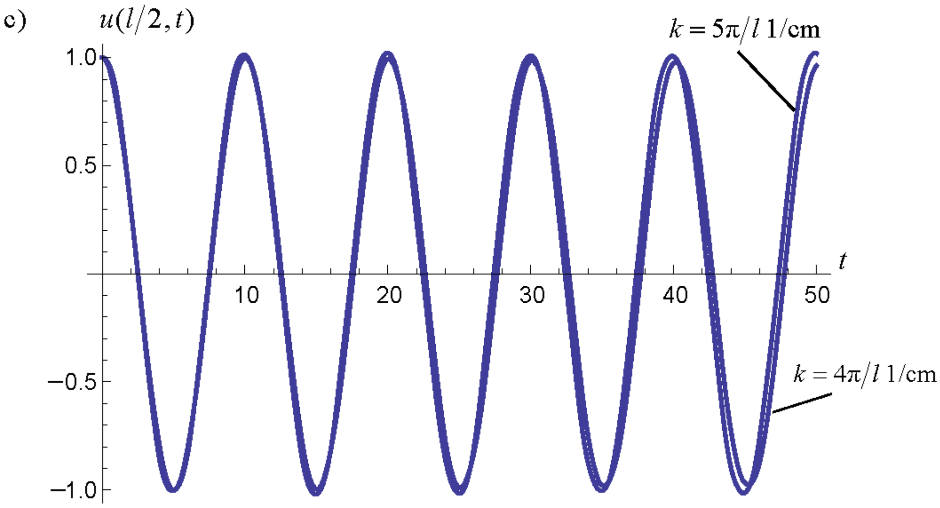


Figure 7: The dependences of the deflection of the middle of the string $u(l/2,t)$ on time t a) $k=4\pi/l$ 1/cm b) $k=5\pi/l$ 1/cm c) $k=4\pi/l$ 1/cm and $k=5\pi/l$ 1/cm ; solid line is the numerical solution, dashed line is the analytical solution.

As is seen from Figure 7, there is a good agreement between the analytical and numerical results. In particular, value of the eigenfrequency Ω_1 changes due to modulation, as it was predicted.

As follows from the analytical results, at considered initial conditions the additional high-frequency component of the inhomogeneous string oscillations is much smaller than the low-frequency one. In order to check whether this component is present or not, and to verify the value of the corresponding eigenfrequency we study a residual between the numerical solution and the analytically obtained low-frequency component. The dependence of this residual $u(l/2,t)-u_{lf}^{an}(l/2,t)$ on time t is shown in Figure 8 for $\rho S_0/T=1\text{s}^2/\text{cm}^2, l=5\,\text{cm}$, $\alpha=0.3$ and $k=20\pi/l$ 1/cm.

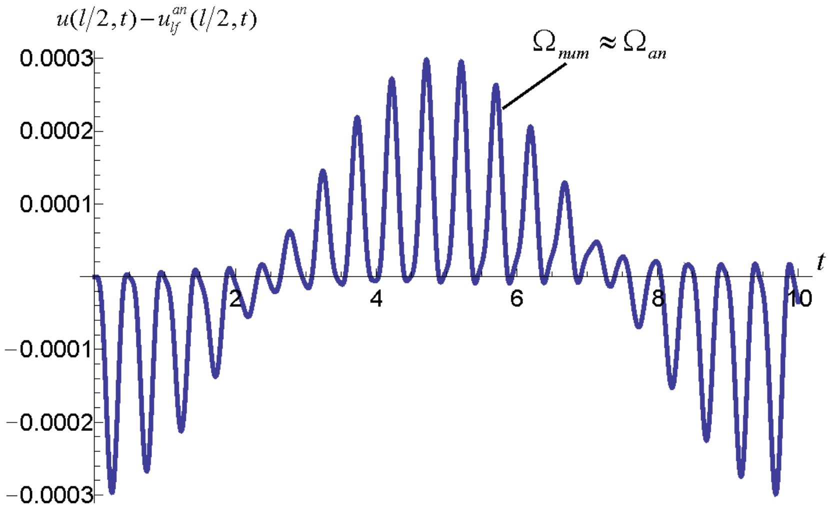


Figure 8: The dependence of the residual between the numerical solution and the analytically obtained low-frequency component of the inhomogeneous string oscillations on time.

As is seen from Figure 8, the conclusion regarding the emergence of a spectrum of additional high eigenfrequencies, which correspond to large wave lengths, is confirmed by the numerical experiments. It should be noted that values of these frequencies Ω_{an}, predicted analytically, almost coincide with the numerical ones Ω_{num} (for $k > 5\lambda_{1(0)}$). However, numerical experiments have shown that the dependence of the high-frequency component of the string oscillations on time is more complex, than it was predicted. This may be explained by the fact that the equation for variable ψ was solved only approximately, and so the eigenmodes, corresponding to additional high eigenfrequencies, are determined with some degree of error.

It should be noted that structural damping, which is present in every real system, affects the high-frequency motion significantly - it suppresses high-frequency components. However its influence on the low-frequency motion is relatively weak. So, the revealed effect of the emergence of a spectrum of additional high eigenfrequencies, which correspond to large wave lengths, should be less pronounced in real mechanical systems.

8 ON THE EFFECT OF SUPPRESSION OF HIGH-FREQUENCY COMPONENTS OF STRING OSCILLATIONS

If initial deflection of the inhomogeneous string is set as one of its eigenmodes (25), then it will oscillate with corresponding frequency Ω (e.g. with the low frequency Ω_1). However, oscillations of a homogeneous string at such complex initial conditions will involve many high-frequency components. So, we may say that at this initial conditions the inhomogeneous string "behaves better", than the homogeneous one. Thereby, we can control the character of string oscillations by modulation of its cross-section. In particular, for given initial conditions a modulation may be introduced that will compel string to oscillate on certain (e.g. low) frequency. So, the effect of suppression of high-frequency components of string oscillations may be achieved.

To illustrate this effect the dependencies of the accelerations of the inhomogeneous and homogeneous strings on time are presented in Figure 9 for $\rho S_0/T = 1\text{s}^2/\text{cm}^2, l = 5\,\text{cm}$, $\alpha = 0.8$ and $k = 16.9\frac{\pi}{l}1/\text{cm}$. Initial deflection is the eigenmode of the inhomogeneous string (25), which corresponds to the first eigenfrequency Ω_1.

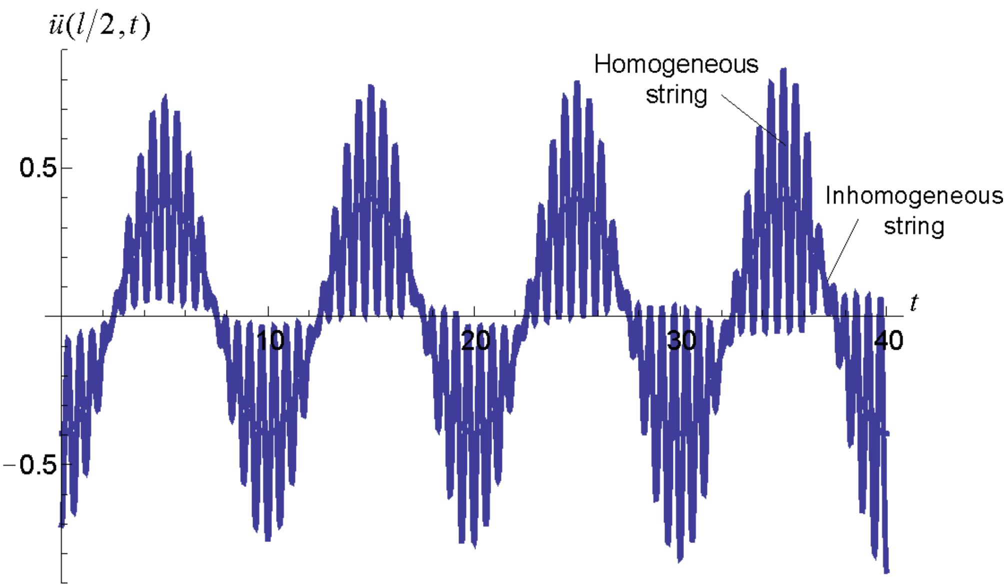


Figure 9: The dependencies of the accelerations of the inhomogeneous and homogeneous strings on time.

As is seen from Figure 9, at considered initial conditions the inhomogeneous string indeed behaves better, than the homogeneous one: there are no high-frequency components, and the amplitude of acceleration is much smaller. Thus, the character of string oscillations may be controlled by modulation of its cross-section, in particular, the effect of high-frequency components suppression may be achieved. This effect, revealed for the considered simple structure, may have various important applications; it may be employed, e.g. to control sound and vibration in spatially periodic systems used in engineering.

9 CONCLUSIONS

In the present paper a new approach for the analysis of oscillations of spatially periodic structures, which is based on the MDSM, is proposed. It is noted that application of the MSM and other classical asymptotic methods for solving the considered equations leads to the erroneous results. The fundamental difference between the MDSM and these methods, which enables to employ the MDSM in the considered case, is revealed. Thereby it is shown that the applicability range of the MDSM is broader than the one of the classical asymptotic methods.

Study of oscillations of a string with variable cross-section is conducted by means of the proposed approach. As the result, analytical expressions for the eigenmodes and the eigenfrequencies of the system are obtained. It is shown that modulation of the string cross-section leads to a change of the eigenfrequencies as compared with their non-modulated values, and to the emergence of a spectrum of additional high eigenfrequencies, which correspond to large wave lengths. A simple physical explanation of the latter effect, which is noted, apparently for the first time, is proposed. It is shown that character of string oscillations may be controlled by modulation of its cross-section, in particular, for given initial conditions the effect of high-frequency components suppression may be achieved.

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REFERENCES

[1] L. Brillouin, *Wave propagation in periodic structures, Second edition*. Dover Publications, 1953.

[2] A.H. Nayfeh, D.T. Mook, *Nonlinear Oscillations*. Wiley-Interscience, 1979.

[3] N.N. Bogoliubov, Y.A. Mitropolski, *Asymptotic Methods in the Theory of Nonlinear Oscillations*. Gordon and Breach, 1961.

[4] N.S. Bakhvalov, G.P. Panasenko, *Homogenization: Averaging Processes in Periodic Media*. Kluwer, 1989.

[5] I.I. Blekhman, *Vibrational Mechanics.* World Scientific, 2000.

[6] I.I. Blekhman, *Selected Topics in Vibrational Mechanics*. World Scientific, 2004.

[7] D.J. Mead, A General Theory of Harmonic Wave Propagation in Linear Periodic Systems with Multiple Coupling. *Journal of Sound and Vibration*, **27**, 235-260, 1973.

[8] D.J. Mead, Wave Propagation in Continuous Periodic Structures: Research Contributions from Southampton. *Journal of Sound and Vibration*, **190** (3), 495-524, 1996.

[9] B.R. Mace, D. Duhamel, M.J. Brennan, L. Hinke, Finite Element Prediction of Wave Motion in Structural Waveguides. *Journal of the Acoustical Society of America*, **117**, 2835–2843, 2005.

[10] G.P. Panasenko, *Multi-Scale Modelling for Structures and Composites*. Springer, 2005.

[11] K.A. Lurie, *Applied Optimal Control Theory of Distributed Systems*. Plenum Press, 1993.

[12] A.D. Pierce, Physical Interpretation of the WKB or Eikonal Approximation for Waves and Vibrations in Inhomogeneous Beams and Plates. *Journal of the Acoustical Society of America*, **48** (1), 275-284, 1970.

[13] V.S. Sorokin, A new approach to the analysis of oscillations of one-dimensional spatially periodic structures. *Journal of Sound and Vibration*, http://dx.doi.org/10.1016/j.jsv.2013.01.037, 2013.

[14] I.I. Blekhman, V.S. Sorokin, On the Separation of Fast and Slow Motions in Mechanical Systems with High-frequency Modulation of the Dissipation Coefficient. *Journal of Sound and Vibration*, **329**, 4936-4949, 2010.

[15] V.S. Sorokin, On Broadening of the Applicability Range of the Method of Direct Separation of Motions. *Proceedings of the 7th European Nonlinear Dynamics Conference (ENOC 2011)*, Rome, Italy, July, 2011.

[16] C.J. Chapman, S.V. Sorokin, The Finite-product Method in the Theory of Waves and Stability. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, **466**, 471-491, 2010.