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THE STABILITY OF THE GENERALISED-Α METHOD IN SOIL-PORE FLUID COUPLED FORMULATION

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Abstract. *This paper investigates the stability of the Generalised-α time integration method (the CH method) for a fully coupled solid-pore fluid formulation. Theoretical stability conditions are derived, which are shown to simplify to the existing ones of the CH method for the one–phase formulation when the solid-fluid coupling is ignored. Finite Element (FE) analyses considering a range of loading conditions are conducted to validate the analytically derived stability condition, showing that the numerical results are in agreement with the theoretical investigation. The CH method is a generalisation of a number of other time integration schemes and hence the derived stability conditions are relevant for most of the commonly utilised time integration methods for fully coupled two-phase formulation.*

1 INTRODUCTION

In dynamic Finite Element (FE) analysis, the time integration method has been widely and successfully used to solve the second order governing equation of motion, due to its ability of reproducing both material and geometric nonlinearities. Since 1950s, various time integration methods have been proposed, like the Newmark method [16] and the Generalised-α method (CH method) of Chung & Hulbert [6], which have been widely implemented into FE programs and successfully utilised to solve numerically the equation of motion for one-phase materials. The CH method satisfies the main requirements for an efficient time marching algorithm, which include unconditional stability for linear problems, second order accuracy and controllable numerical dissipation in the high frequency range [11]. The role of numerical damping is to eliminate spurious high frequency oscillations that are introduced into the solution due to poor spatial representation of the high-frequency modes.

Depending on the soil permeability, the rate of loading and the hydraulic boundary conditions, it is often necessary to employ coupled analysis to accurately model the two phase behaviour of the soil. Dynamic analyses are further complicated by the presence of the inertia forces of the different phases (i.e. solid skeleton and pore water) and the coupling between them. Hence, the formulation of the CH method was extended by Kontoe [13] and Kontoe et al [14] to enable the solution of dynamic coupled consolidation problems and was then implemented in the FE program ICFEP (Imperial College Finite Element Program) [17]. The key feature of unconditional stability of this method has been comprehensively investigated by previous studies, both analytically and numerically, but only for the one-phase formulation. The two-phase coupled FE formulation requires an additional equation (the dynamic consolidation equation) and an additional unknown (pore water pressure). Aiming to solve two dynamic coupled equations (i.e. the equation of motion and the consolidation equation), the time integration method is applied to both equations. This not only increases the complexity of the implementation, but it also changes the numerical features of the time integration method. Therefore, the numerical stability of the CH method needs to be investigated rigorously for two-phase coupled problems.

In this paper, the stability of the CH method for the coupled formulation is analytically investigated and the corresponding theoretical stability conditions are derived. The analytically derived stability conditions are validated by various FE analyses considering a range of loading conditions. Furthermore, since the CH method is a generalisation of the Newmark [16], HHT [10] and WBZ [19] methods, the stability investigation of the CH method for the two-phase coupled formulation is relevant for most of the commonly used time integration methods.

2 THE CH METHOD FOR THE COUPLED DYNAMIC FORMULATION

The coupled FE formulation for saturated porous materials is based on Biot's consolidation theory, proposed in 1956 and 1962 [2][3][4], which was firstly implemented numerically by Ghaboussi and Wilson [8]. Zienkiewicz and Shiomi [20] identified three categories of dynamic coupled formulations, namely the u-p formulation, the u-p-w formulation and the u-U formulation, where u, w, U and p denote the solid displacement, the fluid velocity relative to the solid component, the fluid displacement and the pore water pressure respectively. According to Zienkiewicz and Shiomi [20], the fluid acceleration relative to the solid and its convective terms are insignificant in the frequency range that is of concern in earthquake engineering problems. Therefore, by ignoring the relative fluid acceleration and its convective terms, Biot's theory can be simplified to the u-p formulation. In ICFEP, the u-p formulation was adopted by Potts and Zdravkovic [17] and was later extended to deal with dynamic loading by Hardy [9] and Kontoe [13].

The principle of the CH method is that the acceleration, velocity, displacement and pore water pressure terms are evaluated at different instants within the time step, controlled by integration parameters α_m and α_f (shown in Equation (1)). The variations of velocity and displacement within the time step for the CH method are approximated by Newmark's equations, governed by the integration parameters α and δ (shown in Equation (2)). Kontoe [13] and Kontoe et al [14] showed that the final dynamic FE coupled consolidation formulation for the CH method can be described by Equation (3), where five integration parameters, α_m, α_f, α, δ and β, are involved. It should be noted that β is the integration parameter of a time marching scheme which is employed in ICFEP to solve the integrals of the consolidation equation, which are shown in Equation (4).

$$\begin{Bmatrix} \ddot{u}_{t_{k+1-\alpha_m}} &= (1-\alpha_m)\ddot{u}_{t_{k+1}} + \alpha_m \ddot{u}_{t_k} \\ \dot{u}_{t_{k+1-\alpha_f}} &= (1-\alpha_f)\dot{u}_{t_{k+1}} + \alpha_f \dot{u}_{t_k} \\ u_{t_{k+1-\alpha_f}} &= (1-\alpha_f)u_{t_{k+1}} + \alpha_f u_{t_k} \\ p_{t_{k+1-\alpha_f}} &= (1-\alpha_f)p_{t_{k+1}} + \alpha_f p_{t_k} \\ t_{k+1-\alpha_m} &= (1-\alpha_m)t_{k+1} + \alpha_m t_k \\ t_{k+1-\alpha_f} &= (1-\alpha_f)t_{k+1} + \alpha_f t_k \end{Bmatrix} \tag{1}$$

$$\begin{aligned} \Delta\dot{u} &= \frac{\delta}{\alpha\Delta t}\Delta u - \frac{\delta}{\alpha}\dot{u}_{t_k} + \left(1 - \frac{\delta}{2\alpha}\right)\Delta t\ddot{u}_{t_k} \\ \Delta\ddot{u} &= \frac{1}{\alpha\Delta t^2}\Delta u - \frac{1}{\alpha\Delta t}\dot{u}_{t_k} - \frac{1}{2\alpha}\ddot{u}_{t_k} \end{aligned} \tag{2}$$

$$\begin{bmatrix} \left(\frac{1-\alpha_m}{\alpha\Delta t^2}\right)[M] + \left(\frac{(1-\alpha_f)\delta}{\alpha\Delta t}\right)[C] + (1-\alpha_f)[K] & (1-\alpha_f)[L] \\ \frac{(1-\alpha_m)\beta}{\alpha\Delta t}[G] + (1-\alpha_f)[L]^T & (1-\alpha_f)(-\beta\Delta t[\phi] - [S]) \end{bmatrix} \begin{bmatrix} \{\Delta u\} \\ \{\Delta p\} \end{bmatrix} = \begin{bmatrix} \Delta\bar{R} \\ \Delta\bar{F} \end{bmatrix} \tag{3}$$

$$\begin{aligned} \int_{t_k}^{t_k+\Delta t} \{p\} dt &= \left[\{p_{t_k}\} + \beta\{\Delta p\}\right]\Delta t \\ \int_{t_k}^{t_k+\Delta t} \{\ddot{u}\} dt &= \left[\{\ddot{u}_{t_k}\} + \beta\{\Delta\ddot{u}\}\right]\Delta t \end{aligned} \tag{4}$$

The coupled dynamic formulation (Equation (3)) was derived based on the dynamic equilibrium of the solid-fluid mixture, the continuity equation of the pore fluid flow and the generalised Darcy's law. In the above equations, $[M]$, $[C]$, $[K]$, $[L]$, $[\phi]$ and $[S]$ are the global mass, damping, stiffness, coupling, permeability and water compressibility matrices respectively, and $\ddot{u}$, $\dot{u}$, u and p represent the nodal acceleration, velocity, displacement and pore water pressure variables respectively. It should be noted that the matrix $[G]$ in Equation (3) represents the impact of the inertia of the solid on the pore water pressure. It has been though suggested that the influence of the matrix $[G]$ on the dynamic response is insignificant for the frequency range within which the "u-p" formulation is valid [5]. Therefore, the matrix $[G]$ is not taken into account in the dynamic coupled formulation for the work presented herein. Furthermore, the matrices and the right hand side terms of Equation (3) are detailed in the Appendix.

3 STABILITY ANALYSIS OF THE CH METHOD FOR THE COUPLED FORMULATION

A time integration method is considered stable when it produces a numerical solution which remains always bounded [1]. The fundamental assumption of a time integration method is that the acceleration, velocity and displacement at a specific increment can be expressed as a function of these variables at previous increments, as follows:

$$\begin{Bmatrix} \ddot{u}_{t_{k+1}} \\ \dot{u}_{t_{k+1}} \\ u_{t_{k+1}} \end{Bmatrix} = [A] \begin{Bmatrix} \ddot{u}_{t_k} \\ \dot{u}_{t_k} \\ u_{t_k} \end{Bmatrix} \tag{5}$$

where $[A]$ is the amplification matrix controlling the stability, accuracy and other numerical features of the considered time integration method. Based on the definition of numerical stability, for a time integration method to be stable, the matrix $[A]$ should be bounded. Therefore, the modulus of the eigenvalues of matrix $[A]$ should be less than one, expressed by the following equation:

$$\rho(A) = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} \leq 1 \tag{6}$$

where ρ is the spectral radius of the matrix $[A]$, and λ_1, λ_2 and λ_3 are the eigenvalues of the matrix $[A]$.

Considering the above-mentioned stability condition, Routh [18] and Hurwitz [12] suggested that, if λ is substituted by a complex number z (as shown in Equation (7)), the condition of $|\lambda| \leq 1$ can be converted to the condition of $\mathrm{Re}(z) \leq 0$.This indicates that the real part of z should be less or equal to zero in order to satisfy the stability condition.

$$\lambda = \frac{1+z}{1-z} \tag{7}$$

Furthermore, for a general polynomial of the form shown in Equation (8) to satisfy the condition of $\mathrm{Re}(z) \leq 0$, the conditions of Equation (9) should be also satisfied. Therefore, the stability condition of $|\lambda| \leq 1$ is finally converted to the condition given by Equation (9) which will be utilised for the following stability analysis of the CH method.

$$a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots a_n = 0 \tag{8}$$

$$a_0 > 0 \quad and \quad a_i \geq 0 \quad for \quad i > 0 \tag{9}$$

Expressing the acceleration, velocity, displacement and pore water pressure at time increment t_{k+1} as a function of their historic values and employing the Routh-Hurwitz condition results to:

$$\begin{Bmatrix} \ddot{u}_{t_{k+1}} \\ \dot{u}_{t_{k+1}} \\ u_{t_{k+1}} \\ p_{t_{k+1}} \end{Bmatrix} = \frac{1+z}{1-z} \begin{Bmatrix} \ddot{u}_{t_k} \\ \dot{u}_{t_k} \\ u_{t_k} \\ p_{t_k} \end{Bmatrix} \tag{10}$$

In addition, by employing Newmark's expressions (Equation (2)) and the CH method (Equation (1)), the acceleration, velocity, displacement and pore water pressure at different instants within a time step can be expressed as follows:

$$\begin{Bmatrix} \ddot{u}_{t_{k+1-\alpha_m}} \\ \dot{u}_{t_{k+1-\alpha_f}} \\ u_{t_{k+1-\alpha_f}} \\ p_{t_{k+1-\alpha_f}} \end{Bmatrix} = \begin{Bmatrix} \dfrac{(1-2\alpha_m)z+1}{2z}\Delta\ddot{u} \\ \dfrac{(2\delta-1)(1-2\alpha_f)z^2+(2\delta-2\alpha_f)z+1}{4z^2}\Delta\ddot{u}\Delta t \\ \dfrac{(4\alpha-2\delta)(1-2\alpha_f)z^3+(4\alpha-4\delta\alpha_f+2\alpha_f-1)z^2+(2\delta-2\alpha_f)z+1}{8z^3}\Delta\ddot{u}\Delta t^2 \\ \dfrac{(1-2\alpha_f)z+1}{2z}\Delta p \end{Bmatrix} \tag{11}$$

Equation (11) can be then substituted into the scalar forms of the coupled dynamic FE formulation (Equations (12) and (13)). This allows the equation of motion and the dynamic consolidation equation to be expressed in terms of only two unknowns ($\Delta\ddot{u}$ and Δp) as shown in Equations (14) and (15).

$$m\ddot{u}_{t_{k+1-\alpha_m}} + c\dot{u}_{t_{k+1-\alpha_f}} + ku_{t_{k+1-\alpha_f}} + lp_{t_{k+1-\alpha_f}} = 0 \tag{12}$$

where m, c, k and l are the scalar forms of the matrices $[M]$, $[C]$, $[K]$ and $[L]$ respectively.

$$-(1-\alpha_f)\varphi(p_k+\beta\Delta p)-(1-\alpha_f)s\frac{\Delta p}{\Delta t}+(1-\alpha_f)l^T\frac{\Delta u}{\Delta t}=0 \tag{13}$$

where φ and s are the scalar forms of the matrices $[\phi]$ and $[S]$ respectively.

$$\begin{Bmatrix} [4m(1-2\alpha_m)+2c\Delta t(2\delta-1)(1-2\alpha_f)+k\Delta t^2(4\alpha-2\delta)(1-2\alpha_f)]z^3 \\ +[4m+2c\Delta t(2\delta-2\alpha_f)+k\Delta t^2(4\alpha-4\delta\alpha_f+2\alpha_f-1)]z^2 \\ +[2c\Delta t+k\Delta t^2(2\delta-2\alpha_f)]z \\ +k\Delta t^2 \end{Bmatrix}\Delta\ddot{u} + \begin{Bmatrix} 4l(1-2\alpha_f)z^3 \\ +(4l)z^2 \end{Bmatrix}\Delta p = 0 \tag{14}$$

$$\begin{Bmatrix} l^T\Delta t(4\alpha-2\delta)(1-\alpha_f)z^2 \\ +l^T\Delta t(2\delta-1)(1-\alpha_f)z \\ +l^T\Delta t(1-\alpha_f) \end{Bmatrix}\Delta\ddot{u} - \begin{Bmatrix} \varphi(4\beta-2)(1-\alpha_f)+4\dfrac{s}{\Delta t}(1-\alpha_f)z^2 \\ +2\varphi(1-\alpha_f)z \end{Bmatrix}\Delta p = 0 \tag{15}$$

For convenience, applying A, B, C and D to represent the multipliers in front of $\Delta\ddot{u}$ and Δp, Equations (14) and (15) can be written as the following Equation (16):

$$\begin{bmatrix} A & B \\ C & -D \end{bmatrix}\begin{Bmatrix} \Delta\ddot{u} \\ \Delta p \end{Bmatrix} = 0 \tag{16}$$

For a non-trivial solution, the determinant of matrix $\begin{bmatrix} A & B \\ C & -D \end{bmatrix}$ must be zero which leads to a polynomial of z expressed by Equation (17):

$$F(z)=\begin{vmatrix} A & B \\ C & -D \end{vmatrix}=A(-D)-BC=0 \tag{17}$$

where $F(z)$ is the polynomial of z. After substituting the original forms of A, B, C and D into Equation (17), $F(z)$ is expressed by the following equation:

$$F(z)=a_0 z^5+a_1 z^4+a_2 z^3+a_3 z^2+a_4 z=0 \tag{18}$$

where

$$a_0=(a+b+c)(o+p)+lj$$
$$a_1=(a+b+c)q+(d+e+f)(o+p)+lk+mj$$
$$a_2=(d+e+f)q+(g+h)(o+p)+mk+nj$$
$$a_3=(g+h)q+i(o+p)+nk$$
$$a_4=iq$$

where *a*, *b*, …,*q* are illustrated in Figure 1.

$$\left\{\begin{array}{c} \Big[\underbrace{4m(1-2\alpha_m)}_{a}+\underbrace{2c\Delta t(2\delta-1)(1-2\alpha_f)}_{b}+\underbrace{k\Delta t^2(4\alpha-2\delta)(1-2\alpha_f)}_{c}\Big]z^3 \\ +\Big[\underbrace{4m}_{d}+\underbrace{2c\Delta t(2\delta-2\alpha_f)}_{e}+\underbrace{k\Delta t^2(4\alpha-4\delta\alpha_f+2\alpha_f-1)}_{f}\Big]z^2 \\ +\Big[\underbrace{2c\Delta t}_{g}+\underbrace{k\Delta t^2(2\delta-2\alpha_f)}_{h}\Big]z \\ +\underbrace{k\Delta t^2}_{i} \end{array}\right\}\Delta\ddot{u}+\left\{\begin{array}{c} \underbrace{4l(1-2\alpha_f)}_{j}z^3 \\ +\underbrace{(4l)}_{k}z^2 \end{array}\right\}\Delta p=0$$

$$\left\{\begin{array}{c} \underbrace{l^T\Delta t(4\alpha-2\delta)(1-\alpha_f)}_{l}z^2 \\ +\underbrace{l^T\Delta t(2\delta-1)(1-\alpha_f)}_{m}z \\ +\underbrace{l^T\Delta t(1-\alpha_f)}_{n} \end{array}\right\}\Delta\ddot{u}-\left\{\begin{array}{c} \Big[\underbrace{\varphi(4\beta-2)(1-\alpha_f)}_{o}+\underbrace{4\frac{s}{\Delta t}(1-\alpha_f)}_{p}\Big]z^2 \\ +\underbrace{2\varphi(1-\alpha_f)}_{q}z \end{array}\right\}\Delta p=0$$

Figure 1: Illustration of the parameters in the characteristic polynomial

Based on the Routh-Hurwitz condition for stability, in order for the CH method to be stable, the coefficients in Equation (18) should obey the conditions of $a_0>0 \; and \; a_i \geq 0 \; for \; i>0$. However, based on the expressions of a_0 to a_4, the time step Δt is included in the terms a_0 to a_4, which means that the stability conditions depend on the time step and therefore the CH method is only conditionally stable. In order to obtain unconditional stability for the CH method, the time step should be separated from the stability conditions. Hence, every single term of a_0 to a_4 (i.e. a, b, …,q) should be equal or larger than zero, which leads to the following unconditional stability conditions for the CH method:

$$a,b,\ldots,q \geq 0 \tag{19}$$

Based on Chung and Hulbert [6], the CH method achieves second accuracy and maximum high frequency dissipation when $\delta=0.5-\alpha_m+\alpha_f$ and $\alpha=0.25(1-\alpha_m+\alpha_f)^2$. These two expressions for δ and α are employed herein to derive the unconditional stability conditions only in terms of α_m, α_f and β. Hence, after substituting the two expressions into Equation (19), the final unconditional stability conditions of the CH method for the coupled formulation are expressed by Equation (20).

$$\alpha_m \le \alpha_f \le 0.5 \quad and \quad \beta \ge 0.5 \tag{20}$$

In the proposed stability conditions, the former condition is based on the equation of motion and the latter one is based on the dynamic consolidation equation. It is obvious that the stability conditions can reduce to $\alpha_m \le \alpha_f \le 0.5$ by ignoring the hydraulic coupling, which are identical to the unconditional stability conditions of the CH method for the one-phase formulation proposed by Chung and Hulbert [6].

4 VALIDATION OF STABILITY CONDITIONS

A one-dimensional (1-D) soil column, assuming linear elastic soil behaviour and plane strain conditions, is analysed considering a range of dynamic loading conditions. The CH method is utilised for the simulations and the numerical results are firstly compared with either a closed form solution or an existing numerical investigation. Furthermore, parametric studies of varying integration parameters are conducted, in order to validate the proposed stability conditions. It should be noted that zero material damping is employed for all the numerical simulations in this paper, for the consistency with the assumptions of the theoretical stability analysis.

4.1 1-D Column Subjected to Step Loading

The first example concerns a soil column subjected to step loading. The finite element mesh (consisting of 500 4-noded elements) and the boundary conditions are shown in Figure 2. For the hydraulic boundary conditions, pore water pressure is prescribed as zero at the top of the mesh and the remaining boundaries are considered to be impermeable. Furthermore, a uniformly distributed load is applied on the top boundary as illustrated in Figure 3. The soil properties are shown in Table 1. Lastly, the values for the integration parameters of the CH method are listed in Table 2, which satisfy the proposed unconditional stability conditions.

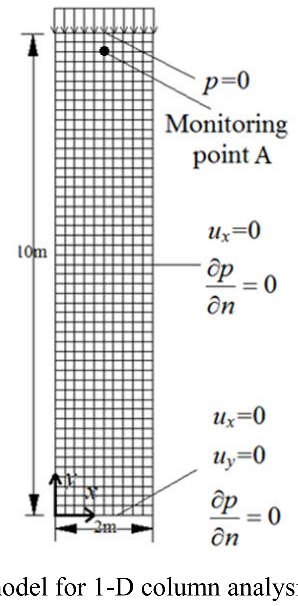


Figure 2: Finite element model for 1-D column analysis subjected to step loading

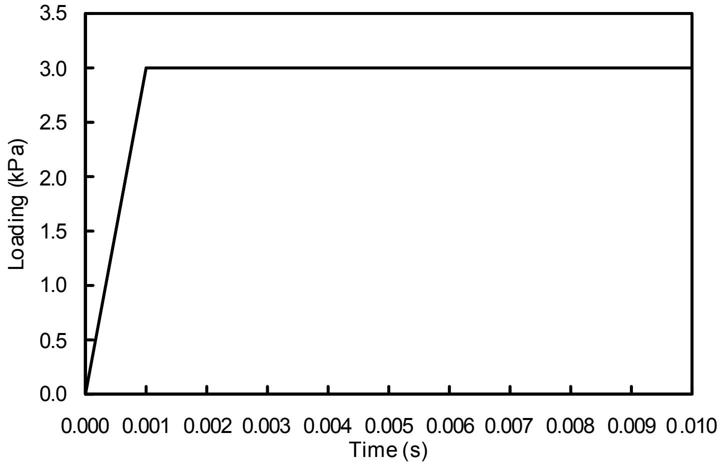


Figure 3: The loading history for 1-D column analysis subjected to step loading

Table 1 Soil properties for 1-D column analysis subjected to step loading

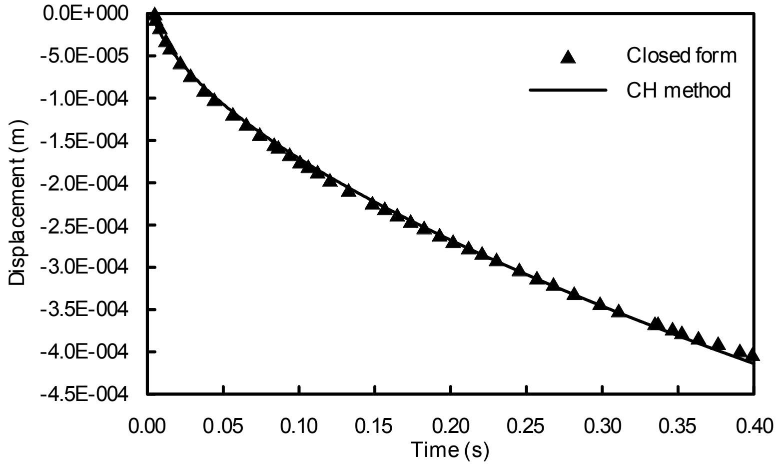
Parameter	Value
Yong's modulus E (kPa)	3.0E+04
Density ρ (g/cm^3)	1.67
Poisson ratio ν	0.2
Void ratio e	0.5
Permeability k (m/s)	1.0E-02
Time step Δt (s)	1.0E-03

Table 2 Integration parameters for the CH method

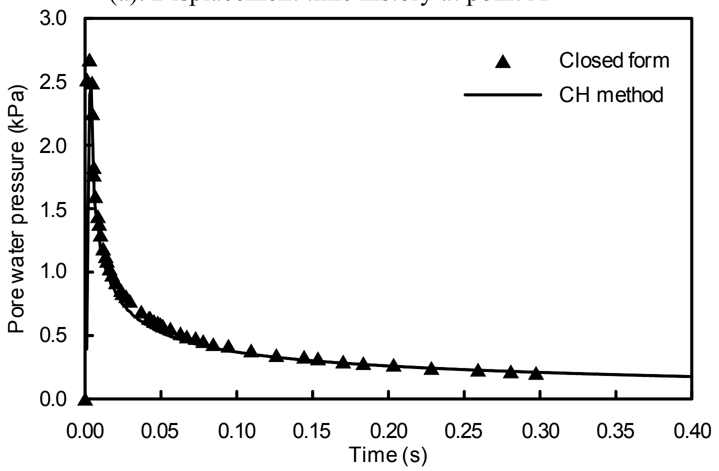
Parameter	δ	α	α_m	α_f	β
CH method	0.6	0.3025	0.35	0.45	0.8

The displacement and pore water pressure time histories[a] are calculated at a monitoring point A (0.4m below the top boundary as shown in Figure 2). These two sets of results are compared with a closed form solution, which was proposed by de Boer et al. [7] to simulate the dynamic response of saturated porous media. Figure 4 shows the comparison between numerical analysis results and the closed form solution. It can be seen that both the displacement and pore water pressure time histories are accurately predicted by the numerical analysis.

a: Downward displacements are shown as negative values and compressive pore water pressures are shown as positive values in the results of this paper.



(a): Displacement time history at point A

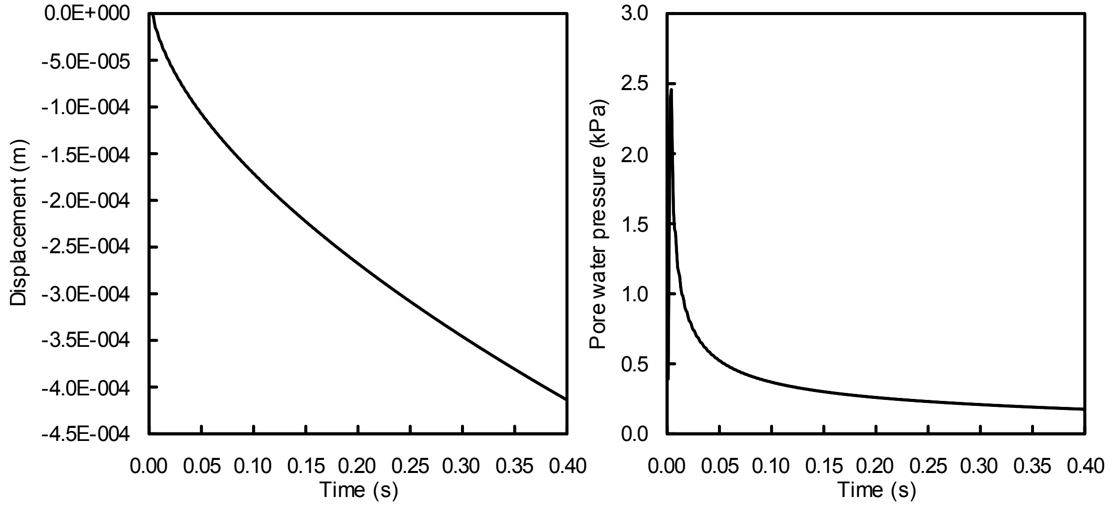


(b): Pore water pressure time history at point A

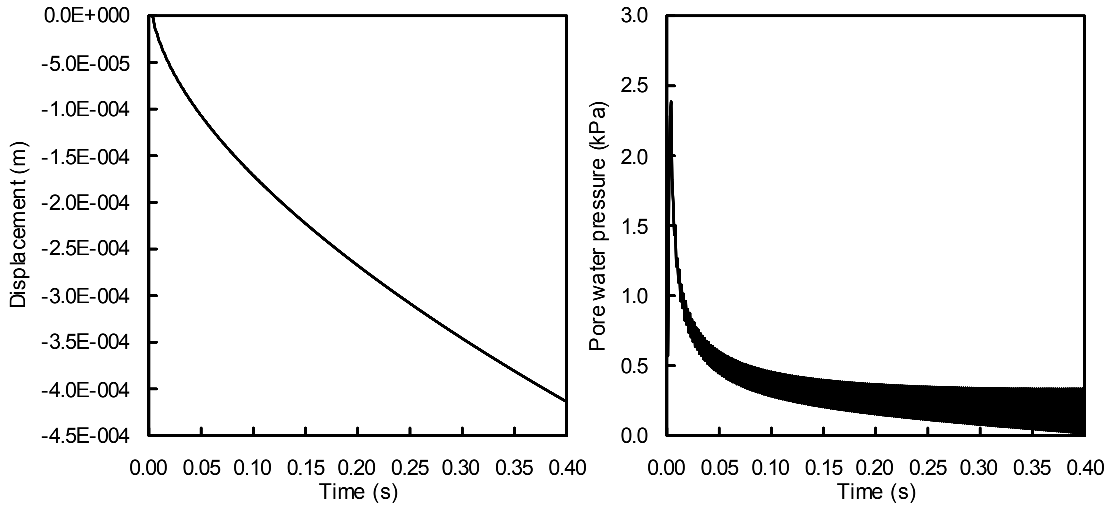
Figure 4: Comparison of numerical results and the closed form solution of de Boer at al.

In the following analyses, the integration parameters are varied parametrically, in order to validate the stability conditions introduced in the previous section. As it was mentioned before, the parameters α_m, α_f and β govern the stability conditions for the CH method, which are $\alpha_m \leq \alpha_f \leq 0.5 \; and \; \beta \leq 0.5$. Since the stability of the CH method for the one-phase formulation has been previously thoroughly checked by other studies, the focus of the present investigation is only on the stability for the coupled formulation. Therefore, for the following validation analyses, only the effect of parameter β is examined. In particular, β is gradually reduced in order to reach numerically the stability limit for the CH method.

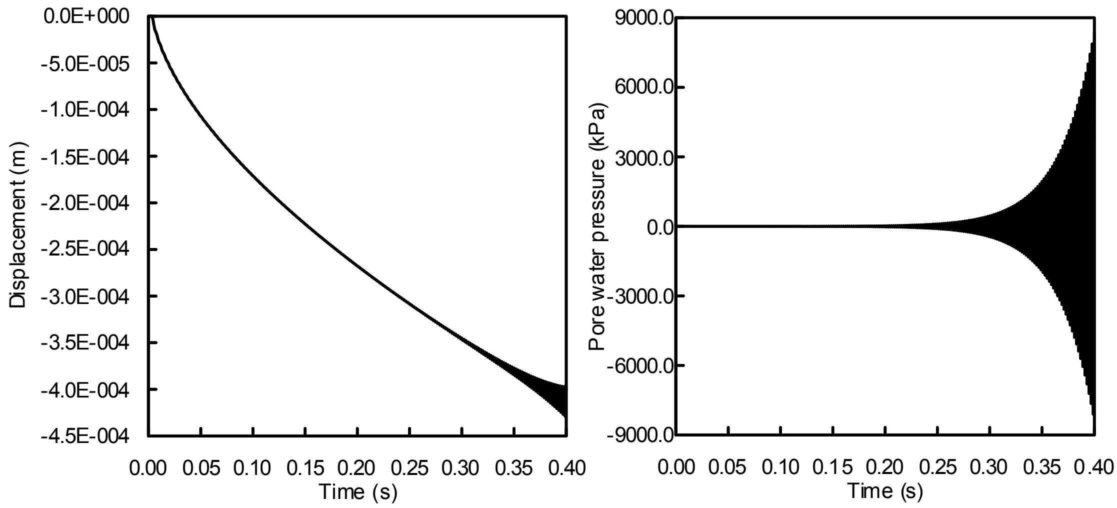
Figure 5 shows the displacement and pore water pressure time histories recorded at the monitoring point A for various values of β. Clearly, when β =0.8 (Figure 5a), both the displacement and pore water pressure responses are stable. When, however, β is reduced to a value slightly less than 0.5 (0.499), an instability in the pore water pressure response is observed immediately. Nevertheless, the displacement still remains stable (Figure 5b). Finally, when β is reduced further to 0.492 , the instability in terms of pore water pressure aggravates, while the displacement response also starts to be unstable (Figure 5c). Consequently, based on this example, in order to ensure stability of the numerical solution when using the CH method, β should be equal or larger than 0.5, which is in agreement with the proposed theoretical unconditional stability conditions. Furthermore, the pore water pressure response seems to be more sensitive to the β value than the displacement response.



(a): Displacement and pore water pressure time histories (β=0.8)



(b): Displacement and pore water pressure time histories (β=0.499)



(c): Displacement and pore water pressure time histories (β=0.492)

Figure 5: Displacement and pore water pressure time histories at point A for various β values

4.2 1-D Column Subjected to Harmonic Loading

To validate the stability conditions of the CH method when subjected to more complicated loading, 1-D soil column analyses under harmonic loading are carried out in this part. The finite element mesh (consisting of 80 8-noded elements) and the boundary conditions are shown in Figure 6. The displacement and hydraulic boundary conditions are identical to the ones used in the previous section. Furthermore, a uniformly distributed sinusoidal load is applied on the top boundary of the model, which is described by Equation (21) and is illustrated in Figure 7. Lastly, the soil properties of the model are listed in Table 3.

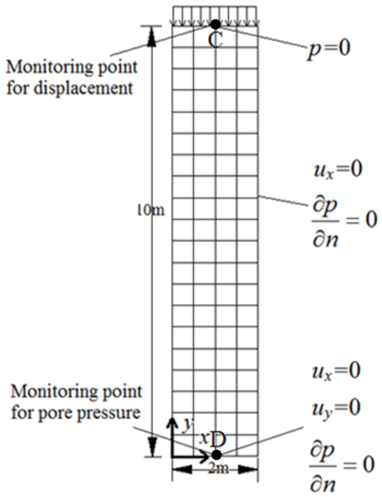


Figure 6: Finite element model for 1-D column analysis subjected to harmonic loading

$$q = \begin{cases} t/0.02 & 0 \le t \le 0.02s \\ 1 + 0.25\sin 20\pi(t - 0.02) & t > 0.02s \end{cases} \tag{21}$$

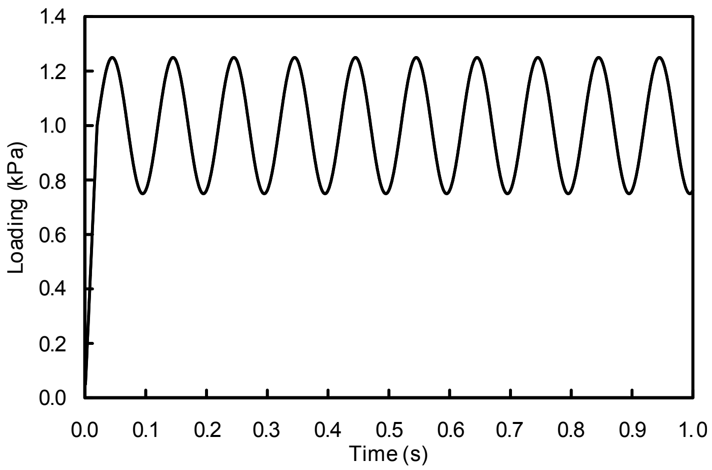


Figure 7: The loading history for 1-D column analysis subjected to harmonic loading

Table 3 Soil properties for 1-D column analysis subjected to harmonic loading

Parameter	Case1
Yong's modulus E (kPa)	1.0E+04
Density ρ (g/cm^3)	2.0
Poisson ratio ν	0.2
Void ratio e	0.538
Permeability k (m/s)	1.0E-15
Time step Δt (s)	1.0E-03

Firstly, the numerical predictions of this study are compared with the results presented by Li et al. [15], where an iterative stabilised fractional step algorithm was used. It should be noted that an impermeable soil column was simulated by Li et al. In the present study, a very low permeability value (k=1.0E-15 m/s) is adopted to represent an impermeable material but still utilise the two-phase coupled formulation. Figure 8 shows the comparison of the pore water pressure time history at the point D, which shows good agreement between the numerical prediction and Li et al.'s results.

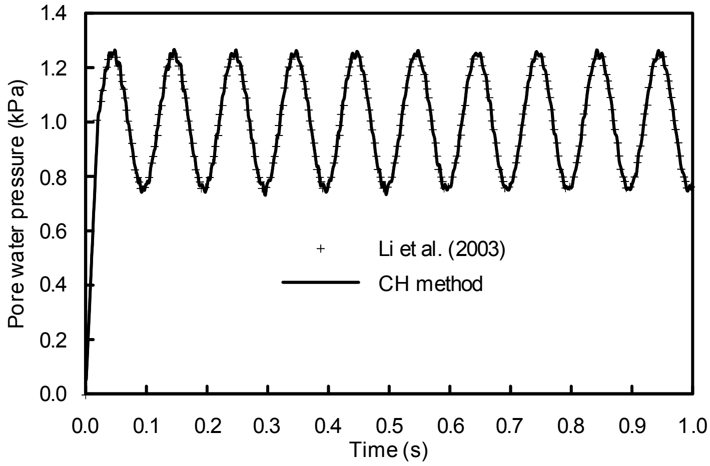
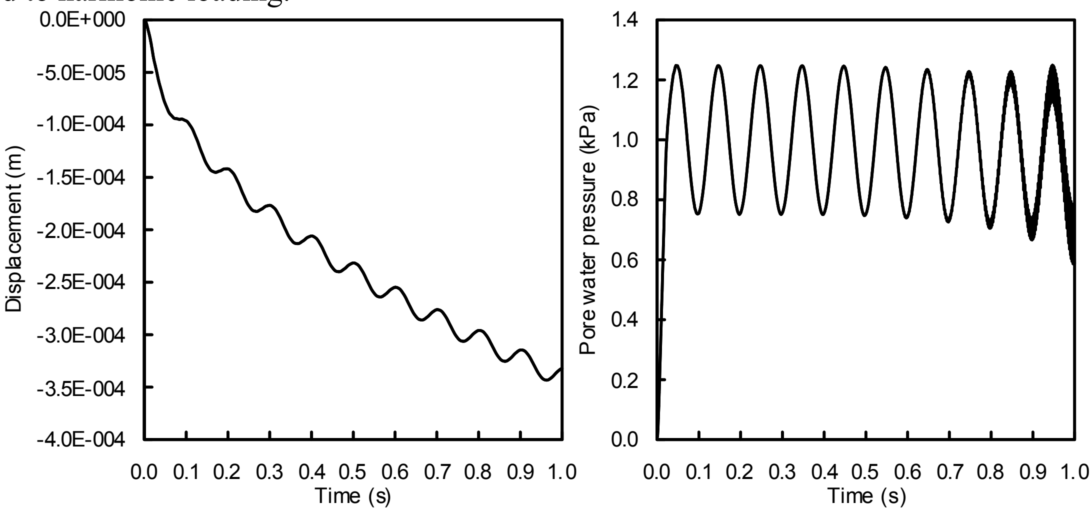
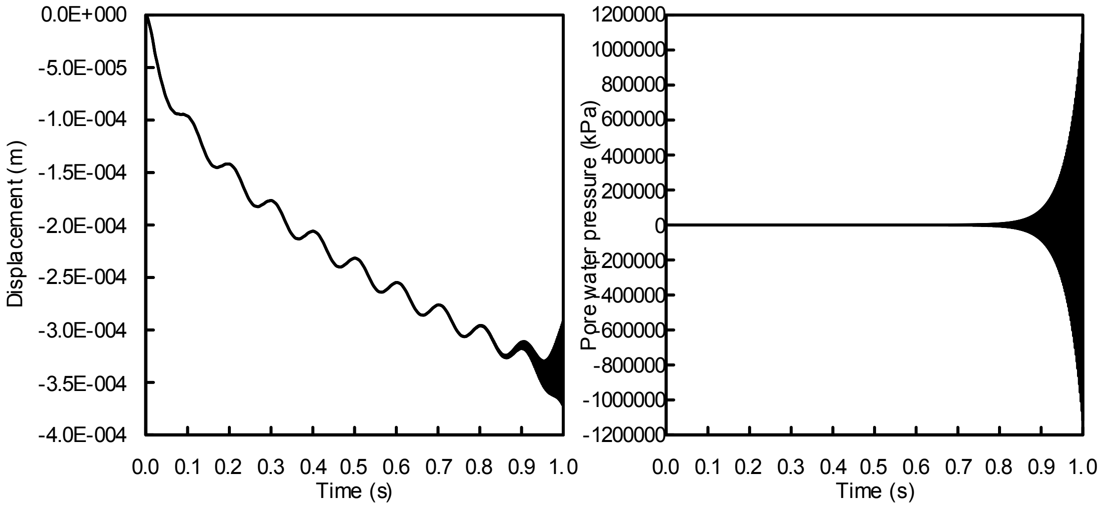


Figure 8: Comparison of the pore water pressure time history at point D

In order to highlight the effect of the parameter β on the stability, a higher permeability value (k=1.0E-2 m/s) is adopted for the soil column in the following parametric investigations. Similar to the analyses conducted in the previous section, β is gradually reduced to reach numerically the stability limit. The analysis results in terms of displacement (at point C) and pore water pressure (at point D) time histories are shown in Figure 9. It is obvious that when β is reduced to 0.497, the pore water pressure response starts to be unstable immediately, while the displacement response is still stable (Figure 9a). Furthermore, when β is reduced to 0.493, the instability of the pore water pressure aggravates and this triggers the instability of the displacement response (Figure 9b). Therefore, good agreement is observed between the numerical results and the theoretical stability conditions for the 1-D column analysis subjected to harmonic loading.



(a): Displacement and pore water pressure time histories (β=0.497)



(b): Displacement and pore water pressure time histories (β=0.493)

Figure 9: Displacement (at point C) and pore water pressure (at point D) time histories for various β values

5 CONCLUSIONS

The stability of the CH method for a fully coupled dynamic consolidation formulation is analytically investigated and unconditional stability conditions are proposed. It is observed that the proposed stability conditions can simplify to the ones of the CH method for the one-phase formulation by ignoring the hydraulic coupling. In addition, since the CH method is a generalisation of the Newmark, HHT and WBZ methods, the proposed stability conditions are relevant for most of the commonly used time integration methods for two-phase coupled formulation.

The analytically derived stability conditions are validated with FE analyses considering a range of loading conditions. According to the numerical results, in order to ensure the stability for the coupled dynamic analysis, the integration parameter for the hydraulic coupling formulation, β, should be equal or larger than 0.5, which agrees with the proposed theoretical unconditional stability conditions of the CH method for the solid-pore fluid coupled formulation.

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APPENDIX

$$[M] = \sum_{i=1}^{N} \left(\int_{Vol} [N]^T \rho [N] dVol \right)$$

$$[C]=\sum_{i=1}^{N}\left(\int_{Vol}[N]^{T}c[N]dVol\right)$$

$$[K]=\sum_{i=1}^{N}\left(\int_{Vol}[B]^{T}[D][B]dVol\right)$$

$$[L]=\sum_{i=1}^{N}\left(\int_{Vol}\{m\}[B]^{T}[N_p]dVol\right)$$

$$\{m\}^{T}=\{1\quad 1\quad 1\quad 0\quad 0\quad 0\}$$

$$[\phi]=\sum_{i=1}^{N}\left(\int_{Vol}\frac{1}{\gamma_f}[E]^{T}[k][E]dVol\right)$$

$$[G]=\sum_{i=1}^{N}\left(\int_{Vol}\frac{1}{g}[N]^{T}[k][E]dVol\right)$$

$$[S]=\sum_{i=1}^{N}\left(\int_{Vol}\frac{n}{K_f}[N_p]^{T}[N_p]dVol\right)$$

$$[n]=\sum_{i=1}^{N}\left(\int_{Vol}[E]^{T}[k]\{i_G\}dVol\right)$$

$$[E]=\begin{bmatrix}\frac{\partial N_p}{\partial x} & \frac{\partial N_p}{\partial y} & \frac{\partial N_p}{\partial z}\end{bmatrix}^{T}$$

$$\Delta\overline{R}=(1-\alpha_f)\{\Delta R\}+(1-\alpha_m)[M]\left(\frac{1}{\alpha\Delta t}\{\dot{u}_{t_k}\}+\frac{1}{2\alpha}\{\ddot{u}_{t_k}\}\right)+(1-\alpha_f)[C]\left(\frac{\delta}{\alpha}\{\dot{u}_{t_k}\}-\left(1-\frac{\delta}{2\alpha}\right)\Delta t\{\ddot{u}_{t_k}\}\right)+\{\Delta R\}_{t_k}$$

$$\Delta\overline{F}=(1-\alpha_f)\Delta t\left(-[n]-\{Q\}+[\phi]\{p_{t_k}\}-[G]\{\ddot{u}_{t_k}\}\right)+(1-\alpha_m)\beta\Delta t[G]\left(\frac{1}{\alpha\Delta t}\{\dot{u}_{t_k}\}+\frac{1}{2\alpha}\{\ddot{u}_{t_k}\}\right)+\{\Delta F\}_{t_k}$$

where $[N]$ is the shape function matrix, $[B]$ is the derivative matrix of the shape function, $[D]$ is the constitutive relation matrix, $[N_p]$ *is* the shape function matrix for the pore fluid, ρ and c are the material density and damping ratio of the material, $[k]$ is the permeability matrix, n and K_f are the porosity of the soil and the bulk modulus of the pore fluid, i_G are the unit vectors, which is opposite to the gravity, γ_f is the bulk unit weight for the pore fluid, Q is any sink or/and sources, and $\{\Delta R_{t_k}\}$ and $\{\Delta F_{t_k}\}$ are the out-of-balance force at the previous time increment, which ideally should be zero.