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FREE VIBRATIONS OF A CYLINDRICAL SHELL MADE OF AN ANISOTROPIC MATERIAL

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Abstract. *Two-dimensional model of shell made of an anisotropic material described by 21 elastic moduli is delivered. It is shown that, in the general case (in contrary to isotropic or orthotropic materials), both the Kirchhoff-Love and the Timoshenko-Reissner hypotheses result in 2D models, which are inconsistent in the major terms with respect to the small thickness parameter equal to the ratio between the thickness of shell and the wavelength. For this aim the generalized Timoshenko-Reissner hypothesis is proposed. The asymptotic analysis of the obtained 2D shell model is used to investigate the cylindrical shell free vibrations. The main types of the vibration modes are described and they are compared with the modes of the isotropic shell. The membrane modes are investigated. The system of Donnell type is delivered. This system is used to investigate free vibrations of a circular cylindrical shell. The example of material with the general anisotropy is given.*

1 INTRODUCTION

Numerous investigations including monographs [1–6] are devoted to derivation of two-dimensional (2D) models of plates and shells from three-dimensional (3D) equations of the theory of elasticity. The presence of anisotropy of material introduces the additional difficulties. The 2D models for isotropic and transversely isotropic materials [2–8] have been investigated in detail. The general anisotropy described by 21 elastic moduli have not been sufficiently investigated [9]. The problem of anisotropic beam in the case of the general anisotropy is considered in [10], and the 3D stress state is studied in [11]. It is shown that, in the general case (in contrast to isotropic or orthotropic materials), both the Kirchhoff–Love (KL) and Timoshenko–Reissner (TR) hypotheses lead to 2D models, which are incorrect in the principal terms with respect to the small thickness parameter. The generalized TR hypothesis is used in [10,11] for deriving the 2D model.

The hypotheses accepted in [10,11] should be defined more accurately because some boundary conditions at the upper and lower planes of the plate are not satisfied. More accurate hypotheses, free from this imperfection, are introduced in [12] for a beam and these hypotheses are used for a shell in the present paper. This definition is essential in the case of comparatively small elastic moduli in the transversal directions and also in construction of a boundary layer. As is well known (see [1,2,8]) in the TR model the effective transverse shear moduli kG contains the correcting factor k which depends on the studied problem (here $k = 5/6$). The hypotheses [10,11] lead to $k = 1$.

Under the assumption that the transverse elastic moduli are of the same order as the tangential ones the asymptotic analysis and simplification of the obtained system are made. Our aim is to deliver the system with a minimal exactness however such that the principal terms with respect to the small thickness parameter are correct. We assume that the external load acting on the shell varies slowly. Then (see [7,8]) the stress-strain state (SSS) of the shell consists of three parts: the main membrane state with the small variability, the edge effect with the intermediate variability which is standard in the KL model, and the boundary layer with the large variability which appears only in the TR model. The wave length in the boundary layer is of the order of shell thickness h and the corresponding SSS quickly decreases away from the edges. The boundary layer SSS is poorly described by 2D models and it is necessary to use 3D equations [6]. One of the aims of the presented simplification is to exclude the boundary layer from the consideration. In this case the differential order of the system decreases from 10 to 8, and the number of boundary conditions decreases from 5 to 4.

The asymptotic analysis of the obtained 2D shell model is used to investigate the cylindrical shell free vibrations. The main types of the vibration modes are described and they are compared with the modes of the isotropic shell. The membrane modes are investigated. The system of Donnell type is delivered, and this system is used to investigate the low-frequency free vibrations of a circular cylindrical shell. The example of material with the general anisotropy is given.

2 THE MAIN ASSUMPTIONS AND ELASTICITY RELATIONS

Consider a thin circular cylindrical shell of radius R and of the constant thickness h. We introduce curvilinear co-ordinates $x_1 = Rs,\ x_2 = R\theta$ in the midsurface coinciding with the longitudinal and circumferential directions. The third co-ordinate $x_3 = z$ ($|z| \leq h/2$) coincides with the midsurface normal (Fig. 1). The Lame coefficients of the shell body are $H_1 = R,\ H_2 = R + z,\ H_3 = 1$. We neglect h/R compared with 1, and put $H_2 \simeq R$.

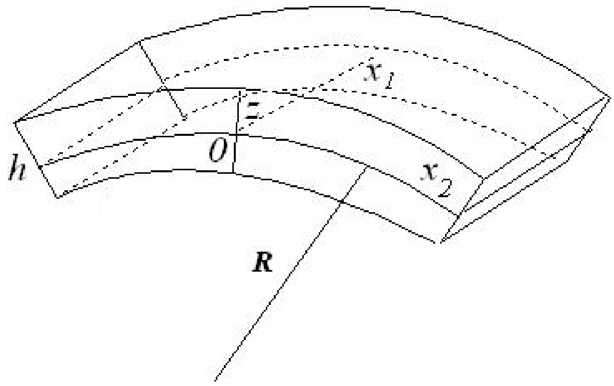

Figure 1: The shell.

Consider a material with the general anisotropy containing 21 elastic moduli. To describe elasticity relations it is more convenient to use matrix designations instead of the tensor of 4th rank. Let us introduce strains ε_{ij} in cylindrical co-ordinate system as

$$\begin{aligned} &\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \qquad &&\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}, \qquad &&\varepsilon_{13} = \frac{\partial u_1}{\partial z} + \frac{\partial w}{\partial x_1}, \\ &\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} + \frac{w}{R}, \qquad &&\varepsilon_{23} = \frac{\partial u_2}{\partial z} + \frac{\partial w}{\partial x_2} - \frac{u_2}{R}, \qquad &&\varepsilon_{33} = \frac{\partial w}{\partial z}. \end{aligned} \tag{1}$$

Writing the relations between the stresses σ_{ij} and the strains ε_{ij} in the shell body we divide them into groups of tangential $\boldsymbol{\sigma}_t,\ \boldsymbol{\varepsilon}_t$ and the non-tangential $\boldsymbol{\sigma}_n,\ \boldsymbol{\varepsilon}_n$ stresses and strains,

$$\begin{aligned} &\boldsymbol{\sigma}_t = (\sigma_{11},\, \sigma_{12},\, \sigma_{22})^T, \qquad &&\boldsymbol{\sigma}_n = (\sigma_{13},\, \sigma_{23},\, \sigma_{33})^T, \\ &\boldsymbol{\varepsilon}_t = (\varepsilon_{11},\, \varepsilon_{12},\, \varepsilon_{22})^T, \qquad &&\boldsymbol{\varepsilon}_n = (\varepsilon_{13},\, \varepsilon_{23},\, \varepsilon_{33})^T; \end{aligned} \tag{2}$$

the superscript ^{T} designates the transposition. Then the relations between stresses and strains are written in the matrix form

$$\boldsymbol{\sigma}_t = \mathbf{A}\cdot\boldsymbol{\varepsilon}_t + \mathbf{B}\cdot\boldsymbol{\varepsilon}_n, \quad \boldsymbol{\sigma}_n = \mathbf{B}^T\cdot\boldsymbol{\varepsilon}_t + \mathbf{C}\cdot\boldsymbol{\varepsilon}_n, \tag{3}$$

where $\mathbf{A} = \{A_{ij}\},\ \mathbf{B} = \{B_{ij}\},\ \mathbf{C} = \{C_{ij}\}$ are the elastic-moduli matrices, the matrices $\mathbf{A}$ and $\mathbf{C}$ being symmetric. Relations (3) contain 21 elastic moduli. It is assumed that the matrix

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{pmatrix}$$

of the 6th order is positively determined.

Assume that the face surfaces $z = \pm h/2$ are free

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0, \quad z = \pm h/2. \tag{4}$$

As in the KL and the TR models we accept that the normal stresses can be neglected $\sigma_{33} \equiv 0$, and that the normal deflection w does not depend on z. Using the relation $\sigma_{33} = 0$ we exclude the strain ε_{33} from the relations (2). Then they become

$$\boldsymbol{\sigma}_t = \mathbf{A}^* \cdot \boldsymbol{\varepsilon}_t + \mathbf{B}^* \cdot \boldsymbol{\varepsilon}_n^*, \quad \boldsymbol{\sigma}_n^* = \mathbf{B}^{*T} \cdot \boldsymbol{\varepsilon}_t + \mathbf{C}^* \cdot \boldsymbol{\varepsilon}_n^*, \tag{5}$$

where in contrast to (2) $\boldsymbol{\sigma}_n^* = (\sigma_{13},\, \sigma_{23})^T$, $\boldsymbol{\varepsilon}_n^* = (\varepsilon_{13},\, \varepsilon_{23})^T$. The elements of matrices $\mathbf{A}^*(3\times 3), \mathbf{B}^*(3\times 2), \mathbf{C}^*(2\times 2)$ are as follows

$$A_{ij}^* = A_{ij} - \frac{B_{i3}B_{j3}}{C_{33}}, \quad B_{ij}^* = B_{ij} - \frac{B_{i3}C_{j3}}{C_{33}}, \quad C_{ij}^* = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}. \tag{6}$$

3 THE GENERALIZED TIMOSHENKO–REISSNER MODEL

In order to obtain the 2D shell model one should accept some assumptions about the distribution of the transversal shear strains $\varepsilon_{i3},\ i=1,2$ in the normal direction. According to the KL model the relation $\varepsilon_{i3}=0$ is accepted. According to the TR model the relation $\varepsilon_{i3}=\gamma_i$ or the more exact relation $\varepsilon_{i3}=\gamma_i(1-4z^2/h^2)$ is accepted. In the case of the latter relation for the orthotropic material the boundary conditions (4) are satisfied.

Both the KL and the TR hypotheses in the case of the general anisotropy do not lead to correct principal terms with respect to the small thickness parameter (see [10,11]). Therefore as in [12] we accept

$$\varepsilon_{i3} = \gamma_i + \delta_i z + P_2^0(z)\beta_i, \quad i=1,2, \quad P_2^0(z) = z^2/h^2 - 1/12, \tag{7}$$

where the functions $\gamma_i, \delta_i, \beta_i$ do not depend on z. The functions γ_i are the average shear angles, and the functions δ_i, β_i will be chosen to satisfy the conditions (4).

Equating expressions (1) and (7) for ε_{13} and ε_{23}, we find the tangential displacements

$$u_i(z) = u_i^0 + \varphi_i z + P_2(z)\delta_i + P_3(z)\beta_i, \quad P_2 = \frac{z^2}{2} - \frac{h^2}{24}, \quad P_3 = \frac{z^3}{3h^2} - \frac{z}{12}, \tag{8}$$

where u_i^0 are the average displacements and $\varphi_1 = \gamma_1 - \partial w/\partial x_i,\ \varphi_2 = \gamma_2 + u_2/R - \partial w/\partial x_2$ are the average angles of the normal fibers rotation.

The strains $\varepsilon_{ij},\ i,j=1,2$ are

$$\begin{aligned}
&\boldsymbol{\varepsilon}_t = (\varepsilon_{11},\varepsilon_{12},\varepsilon_{22})^T = \boldsymbol{\varepsilon}_t^0 + z\mathbf{K}(\boldsymbol{\varphi}) + P_2(z)\mathbf{K}(\boldsymbol{\delta}) + P_3(z)\mathbf{K}(\boldsymbol{\beta}),\\
&\mathbf{K}(\boldsymbol{\xi}) \equiv \left(\frac{\partial\xi_1}{\partial x_1},\ \frac{\partial\xi_1}{\partial x_2}+\frac{\partial\xi_2}{\partial x_1},\ \frac{\partial\xi_2}{\partial x_2}\right)^T, \qquad \boldsymbol{\varepsilon}_t^0 = \mathbf{K}(\boldsymbol{u}^0) + \left(0,0,\frac{w}{R}\right)^T.
\end{aligned} \tag{9}$$

Here $\boldsymbol{\varepsilon}_t^0$ are the tangential strains of the midsurface. The operator $\mathbf{K}(\cdot)$ will be applied to the 2D vectors $\boldsymbol{\gamma}=(\gamma_1,\gamma_2)^T$, $\boldsymbol{\varphi}=(\varphi_1,\varphi_2)^T$, $\boldsymbol{\delta}=(\delta_1,\delta_2)^T$, $\boldsymbol{\beta}=(\beta_1,\beta_2)^T$. According to relations (7) and (9) the boundary conditions (4) are satisfied if

$$\mathbf{B}^{*T}\cdot\mathbf{K}(\boldsymbol{\varphi}) + \mathbf{C}^*\cdot\boldsymbol{\delta} = 0, \qquad \mathbf{B}^{*T}\cdot\left(\boldsymbol{\varepsilon}_t^0 + h^2\mathbf{K}(\boldsymbol{\delta})/12\right) + \mathbf{C}^*\cdot(\boldsymbol{\gamma}+\boldsymbol{\beta}/6) = 0.$$

Omitting the small term $h^2\mathbf{K}(\boldsymbol{\delta})/12$ we find

$$\boldsymbol{\delta} = -(\mathbf{C}^*)^{-1}\cdot\mathbf{B}^{*T}\cdot\mathbf{K}(\boldsymbol{\varphi}), \quad \boldsymbol{\beta} = -6\left(\boldsymbol{\gamma} + (\mathbf{C}^*)^{-1}\cdot\mathbf{B}^{*T}\cdot\boldsymbol{\varepsilon}_t^0\right). \tag{10}$$

Then we re-write the expressions (7) and (9) for the strains in the form

$$\begin{aligned}
&\boldsymbol{\varepsilon}_n^* = \boldsymbol{\gamma} - z(\mathbf{C}^*)^{-1}\cdot\mathbf{B}^{*T}\cdot\mathbf{K}(\boldsymbol{\varphi}) - 6P_2^0(z)\left(\boldsymbol{\gamma} + (\mathbf{C}^*)^{-1}\cdot\mathbf{B}^{*T}\cdot\boldsymbol{\varepsilon}_t^0\right),\\
&\boldsymbol{\varepsilon}_t = \boldsymbol{\varepsilon}_t^0 + z\mathbf{K}(\boldsymbol{\varphi}) - P_2(z)\mathbf{K}\left((\mathbf{C}^*)^{-1}\cdot\mathbf{B}^{*T}\cdot\mathbf{K}(\boldsymbol{\varphi})\right) - 6P_3(z)\mathbf{K}\left(\boldsymbol{\gamma} + (\mathbf{C}^*)^{-1}\cdot\mathbf{B}^{*T}\cdot\boldsymbol{\varepsilon}_t^0\right).
\end{aligned} \tag{11}$$

Using strains (11) we calculate stresses $\boldsymbol{\sigma}_t$ and $\boldsymbol{\sigma}_n^*$ by relations (5). The stress-resultants T_{ij}, Q_i and the stress-couples M_{ij} are to be found over integration in the thickness direction

$$\{T_{ij}, Q_i\} = \int_{-h/2}^{h/2} \{\sigma_{ij}, \sigma_{i3}\}\, dz, \quad \{M_{ij}\} = \int_{-h/2}^{h/2} \{\sigma_{ij}\}\, z dz, \quad i,j = 1,2. \tag{12}$$

In the case when the elastic moduli **A**, **B**, **C** do not depend on z relations (12) can be rewritten as

$$\begin{aligned}
&\mathbf{T} = h\left(\mathbf{A}^* \cdot \boldsymbol{\varepsilon}_t^0 + \mathbf{B}^* \cdot \boldsymbol{\gamma}\right), && \mathbf{T} = (T_{11}, T_{12}, T_{22})^T, \\
&\mathbf{Q} = h\left(\mathbf{B}^{*T} \cdot \boldsymbol{\varepsilon}_t^0 + \mathbf{C}^* \cdot \boldsymbol{\gamma}\right), && \mathbf{Q} = (Q_1, Q_2)^T, \\
&\mathbf{M} = J\left(\mathbf{A}^{**} \cdot \mathbf{K}(\boldsymbol{\varphi}) + \mathbf{A}^* \cdot \mathbf{K}(\boldsymbol{\gamma} + (\mathbf{C}^*)^{-1} \cdot \mathbf{B}^{*T} \cdot \boldsymbol{\varepsilon}_t^0)/5\right), && \mathbf{M} = (M_{11}, M_{12}, M_{22})^T,
\end{aligned} \tag{13}$$

where

$$J = \frac{h^3}{12}, \quad \boldsymbol{\gamma} = \boldsymbol{\varphi} + \nabla w, \quad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)^T,$$

$$\mathbf{A}^{**} = \mathbf{A}^* - \mathbf{B}^* \cdot \mathbf{C}^{*-1} \cdot \mathbf{B}^{*T} = \mathbf{A} - \mathbf{B} \cdot \mathbf{C}^{-1} \cdot \mathbf{B}^T. \tag{14}$$

Therefore the 2D stress-resultants and the stress-couples are expressed as functions of the main unknown variables, namely of the deflections of a midsurface u_1^0, u_2^0, w and of the average angles of the normal fibers rotation φ_1, φ_2. Further we omit the index ^{0} at u_i^0.

4 THE 2D EQUATIONS AND BOUNDARY CONDITIONS

The 2D equations are exactly the same as those in the KL and the TR models. The difference lies in the elasticity relations (13).

$$\begin{aligned}
&\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \rho h \omega^2 u_1 = 0, && \frac{\partial T_{12}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{Q_2}{R} + \rho h \omega^2 u_2 = 0, \\
&\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} - \frac{T_{22}}{R} + \rho h \omega^2 w = 0, && \\
&\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - Q_1 + \rho J \omega^2 \varphi_1 = 0, && \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - Q_2 + \rho J \omega^2 \varphi_2 = 0.
\end{aligned} \tag{15}$$

System (15) is used for the investigation of free vibrations. It is obtained after the separation of variables of type $w(x_1, x_2, t) = w(x_1, x_2)e^{i\omega t}$, where t is the time, ω is the natural frequency.

System (15) is of the 10th order. The simplest boundary conditions at the egdes $x_1 = 0$ and $x_1 = L_0$ are

$$\begin{aligned}
u_1 &= 0 \quad && or \quad T_1 = 0, \\
u_2 &= 0 \quad && or \quad T_{12} = 0, \\
w &= 0 \quad && or \quad Q_1 = 0, \\
\varphi_1 &= 0 \quad && or \quad M_1 = 0, \\
\varphi_2 &= 0 \quad && or \quad M_{12} = 0,
\end{aligned} \tag{16}$$

where from the each line of table (16) is to be taken only one condition. The shell edge element in TR model has 5 degrees of freedom with the generalized co-ordinates $u_1, u_2, w, \varphi_1, \varphi_2$. The left side of conditions (16) corresponds to the clamped co-ordinate and the right side corresponds to the free coordinate. For clamped edge

$$u_1 = u_2 = w = \varphi_1 = \varphi_2 = 0 \tag{17}$$

and for free edge

$$T_{11} = T_{12} = Q_1 = M_{11} = M_{12} = 0. \tag{18}$$

Rewrite system (15) in the dimensionless form. For this aim we put

$$h_* = \frac{h}{R}, \quad \Lambda = \frac{\rho h^2 \omega^2}{E}, \quad \{u_i, w\} = h\{\hat{u}_i, \hat{w}\}, \quad \{T_{ij}, Q_i\} = Eh\{\hat{T}_{ij}, \hat{Q}_i\}, \quad M_{ij} = ERh\hat{M}_{ij},$$

where h_* is the main small parameter equal to the relative shell thickness, Λ is the unknown frequency parameter, E is the typical value of elastic moduli in (3). After omitting superscript $\hat{}$ we get

$$\begin{aligned}
&\frac{\partial T_{11}}{\partial s} + \frac{\partial T_{12}}{\partial \theta} + \Lambda u_1 = 0, \qquad && \frac{\partial T_{12}}{\partial s} + \frac{\partial T_{22}}{\partial \theta} + Q_2 + \Lambda u_2 = 0, \\
&\frac{\partial Q_1}{\partial s} + \frac{\partial Q_2}{\partial \theta} - T_{22} + \Lambda w = 0, && \\
&\frac{\partial M_{11}}{\partial s} + \frac{\partial M_{12}}{\partial \theta} - Q_1 + \frac{h_*^2}{12}\Lambda \varphi_1 = 0, && \frac{\partial M_{12}}{\partial s} + \frac{\partial M_{22}}{\partial \theta} - Q_2 + \frac{h_*^2}{12}\Lambda \varphi_2 = 0
\end{aligned} \tag{19}$$

with

$$\begin{aligned}
&\mathbf{T} = \mathbf{A}^* \cdot \boldsymbol{\varepsilon}_t^0 + \mathbf{B}^* \cdot \boldsymbol{\gamma}, \qquad \mathbf{Q} = \mathbf{B}^{*T} \cdot \boldsymbol{\varepsilon}_t^0 + \mathbf{C}^* \cdot \boldsymbol{\gamma}, \\
&\mathbf{M} = \frac{h_*^2}{12}\left(\mathbf{A}^{**} \cdot \mathbf{K}(\boldsymbol{\varphi}) + \frac{\mathbf{A}^*}{5} \cdot \mathbf{K}\left((\mathbf{C}^*)^{-1} \cdot \mathbf{Q}\right)\right),
\end{aligned} \tag{20}$$

where the elastic moduli are related to E.

The 2D potential energy density

$$\Pi_2 = \frac{1}{2}\left(\mathbf{T}^T \cdot \boldsymbol{\varepsilon}_t^0 + \mathbf{Q}^T \cdot \boldsymbol{\gamma} + \mathbf{M}^T \cdot \mathbf{K}(\boldsymbol{\varphi})\right)$$

corresponds to the system (19).

5 EXAMPLE OF A MATERIAL WITH GENERAL ANISOTROPY

As in [10], we study the shell of the anisotropic material whose parameters are obtained by the averaging of the isotropic material (matrix) reinforced by the system of small extensible fibers. The angle between the axis z of the fiber is equal to β, and the angle between the axis x_1 and the plane containing axis z and fiber is equal to α (Fig. 2).

If the stiffness of fibers is mach larger than the stiffness of matrix, then the potential energy density Π for this anisotropic material may be accepted in the form

$$\begin{aligned}
\Pi = {} & \frac{1-b}{2}\left(\lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})^2 + 2\mu\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2\right) + \mu\left(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2\right)\right) + \\
& + \frac{1}{2} b E_f (\varepsilon_{11} a_1^2 + \varepsilon_{22} a_2^2 + \varepsilon_{33} a_3^2 + \varepsilon_{12} a_1 a_2 + \varepsilon_{13} a_1 a_3 + \varepsilon_{23} a_2 a_3)^2,
\end{aligned} \tag{21}$$

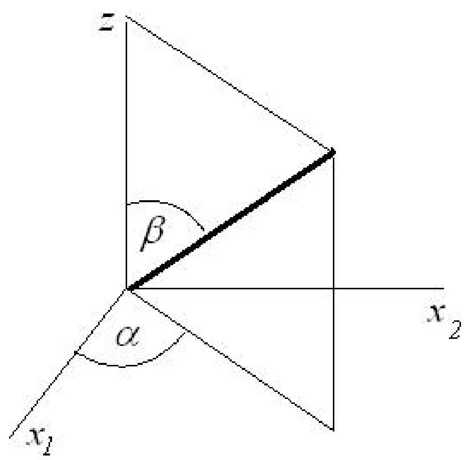

Figure 2: The direction of fibers.

with $a_1 = \cos\alpha\sin\beta,\ a_2 = \sin\alpha\sin\beta,\ a_3 = \cos\beta$, where $\lambda = E\nu/[(1+\nu)(1-2\nu)]$ and $\mu = E/[2(1+\nu)]$ are the Lame coefficients of the matrix, E_f is the Young modulus for fibers, and b is the part of volume occupied by fibers.

We assume that the stresses σ_{ij} are obtained from the relation

$$\sigma_{ij} = \frac{\partial \Pi}{\partial \varepsilon_{ij}}$$

and calculate the elements of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$

$$\begin{array}{lll}
A_{11} = E_1 + E_4 a_1^4, & A_{12} = E_4 a_1^3 a_2, & A_{13} = E_2 + E_4 a_1^2 a_2^2, \\
A_{22} = E_3 + E_4 a_1^2 a_2^2, & A_{23} = E_4 a_1 a_2^3, & A_{33} = E_1 + E_4 a_2^4, \\
B_{11} = E_4 a_1^3 a_3, & B_{12} = E_4 a_1^2 a_2 a_3, & B_{13} = E_2 + E_4 a_1^2 a_3^2, \\
B_{21} = E_4 a_1^2 a_2 a_3, & B_{22} = E_4 a_1 a_2^2 a_3, & B_{23} = E_4 a_1 a_2 a_3^2, \\
B_{31} = E_4 a_1 a_2^2 a_3, & B_{32} = E_4 a_2^3 a_3, & B_{33} = E_2 + E_4 a_2^2 a_3^2, \\
C_{11} = E_3 + E_4 a_1^2 a_3^2, & C_{12} = E_4 a_1 a_2 a_3^2, & C_{13} = E_4 a_1 a_3^3, \\
C_{22} = E_3 + E_4 a_2^2 a_3^2, & C_{23} = E_4 a_2 a_3^3, & C_{33} = E_1 + E_4 a_3^4,
\end{array} \tag{22}$$

where

$$E_1 = \frac{E(1-b)(1-\nu)}{(1+\nu)(1-2\nu)}, \quad E_2 = \frac{\nu E_1}{1-\nu}, \quad E_3 = \frac{E(1-b)}{2(1+\nu)}, \quad E_4 = bE_f. \tag{23}$$

In the next sections we study some examples and take the following values of parameters in (22), (23)

$$b = 0.05,\ \nu = 0.3,\ E_f = 20\,E,\ \alpha = \pi/6,\ \beta = \pi/4.$$

The matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are not used directly, and we take the numerical values of matrices

$\mathbf{A}^*$, $\mathbf{B}^*$, $\mathbf{C}^*$, $\mathbf{A}^{**}$ (see (6),(14))

$$\begin{aligned}
&\frac{\mathbf{A}^*}{E} = \begin{pmatrix} 1.0656 & 0.0291 & 0.3012 \\ 0.0291 & 0.4146 & -0.0162 \\ 0.3012 & -0.0162 & 1.0506 \end{pmatrix}, \qquad \frac{\mathbf{B}^*}{E} = \begin{pmatrix} 0.0582 & 0.0336 \\ 0.0784 & 0.0453 \\ -0.0323 & -0.0187 \end{pmatrix}, \\
&\frac{\mathbf{A}^{**}}{E} = \begin{pmatrix} 1.0577 & 0.0185 & 0.3056 \\ 0.0185 & 0.3903 & -0.0103 \\ 0.2056 & -0.0103 & 1.0482 \end{pmatrix}, \qquad \frac{\mathbf{C}^*}{E} = \begin{pmatrix} 0.5222 & 0.0906 \\ 0.0906 & 0.4177 \end{pmatrix}.
\end{aligned} \tag{24}$$

The material with moduli (22) is the general anisotropic material, but if the direction of one of the axes x_1, x_2, z coincides with the direction of fibers, then this material becomes transversely isotropic one.

6 THE GENERAL SOLUTION AND ASYMPTOTIC ANALYSIS PRINCIPLES

Investigate system (19), (20) with boundary conditions (16) at the assumptions that all elastic moduli are constant and they are of the same asymptotic orders

$$\{A_{ij},\, B_{ij},\, C_{ij}\} \sim E. \tag{25}$$

The opposite case when the transversal (C_{ij}) and mixed (B_{ij}) elastic moduli are much smaller than the tangential ones (A_{ij}) requires the additional investigations. For anisotropic beams this case is studied in [7, 12]. Also we assume that the length of wave in the tangential direction is much larger than the shell thickness.

These assumptions (for small h_*) lead to the following simplifications of system (19), (20): the inertia terms $h_*^2\Lambda\varphi_i/12$ in the last two equations may be omitted, and the expression for $\mathbf{M}$ (instead of (20)) may be taken in the more simple form

$$\mathbf{M} = \frac{h_*^2}{12}\mathbf{A}^{**}\cdot\mathbf{K}(\boldsymbol{\varphi}). \tag{26}$$

To solve system (19), (20) we fulfill the separation of variables

$$Z(s,\theta) = Z(s)e^{in\theta}, \quad n = 0, 1, 2, \ldots, \quad i = \sqrt{-1}, \tag{27}$$

where n is the number of waves in the circular direction and $Z = (u_1, u_2, w, \ldots)$ is the any unknown function in this system. Then the system becomes one-dimensional and its general solution is

$$Z(s) = \sum_{k=0}^{10} C_k Z_k e^{\lambda_k s}, \tag{28}$$

where C_k are arbitrary constants, Z_k are definite constants, and λ_k are the roots of of the system (19) characteristic equation of 10th order. As usually the frequency equation

$$\Delta_{10}(\Lambda) = 0 \tag{29}$$

has the form of a determinant of the linear homogeneous system of 10th order obtained after substitution of the solution (28) into the boundary condition (16). This way is complex enough as for the qualitative investigation so for the numerical solution. That is why the asymptotic analysis at $h_* << 1$ is used.

Introduce the indexes of variability [14] to describe the asymptotic properties of the stress-strain states (SSS). The partial indexes of variability t_1 and t_2 are introduced by relations

$$\frac{\partial Z}{\partial s} \sim h_*^{-t_1} Z; \quad \frac{\partial Z}{\partial \theta} \sim h_*^{-t_2} Z, \quad or \quad n \sim h_*^{-t_2}, \tag{30}$$

where Z is any unknown function.

We refer to $t = \max(t_1, t_2)$ as to the common index of variability.

We study the following main SSS:

1) the membrane state with

$$\mathbf{T} \neq \mathbf{0}, \quad \{\mathbf{M}, \mathbf{Q}\} = \mathbf{0}, \quad t = 0, \quad \Lambda \sim 1, \tag{31}$$

2) the bending state with

$$\mathbf{T} = \mathbf{0}, \quad \{\mathbf{M}, \mathbf{Q}\} \neq \mathbf{0}, \quad t = 0, \quad \Lambda \sim h_*^2, \tag{32}$$

3) the mixed state of Donnell's type [2] with

$$\{\mathbf{T}, \mathbf{M}, \mathbf{Q}\} \neq \mathbf{0}, \quad \mathbf{T} \sim \mathbf{M}\, h_*^{-1}, \quad t > 0, \tag{33}$$

4) the semi-momentless state [15] as a partial case of Donnell's type with

$$t_1 = 0, \quad t_2 = 1/4, \quad \Lambda \sim h_*, \tag{34}$$

5) the edge effect state [14,15] as a partial case of Donnell's type with

$$t_1 = 1/2, \quad t_2 < 1/2, \tag{35}$$

6) the boundary layer state with

$$t_1 = 1. \tag{36}$$

These SSS are the same as for the isotropic shell, the difference lies in the concrete equations and it is discussed in the next sections. The approximate value of Λ may be found from the states 1–4, and sometimes from the state 5, the states 5 and 6 are used to satisfy all boundary conditions only.

7 THE MEMBRANE SOLUTION

For the membrane solution the essential simplification of system (19), (20) is possible. Due to $\mathbf{Q} = \mathbf{0}$ we find from two first relations (20)

$$\boldsymbol{\gamma} = -\mathbf{C}^{*-1} \cdot \mathbf{B}^{*T} \cdot \boldsymbol{\varepsilon}_t^0, \quad \mathbf{T} = \mathbf{A}^{**} \cdot \boldsymbol{\varepsilon}_t^0, \ \ \boldsymbol{\varepsilon}_t^0 = (u_1', u_2' + in\, u_1, in\, u_2 + u_3)^T, \quad (\,)' = \frac{d(\,)}{ds}, \tag{37}$$

and first three equations (19) are

$$T_{11}' + inT_{12} + \Lambda u_1 = 0, \quad T_{12}' + inT_{22} + \Lambda u_2 = 0, \quad -T_{22} + \Lambda w = 0. \tag{38}$$

System (38) of 4th order contains only unknown functions u_1, u_2, w. Let us study the clamped boundary conditions (17) at the both edges $s = 0$ and $s = l = L_0/R$. Solving system (38) we satisfy only conditions $u_1 = u_2 = 0$ and as a result we find the approximate values of membrane natural frequencies and the corresponding vibration modes. The rest boundary conditions $w = \varphi_1 = \varphi_2 = 0$ may be satisfied by adding to the membrane mode the edge effect and the boundary layer to the membrane mode. As a result the membrane natural frequencies are defined more precisely. To fulfill this definition it is necessary to know the relative orders of all unknown functions

$$\{u_i, w, \gamma_i, \varphi_i, T_{ij}\} \sim 1, \quad \{M_{ij}, Q_i\} \sim h_*^2. \tag{39}$$

As an example we study the simplest case $n = 0$ and find only the approximate membrane frequencies for the elastic moduli (24) and for the shell length $l = 3$. For isotropic shell the problem divides to the longitudinal and to the torsion vibrations, and here these problems are joined. The characteristic equation has the form

$$a_0(\Lambda)\lambda^4 + a_2(\Lambda)\lambda^2 + a_4(\Lambda) = 0, \qquad a_0 = L - \omega_*^2, \quad \omega_* = 0.973. \tag{40}$$

Calculations show that the set of the dimensionless frequencies $\omega = \sqrt{\Lambda}$ is the following $0.42,\ 0.56,\ 0.70,\ 0.83,\ 0.92,\ 0.955,\ 0.963,\ 0.966,\ 0,968,\ 0.9690,\ 0.9692,\ 0.9699,\ 0.9704,$ $0.9706, \ldots,\ 1.04,\ 1.15,\ 1.20,\ 1.29, \ldots$. This set has the point of accumulation $\omega_* = 0.973$ because two roots of equation (40) go to infinity at $\omega \to \omega_*$. For the isotropic shell the presence of the point of accumulation in the spectrum of axisymmetric vibrations of cylindrical and spherical shells is well known [14,16,17]. As for the isotropic shell here in the neighborhood of the point $\omega = \omega_*$ the membrane approximation is incorrect because assumptions (31) and (40) are not fulfilled.

8 THE BENDING STATE

The pure bending state is described by relations (32). The very low frequencies correspond to this state. This state exists only if there exists the non-zero solution of equations $\boldsymbol{\varepsilon}_t^0 = \mathbf{0}$ (the so called pure bending of surface). This state is possible if the shell edges are free or weakly supported [14,18].

Let us study the cylindrical shell with free edges and write more detailed estimates than (3.2)

$$t = 0, \quad \{u_i, w\} \sim 1, \quad \{\varepsilon_{ij}, \gamma_i, M_{ij}, Q_i, \Lambda\} \sim h_*^2. \tag{41}$$

There exist the solution $w = \cos(n\theta),\ u_1 = 0,\ u_2 = -\sin(n\theta)/n$, which satisfies equations $\boldsymbol{\varepsilon}_t^0 = \mathbf{0}$. For these functions the second and the third equations (19) accept the form

$$\frac{\partial T_{22}}{\partial \theta} + Q_2 + \Lambda u_2 = 0, \quad \frac{\partial Q_2}{\partial \theta} - T_{22} + \Lambda w = 0, \qquad Q_2 = -\frac{h_*^2 A_{33}^{**}}{12} n(n^2-1)\sin(n\theta)$$

Excluding T_{22} from these equations we obtain

$$\Lambda = \frac{h_*^2 A_{33}^{**}}{12} \frac{(n^2-1)^2}{n^2+1}, \quad n = 2, 3, \ldots. \tag{42}$$

For $n = 0$ and $n = 1$ the shell moves as a rigid body, and $\Lambda = 0$. Relation (42) describes vibrations of inextensible ring [18], the influence of anisotropy is contained in elastic modulus A_{33}^{**}.

9 THE APPROXIMATION OF THE DONNELL TYPE

For this approximation the value $t = 1/2$ of index of variability is typical. In the general case for $w \sim 1$ the orders of the rest unknown functions are

$$\{u_i, Q_i\} \sim h_*^{1/2}, \quad \{\gamma_i, \varepsilon_{ij}^0, T_{ij}\} \sim 1, \quad \varphi_i \sim h_*^{-1/2}, \quad M_{ij} \sim h_*. \tag{43}$$

Admitting the error of the order $h_*^{1/2}$ we obtain (as in (37)) from relations (20)

$$\boldsymbol{\gamma} = -(\mathbf{C}^*)^{-1} \cdot \mathbf{B}^{*T} \cdot \boldsymbol{\varepsilon}_t^0, \qquad \mathbf{T} = h\mathbf{A}^{**} \cdot \boldsymbol{\varepsilon}_t^0, \tag{44}$$

and system (19) may be rewritten in the form

$$\frac{\partial T_{11}}{\partial s} + \frac{\partial T_{12}}{\partial \theta} = 0, \quad \frac{\partial T_{12}}{\partial s} + \frac{\partial T_{22}}{\partial \theta} = 0, \quad \frac{\partial^2 M_{11}}{\partial s^2} + 2\frac{\partial^2 M_{12}}{\partial s\,\partial \theta} + \frac{\partial^2 M_{22}}{\partial \theta^2} + T_{22} - \Lambda w = 0. \tag{45}$$

In system (45) the shear angles γ_i and shear stress-resultants are excluded, the comparatively small tangential inertia is neglected. Let us put this system in the Donnell form [2] containing two unknown functions which are the normal deflection w and the stress function Φ with

$$T_{11} = \frac{\partial^2 \Phi}{\partial \theta^2}, \quad T_{12} = -\frac{\partial^2 \Phi}{\partial s \partial \theta}, \quad T_{22} = \frac{\partial^2 \Phi}{\partial s^2}. \tag{46}$$

The first and the second equations (45) are satisfied. We accept approximately

$$K(\varphi) = -\left(\frac{\partial^2 w}{\partial s^2}, \frac{\partial^2 w}{\partial s\,\partial \theta}, \frac{\partial^2 w}{\partial \theta^2}\right). \tag{47}$$

Then excluding u_1^0 and u_2^0 from the relation $\boldsymbol{\varepsilon}_t^0 = (h\mathbf{A}^{**})^{-1} \cdot \mathbf{T}$ we obtain the system

$$L_4\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \theta}\right)\Phi - \frac{\partial^2 w}{\partial s^2} = 0, \quad \frac{h_*^2}{12} N_4\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \theta}\right) w + \frac{\partial^2 \Phi}{\partial s^2} - \Lambda\, w = 0, \tag{48}$$

where the differential operators of 4th order are

$$\begin{aligned} L_4(p,q) &= a_{11}q^4 - 2a_{12}pq^3 + (a_{22}+2a_{13})p^2q^2 - 2a_{23}p^3q + a_{33}p^4, \quad \{a_{ij}\} = (\mathbf{A}^{**})^{-1}, \\ N_4(p,q) &= A_{11}^{**}p^4 + 4A_{12}^{**}p^3q + 2(2A_{22}^{**} + A_{13}^{**})p^2q^2 + 4A_{23}^{**}pq^3 + A_{33}^{**}q^4. \end{aligned} \tag{49}$$

For the material described in Section 5 the elements of matrix $(\mathbf{A}^{**})^{-1}$ are

$$(\mathbf{A}^{**})^{-1} = \{a_{ij}\} = \begin{pmatrix} 1.0438 & -0.0583 & -0.3370 \\ -0.0583 & 2.5662 & 0.0466 \\ -0.3370 & 0.0466 & 1.1639 \end{pmatrix}. \tag{50}$$

System (48) is convenient for analytical and numerical investigations. With accuracy of the order h_*^t system (48) may be used also for the index of variability $0 < t < 1/2$.

If we neglect the summand $\partial^2\Phi/\partial s^2$ then the second equation (48) describes the transversal vibrations of an anisotropic plate [11].

The system (19) is of the 10th order, and the orders of system (48) is equal to 8. In order to formulate correctly the boundary conditions for system (48) it is necessary to construct the boundary layer solution.

10 THE SEMI-MOMENTLESS STATE

For this state according of (34) index of variability t_2 in circular direction is larger than index t_1 in longitudinal direction, and as a result $|M_{22}| >> |M_{11}|$ and moment M_{11} may be neglected. That is why this state is named as a "semi-momentless" one [15]. Excluding the cases with free or weakly supported edges the minimal natural frequency have the semi-momentless mode [15].

If the estimates (34) are fulfilled then system (48) may be essentially simplified. In zero approximation (with the error of order $h_*^{1/4}$) we put approximately

$$L_4\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \theta}\right)\Phi = a_{11}\frac{\partial^4 \Phi}{\partial \theta^4}, \qquad N_4\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \theta}\right) w = A_{33}^{**}\frac{\partial^4 w}{\partial \theta^4}, \tag{51}$$

and system (48) accepts the form

$$a_{11}\frac{\partial^4 \Phi}{\partial \theta^4} - \frac{\partial^2 w}{\partial s^2} = 0, \quad \frac{h_*^2 A_{33}^{**}}{12}\frac{\partial^4 w}{\partial \theta^4} + \frac{\partial^2 \Phi}{\partial s^2} - \Lambda\, w = 0. \tag{52}$$

We seek solution of this system in the form

$$w(s,\theta) = w(s)\cos(n\theta), \quad \Phi(s,\theta) = \Phi(s)\cos(n\theta), \quad n = O(h_*^{-1/4}), \tag{53}$$

and excluding $\Phi(s)$ from system (52) reduce it to the ordinary problem of the beam vibrations

$$\frac{d^4 w}{\partial s^4} - p^4 w = 0, \quad p^4 = n^4 a_{11}(\Lambda - n^4\mu^8 A_{33}^{**}), \quad \mu^8 = \frac{h_*^2}{12}. \tag{54}$$

For the given p and n the frequency parameter is equal to

$$\Lambda = n^4\mu^8 A_{33}^{**} + \frac{p^4}{n^4 a_{11}} \tag{55}$$

and find the minimal value $\Lambda_* = \min \Lambda$ and the corresponding value n_* of number of waves in the circular direction are

$$\Lambda_* = 2\mu^4 p^2 \left(\frac{A_{33}^{**}}{a_{11}}\right)^{1/2}, \qquad n_* = \frac{1}{\mu}\left(\frac{p^4}{a_{11}A_{33}^{**}}\right)^{1/8}. \tag{56}$$

Integer n close to n_* is to be taken to find $\min \Lambda$ from (55).

The order of equation (54) allows us to satisfy only two (main) boundary conditions at the each shell edge. For the isotropic shell the way of choosing main boundary condition is discussed in [13,18]. If both edges are clamped ($w = dw/ds = 0$) then $p = 4.73/l$.

For the material described in section 5 with $l = 3,\ R/h = 250$ the dimensionless natural frequencies $\omega = \sqrt{\Lambda}$ close to the minimal one are ($n_* = 6.82$)

$$\begin{array}{lcccccc} n = & 5 & 6 & 6.82 & 7 & 8 & 9 \\ \omega = & 0.1013 & 0.0788 & 0.0740 & 0.0742 & 0.0814 & 0.0959 \end{array} \tag{57}$$

The approximate results (53), (55)–(57) may be made more precise by solving two problems. At first the approximation (51) leads not only to the quantitative error of the order μ, but also contains the qualitative error. If in zero approximation the vibration modes (53) are elongated in the longitudinal direction then the more exact system (48) leads to the slightly inclined modes (Fig. 3).

At second it is necessary to satisfy not only two main boundary conditions, but 5 conditions (17). Two of the additional conditions may be satisfied by the edge effect solutions and the rest one may be satisfied by the boundary layer.

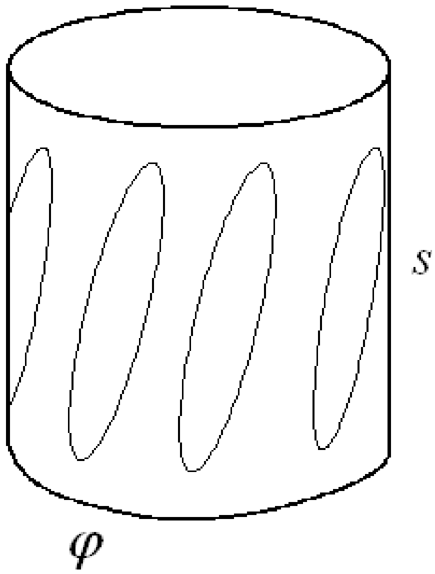

Figure 3: The vibration mode.

11 THE EDGE EFFECT

If $t_1 > t_2$ then the main terms of operators L_4 and N_4 are

$$L_4\left(\frac{\partial}{\partial s},\frac{\partial}{\partial \theta}\right)\Phi = a_{33}\frac{\partial^4\Phi}{\partial s^4}, \qquad N_4\left(\frac{\partial}{\partial s},\frac{\partial}{\partial \theta}\right)w = A_{11}^{**}\frac{\partial^4 w}{\partial s^4}, \tag{58}$$

and system (48) leads to the edge effect equation

$$\frac{\partial^4 w}{\partial s^4} + p_e^4 w = 0, \quad p_e^4 = \frac{12}{h_*^2 a_{33} A_{11}^{**}}(1-\Lambda a_{33}). \tag{59}$$

If $\Lambda < 1/a_{33}$ then due to $|p_e| >> 1$ equation (59) has two solutions $w = C(\theta)e^{\pm p_e x}$ quickly decreasing away from the shell edges. For $\Lambda > 1/a_{33}$ two solutions of equation (59) oscillate, the edge effect degenerates, and the equation (59) gives the natural frequencies with $t = 1/2$.

12 THE BOUNDARY LAYER

As in the TR model for the boundary layer the index of variability $t = 1$. Assuming that $w \sim 1$ we find the following asymptotical orders of the remaining unknown variables

$$u_i \sim 1, \quad \{\gamma_i, \varphi_i, \varepsilon_{ij}^0\} \sim h_*^{-1}, \quad \{T_{ij}, Q_i\} \sim h_*^{-1}, \quad \{M_{ij}\} \sim h_*^{-2}, \quad i,j = 1,2. \tag{60}$$

According to these estimates the boundary layer in zero approximation can be found from the system

$$\frac{\partial T_{i1}}{\partial y_1} + \frac{\partial T_{i2}}{\partial y_2} = 0, \quad \frac{\partial M_{i1}}{\partial y_1} + \frac{\partial M_{i2}}{\partial y_2} + Q_i = 0, \quad i = 1,2, \qquad \frac{\partial Q_1}{\partial y_1} + \frac{\partial Q_2}{\partial y_2} = 0, \tag{61}$$

where in the 2D elasticity relations (44) in contrast to (9) $\boldsymbol{\varepsilon}_t^0 = \mathbf{K}(\boldsymbol{u}^0)$.

To construct the boundary layer near the edge $x_1 = x_1^0$ we suppose that one of the indexes of variability $t_1 = 1$, and the other index of variability $t_2 < t_1$ (for example, $t_2 = 1/2$ if the boundary layer is constructed to formulate the boundary conditions for system (32)).

We seek the boundary layer in the form

$$Z(x_1, x_2) = e^{py_1/h_*} Z^b(x_2), \quad p = p(x_2), \quad y_1 = A_1(x_2)x_1, \tag{62}$$

where Z is any unknown function, and the co-ordinate x_2 is a parameter. Then in zero approximation system (61) yields

$$T_{11} = 0, \quad T_{12} = 0, \quad Q_1 = 0, \quad M_{11} = 0, \quad M_{12} + Q_2 = 0. \tag{63}$$

System (63) is a linear system with respect to $u_1^b, u_2^b, w^b, \gamma_1^b, \gamma_2^b$ with the coefficients depending on p and the elements of matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$. The determinant of this system has the form $\Delta(p) = p^8(ap^2 - b)$. Its non-zero roots $p = \pm\sqrt{b/a}$ correspond to the boundary layer. Notice that the boundary layer in zero approximation does not depend on the curvatures k_1, k_2, and therefore the boundary layer for a shell is same as for a plate.

To illustrate the influence of the boundary layer on the boundary conditions we study the clamped edge $x_1 = x_1^0$ and write the boundary conditions in the form

$$u_i^m + Cu_i^b = 0, \quad \varphi_i^m + C\varphi_i^b = 0, \quad i = 1, 2, \qquad w^m + Cw^b = 0, \tag{64}$$

where index ^{m} corresponds to the solution of system (32) (which we call the main solution), and the constant C is to be chosen to exclude the boundary layer. It follows from estimates (30) and (31) that $C \sim h^{1/2}$. Then $|Cw^b| \ll |w^m|$ and the orders of summands in the rest equations (64) are equal to each other. Then (for example) $C = -\varphi_2^m/\varphi_2^b$, and four boundary conditions for system (51) are

$$u_i^m + C\varphi_2^m = 0, \ i = 1, 2, \quad w^m = 0, \quad \varphi_1^m + C\varphi_2^m = 0. \tag{65}$$

The values Z^b are very unwieldy functions of the elements of matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and for this reason are not given here.

The previously discussed states 1–5 are described by the 2D model with the error which goes to zero at $h_* \to 0$. According to the estimate of the 2D shell theory [13,20]

$$\Delta \sim \max\{h_*, h_*^{2-2t}\}, \tag{66}$$

where t is the index of variability of studied SSS, the error Δ of the boundary layer, obtained from 2D model, is $\Delta \sim 1$. The boundary layer may be found only from the 3D equations. To construct the full 3D solution it is necessary to execute the asymptotic matching of the boundary layer and of the interior solution. This problem is out of frames of the presented paper. Our aim is to find the correct interior SSS. The governing system (19) contains the (inexact) boundary layer, and in its simplifications 1-5 the boundary layer is excluded. The problem is to formulate 4 boundary conditions at the shell edges by excluding the boundary layer. Remind that the well known condition $Q_1 + \partial M_{12}/\partial x_2 = 0$ in the KL theory my be delivered from the TR theory by excluding the boundary layer.

13 CONCLUSIONS

By using the cinematic hypothesis (7) the 2D model of thin elastic shell made of an anisotropic material described by 21 elastic moduli is delivered. The obtained system is comparatively new and insufficiently studies. Here we are restricted with the asymptotic analysis of the case when all elastic moduli in the elasticity relations (3) are of the identical orders. It would be interesting to investigate the case when the moduli in the transversal directions are much smaller than in the tangential directions. It would be interesting to study the various boundary conditions and solve the problem of excluding from them the boundary layer, and as a result to reduce the system of 10th order to the system of 8th order. Also it would be interesting to study the buckling problems for this anisotropic material and to compare the results with the case of the isotropic material.

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