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RIGID BODY DYNAMICS USING A NOVEL APPROACH BASED ON LIE GROUP THEORY

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Abstract. A systematic theoretical approach is presented, in an effort to provide a complete and illuminating study on motion of a rigid body rotating about a fixed point. Since the configuration space of this motion is a differentiable manifold possessing group properties, this approach is based on some fundamental concepts of differential geometry. A key idea is the introduction of a canonical connection, matching the manifold and group properties of the configuration space. This is sufficient for performing the kinematics. Next, following the selection of an appropriate metric, the dynamics is also carried over. The present approach is theoretically more demanding than the traditional treatments but brings substantial benefits. In particular, an elegant interpretation is provided for all the quantities with fundamental importance in rigid body motion. It also leads to a correction of some misconceptions and geometrical inconsistencies in the field. Finally, it provides powerful insight and a strong basis for the development of efficient numerical techniques in problems involving large rotations.

1 INTRODUCTION

The study of rigid body motion has been in the epicenter of many previous investigations due to both the great theoretical importance and the large practical significance of the subject. The main objective of this work is to present a new look into the old but practically significant and still challenging mechanics problem of finite rotations, based on sound geometrical concepts. These concepts are known in the mathematics and physics literature for a long time. However, with a few exceptions (e.g., [1,2]), they are still not fully explored by the engineering community, despite their large value and usefulness. Here, an effort is made to use these concepts in order to create a clear and complete geometrical picture of large rigid body rotation. In this way, more light is thrown into the meaning of some of the most commonly employed quantities in describing rigid body motion. This, in turn, provides an alternative view and better interpretation of the related formulas employed frequently in the classical engineering literature. At the same time, this helps to identify and correct some common misconceptions in the field and achieve another objective of the present work. The latter refers to providing the means for building a reliable theoretical basis for developing better and more effective numerical integration methodologies for studying and investigating dynamics of single or multiple rigid bodies (e.g., [3-5]).

The geometrical route chosen in the present study deviates significantly from that taken in previous studies of mechanics problems. In those studies the underlying manifold is assumed to possess a Riemannian structure from the outset. A typical path is to introduce a set of coordinates and employ a natural coordinate basis in order to define a metric and then produce a metric compatible connection, having as components the classical Christofell symbols [6]. Usually, the same route is also chosen even for dynamics problems [7]. However, a more primitive and natural path is more beneficial and followed in this work. Namely, after creating the manifold corresponding to the configuration space of the motion, a connection operator is first established in an appropriate manner. This proves to be sufficient for performing a complete study of the kinematics. A canonical connection is selected [8], so that one can fully exploit the benefits associated to matching the special curves related to the manifold and the group properties of the configuration space. Then, a study of the dynamics requires the introduction of a metric. In this work, a suitable metric is chosen, which is not compatible with the connection, but allows for a complete, simple and concise treatment of the dynamics.

The classical treatment of spherical motion is briefly summarized in the following section. Then, some useful concepts of differential geometry are presented in Sect. 3. In the fourth section, rigid body kinematics is examined, based on the geometry of an appropriate three dimensional manifold. The basic properties of this manifold are extracted by the special orthogonal group $SO(3)$. Then, a suitable metric is introduced in Sect. 5, providing a useful tool for examining the dynamics of a rigid body. Finally, the most important conclusions are summarized in the last section.

2 A SUMMARY OF THE CLASSICAL APPROACH

Study of the spherical motion about a point O is typically performed in the ordinary Euclidean space $\mathbb{R}^3$ [9]. A basis $\mathfrak{B}$ is introduced, consisting of three fixed orthonormal vectors $\vec{E}_i$ (with $i = 1,2,3$), having point O as origin. These vectors form a right-handed Cartesian inertial frame of reference. This basis will also be denoted $\{\vec{E}_i\}$. On the other hand, another basis, $\mathfrak{B}'$ or $\{\vec{e}_i(t)\}$, is formed by considering a new set of three orthonormal vectors $\vec{e}_i(t)$, having O as origin, but rigidly attached to and following the motion of the rigid body. These vectors form the so-called body frame of reference [10].

The two bases $\mathfrak{B}$ and $\mathfrak{B}'$ coincide originally. During the subsequent motion, they are related through a linear mapping

$$\vec{e}_i(t) = \Re(t)\,\vec{E}_i, \quad i = 1,2,3 \tag{1}$$

Then, the position vector of an arbitrary point of the body is

$$\vec{x}(t) = \sum_{i=1}^{3} x_i(t)\,\vec{E}_i = \sum_{i=1}^{3} X_i\,\vec{e}_i(t)\,,$$

where x_i and X_i represent the components in the spatial and body frame, respectively. Adopting the notational convention of dropping the sum operator for products involving repeated indices [11], the last relations can be rewritten as

$$\vec{x}(t) = x_i(t)\,\vec{E}_i = X_i\,\vec{e}_i(t)\,. \tag{2}$$

Next, if the components of the transformation $\Re$ in the basis $\mathfrak{B}$ are expressed by matrix R, that is $M_{\mathfrak{B}}^{\mathfrak{B}}(\Re) = R = [r_{ij}]$, then, Eq. (1) can be rewritten in the form

$$\vec{e}_i(t) = r_{ji}(t)\vec{E}_j\,.$$

Therefore, from Eqs. (1) and (2) one gets

$$x_i(t) = r_{ij}(t)X_j \quad \text{or} \quad \underline{x}(t) = R(t)\underline{X}\,, \tag{3}$$

with

$$\underline{x} = (x_1 \quad x_2 \quad x_3)^T \quad \text{and} \quad \underline{X} = (X_1 \quad X_2 \quad X_3)^T\,.$$

Then, employing the rigidity assumption of the body leads to

$$R^T R = I\,, \tag{4}$$

where I is the 3×3 identity matrix. This implies that $R(t)$ is orthogonal. Moreover, it easily turns out that the matrix

$$\tilde{\Omega}(t) \equiv R^T(t)\dot{R}(t) \tag{5}$$

is skew-symmetric (as indicated by $\sim$), with general form

$$\tilde{\Omega} = spin(\underline{\Omega}) \equiv \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}. \tag{6}$$

The corresponding axial vector associated with $\tilde{\Omega}$ is

$$\underline{\Omega} = vect(\tilde{\Omega}) \equiv (\Omega_1 \quad \Omega_2 \quad \Omega_3)^T\,.$$

In a similar manner one can show that $R R^T = I$ and by differentiation obtain the new skew-symmetric matrix

$$\tilde{\omega}(t) \equiv \dot{R}(t)R^T(t)\,. \tag{7}$$

Therefore, combination of Eqs. (4), (5) and (7) yields

$$\dot{R} = \tilde{\omega}R = R\tilde{\Omega}\,. \tag{8}$$

Next, direct differentiation of Eq. (3) with a simultaneous application of Eqs. (7) and (8) yields

$$\underline{v} = \dot{\underline{x}} = \dot{R}\underline{X} = \dot{R}R^T\underline{x} = \tilde{\omega}\underline{x}\,. \tag{9}$$

Finally, Eqs. (7), (8) and (4) lead to

$$\tilde{\omega} = R\tilde{\Omega}R^T \quad \Rightarrow \quad \underline{\omega} = R\underline{\Omega}\,. \tag{10}$$

After establishing the kinematics, one can evaluate all the quantities needed in dynamics. First, the (convective) angular momentum of the body relative to the origin O is obtained by

$$\underline{H}_O(t) = J_O\,\underline{\Omega}(t)\,, \tag{11}$$

where

$$J_O = \int_m \tilde{X}^T \tilde{X}\, dm = \int_m [(\underline{X}\cdot\underline{X})I - \underline{X}\,\underline{X}^T]dm \tag{12}$$

is the mass moment of inertia matrix of the body with respect to the origin O [9]. Then, the dynamics of the rigid body is expressed by Euler's law in the form

$$J_O\dot{\underline{\Omega}} + \underline{\Omega}\times\underline{H}_O = \underline{M}_O\,, \tag{13}$$

where $\underline{M}_O$ are the body components of the resultant external moment about O [9]. Finally, by performing straightforward operations the kinetic energy of the motion is found in the form

$$T = \tfrac{1}{2}\underline{\Omega}^T J_O\,\underline{\Omega} = \tfrac{1}{2}\underline{H}_O\cdot\underline{\Omega}\,. \tag{14}$$

3 SOME ELEMENTS OF DIFFERENTIAL GEOMETRY

The set of orthogonal matrices $R(t)$ introduced in Sect. 2 forms a Lie group. Specifically, each matrix $R(t)$ represents a point in the space of 3×3 matrices, coinciding with the Euclidean space $\mathbb{R}^9$. Considering the orthogonality condition (4), this point lies on a three dimensional subset of $\mathbb{R}^9$ [12]. Since these conditions are nonlinear, this subset is not a vector subspace of $\mathbb{R}^9$ but it forms a three dimensional manifold, instead, which can be viewed as a surface in $\mathbb{R}^9$. In addition, since composite rotations are represented by products of orthogonal matrices [13], which are also orthogonal matrices, this subset is a Lie group, having as product the matrix multiplication and as identity element the 3×3 identity matrix I. Moreover, since $R(0) = I$, its determinant at t=0 is equal to +1 and because there can occur no jump to the value -1 during the subsequent motion, this matrix belongs to the special orthogonal group of order three, denoted by $SO(3)$.

Lie groups present some extra mathematical structure. For instance, if g and h are elements of a Lie group G, then one can define the left translation by g through the mapping

$$L_g(h) = gh\,. \tag{15}$$

Likewise, the right translation by g is defined as a mapping

$$R_g(h) = hg\,. \tag{16}$$

These operations are smooth mappings and give rise to useful gradients. Specifically, let $h(t)$ be a curve on G, with $h(0) = h$ and tangent vector $\dot{h}(0)$ at $t = 0$. This vector is denoted by $\underline{X}_h$ and belongs to the tangent space to G at h, denoted by T_hG. Then, according to Eq. (15), the set of points $p(t) = L_g(h(t))$ represents the image curve of $h(t)$ on G, obtained with a left translation by g. The velocity vector of $p(t)$ at point $L_g(h)$ is given by

$$\dot{p} = \tfrac{d}{dt}\{L_g(h(t))\}\big|_{t=0} = L_{g*}\dot{h}(0)\,, \tag{17}$$

where L_{g*} defines a linear transformation from T_hG to T_pG, known as the differential of L_g at h [12]. In this way, one can take a basis $\{\underline{e}_i(e)\}$ of T_eG at the identity element e of the group and left translate it to a basis $\{\underline{e}_i(p)\}$ of T_pG, with

$$\underline{e}_i(p) = L_{p*}\underline{e}_i(e)\,, \tag{18}$$

on all of G. Then, any vector of the vector space T_pG, of dimension n, can be expressed in the form

$$\underline{v}(p) = \sum_{i=1}^{n} v^i(p)\underline{e}_i(p) = v^i(p)\underline{e}_i(p)\,. \tag{19}$$

Moreover, if $\underline{v}$ is a left invariant vector field on G, then

$$\underline{v}(p) = L_{p*}\underline{v}(e) = L_{p*}[v^i(e)\underline{e}_i(e)] = v^i(e)L_{p*}\underline{e}_i(e) = v^i(e)\underline{e}_i(p)\,.$$

Direct comparison of the last result with (19) yields

$$v^i(p) = v^i(e)\,. \tag{20}$$

Left or right translation on a Lie group is important in the study of its one parameter subgroups. These are specific curves, satisfying the group homomorphism

$$g(t+s) = g(t)g(s) = g(s)g(t)\,. \tag{21}$$

It can be shown that the most general monoparametric subgroup of G must pass from the identity element e and is determined by the exponential map, with

$$g(t) = \exp[t\,g'(0)]\,. \tag{22}$$

It can also be shown by using Eq. (21) that

$$g'(t) = L_{g(t)*}g'(0) = R_{g(t)*}g'(0)\,, \tag{23}$$

which reveals that the one parameter subgroup of G, with tangent vector $g'(0)$ at the identity, coincides with the integral curve through the identity of the vector field, which results by a left translation of $g'(0)$ over all of G.

Given a Lie group G, one can construct its Lie algebra. This consists of the vector space T_eG, equipped with a special operator, known as Lie bracket [10]. For Lie groups, this bracket can be defined by employing the idea of left (or right) invariant vector fields. For instance, consider two vectors $\underline{X}_e$ and $\underline{Y}_e$ of T_eG and extend them by left translation to the vector fields $\underline{X}^L$ and $\underline{Y}^L$ on all of G. Then, their Lie bracket is

$$[\underline{X}_e, \underline{Y}_e] \equiv [\underline{X}^L, \underline{Y}^L]_e\,. \tag{24}$$

In fact, it has been proved that there exist three such canonical connections [8]. The first two of them, known as left and right invariant canonical connection, are defined by

$$\nabla^R_{\underline{X}}\underline{Y} = [\underline{X}^L, \underline{Y}] \quad \text{and} \quad \nabla^L_{\underline{X}}\underline{Y} = [\underline{X}^R, \underline{Y}]\,, \tag{25}$$

where $\underline{X}^L$ and $\underline{X}^R$ is a left and a right invariant vector field, respectively, extending an element $\underline{X}$ of the tangent space at a point of a Lie group G to all of G, while $\underline{Y}$ is a vector field on G. Finally, a symmetric canonical connection is defined by

$$\nabla^S_{\underline{X}}\underline{Y} = \tfrac{1}{2}[\underline{X}^L + \underline{X}^R, \underline{Y}]\,. \tag{26}$$

All these connections are described completely by the Lie bracket and left/right translation. Some of their important properties are summarized in the sequel.

First, let $\{\underline{e}_i^L(p)\}$ (or simply $\{\underline{e}_i(p)\}$) be a basis created at point p by left translating a basis $\{\underline{e}_i(e)\}$ at the identity, using Eq. (18). Since $\underline{e}_i(p)$ is L_g-related to $\underline{e}_i(e)$ [12]

$$L_{g*}[\underline{e}_i(e),\underline{e}_j(e)] = [L_{g*}\underline{e}_i(e), L_{g*}\underline{e}_j(e)] = [\underline{e}_i(p),\underline{e}_i(p)].$$

By using the definition of the structure constants c_{ij}^k of a basis $[\underline{e}_i,\underline{e}_j] = c_{ij}^k\underline{e}_k$,this implies that

$$c_{ij}^k(p) = c_{ij}^k(e). \tag{27}$$

Also, the Lie bracket of the vector fields $\underline{X}$ and $\underline{Y}$ is given by

$$[\underline{X},\underline{Y}] = (X^j\partial_j Y^i - Y^j\partial_j X^i + c_{jk}^i X^j Y^k)\underline{e}_i. \tag{28}$$

Then, if both $\underline{X}$ and $\underline{Y}$ are left invariant vector fields on G

$$[\underline{X},\underline{Y}] = (c_{jk}^i X^j Y^k)\underline{e}_i \equiv Z^i\underline{e}_i.$$

Therefore, since the quantity $Z^i = c_{jk}^i X^j Y^k$ remains constant everywhere on the manifold, this shows that the Lie bracket of two left invariant vector fields is a new left invariant vector field. In a similar manner, if $\underline{X}$ is a left invariant vector field while $\underline{Y}$ is a right invariant vector field on G, it can be shown that their Lie bracket vanishes identically [12], that is

$$[\underline{X}^L,\underline{Y}^R] = \underline{0}. \tag{29}$$

Then, it becomes apparent from Eq. (29) that

$$\nabla_{\underline{X}}^R\underline{Y}^R = \underline{0}, \tag{30}$$

for any right invariant vector field $\underline{Y}$ on G. This means that the parallel translation of a vector on a manifold equipped with a left invariant canonical connection is equivalent to a right translation of it.

Next, by employing the definition of the left invariant canonical connection one arrives at

$$\nabla_{\underline{X}}\underline{Y} = (\partial_j Y^i + c_{jk}^i Y^k)X^j\underline{e}_i, \tag{31}$$

since then $\underline{X} = \underline{X}^L$ and $\partial_j X^i = 0$. On the other hand, the covariant differential of $\underline{Y}$ along $\underline{X}$ is evaluated in the form

$$\nabla_{\underline{X}}\underline{Y} = (\partial_j Y^i + \Lambda_{jk}^i Y^k)X^j\underline{e}_i,$$

where the affinities Λ_{jk}^i of the connection ∇ in the basis $\{\underline{e}_i\}$ are introduced by the following definition

$$\nabla_{\underline{e}_j}\underline{e}_k = \Lambda_{jk}^i\underline{e}_i. \tag{32}$$

It can be shown [14] that the affinities and the structure constants provide the components of the torsion tensor through

$$\tau_{jk}^i = \Lambda_{jk}^i - \Lambda_{kj}^i - c_{jk}^i. \tag{33}$$

Therefore, direct comparison of (32) with Eq. (31) reveals that

$$c_{jk}^i = \Lambda_{jk}^i. \tag{34}$$

Consequently, by substitution in (33) it follows that $\tau^i_{jk} = -\Lambda^i_{kj}$. This indicates that the left invariant canonical connection possesses torsion when $\Lambda^i_{jk} \neq 0$. On the other hand, direct evaluation shows that in such a case, all the components of the curvature tensor vanish ($R^i_{jkl} = 0$), which means that the curvature tensor of this connection is zero [8].

Similar results are also available for the right invariant canonical connection $\nabla^L_{\underline{X}} \underline{Y}$. However, the picture obtained for $\nabla^S_{\underline{X}} \underline{Y}$ is different. First, if $\{\underline{e}_i^R(p)\}$ is a basis created at p by a right translation of the basis $\{\underline{e}_i(e)\}$, then

$$c^i_{jk} = 2\Lambda^i_{jk}. \tag{35}$$

Therefore, by direct substitution in Eq. (33) it follows that

$$\tau^i_{jk} = -\Lambda^i_{jk} - \Lambda^i_{kj}.$$

Also, based on (39), the affinities of the connection are found to be anti-symmetric in their lower indices, i.e.,

$$\Lambda^i_{jk} = -\Lambda^i_{kj}. \tag{36}$$

This implies eventually that $\tau^i_{jk} = 0$. On the other hand, direct evaluation shows that the curvature tensor is non-zero.

For a Lie group, the choice of a canonical connection is a natural way to match the special integral curves associated with its properties as a manifold and as a group. Specifically, by definition the structure constants are anti-symmetric [12]. Consequently, Eqs. (34) and (35) reveal that the canonical connections considered lead to anti-symmetric affinities, satisfying Eq. (40). Then, it can be shown that the tangent vector at each point of an autoparallel curve has constant components on a local frame produced by a left translation. Therefore, each canonical connection relates the affinities Λ^i_{jk} (which express a manifold property) to the structure constants c^i_{jk} of the basis, so that the one parameter Lie subgroups and the curves resulting by their left translation (which are related to the group properties only) coincide with autoparallel curves (which are related to the manifold properties only). These results have remarkable implications in rigid body kinematics and dynamics, examined in the following two sections.

4 RIGID BODY KINEMATICS USING LIE GROUPS

In this section, spherical motion is reconsidered, by employing concepts of differential geometry. A new manifold is introduced, drawing its basic properties through a group representation on the classical $SO(3)$ group.

4.1 Introduction of a new manifold *M(3)*

A new manifold is introduced for the description of rigid body rotation. This manifold belongs to the same abstract group as $SO(3)$, but possesses different geometrical properties. For this reason, this manifold is named $M(3)$. Next, the emphasis is placed in identifying its critical geometrical properties.

First, the orientation of a rigid body can be represented by a point, say p, on $M(3)$. After choosing an appropriate local coordinate system, each point on $M(3)$ is described by three coordinates (or rotation parameters). Then, the spherical motion of a rigid body can be viewed as a motion of a point on a single parameter curve, say $\gamma(t):\mathbb{R}\to M(3)$. Moreover, the angular velocity of the body is expressed by the tangent vector to $\gamma(t)$

$$\underline{w}(t)=\sum_{i=1}^{3}w^i(t)\underline{e}_i(t)=w^i(t)\underline{e}_i(t)\,, \tag{37}$$

where $w^i(t)$ are the components of $\underline{w}(t)$ in a local basis $\{\underline{e}_i(t)\}$ (with $i=1,2,3$) of $T_pM(3)$ at the current position.

A useful concept in detecting changes of a vector field $\underline{v}(t)$ on $M(3)$ is the covariant differential of this field along the direction of vector $\underline{w}$ [12]. This quantity is determined by

$$\nabla_{\underline{w}}\underline{v}(t)=(\dot{v}^i+\Lambda^i_{jk}w^jv^k)\underline{e}_i\,.$$

In particular, a parallel translation of vector $\underline{v}$ along the curve $\gamma(t)$, with tangent vector $\underline{w}$, is defined by

$$\nabla_{\underline{w}}\underline{v}=\underline{0}\quad\Rightarrow\quad\dot{v}^i+\Lambda^i_{jk}w^jv^k=0\,. \tag{38}$$

This represents a set of three coupled linear ordinary differential equations in $v^i(t)$, possessing a unique solution for a given set of initial conditions $v^i(0)$. However, in the general case, the affinities Λ^i_{jk} depend on position. Nevertheless, there exists a special occasion where one can select the affinities so that the solution $\underline{v}(t)$ can be obtained in a convenient closed form, in terms of an exponential matrix, as explained next.

First, in the particular case with $\underline{v}=\underline{w}\equiv\underline{n}$, satisfaction of the condition of parallel translation leads to special curves on the manifold, known as autoparallel curves [15]. If, in addition, the affinities of the connection are anti-symmetric, as in Eq. (36), it can be shown that the components of the tangent vector $\underline{n}(t)$ to the autoparallel curve in the local frame remain constant on the whole curve. That is

$$n^i(t)=n^i(0)\equiv n^i\,. \tag{39}$$

Therefore, if the affinities Λ^i_{jk} are also constant everywhere on the manifold and such that

$$[\Lambda^i_{jk}n^j]=[\tilde{n}^i_k]=\tilde{n}=spin(\underline{n})\,, \tag{40}$$

the parallel translation of any vector $\underline{u}(t)$ of $T_pM(3)$ along the autoparallel curve of $M(3)$ connecting point $p(t)$ to any other point of $M(3)$ is described by the following system

$$\begin{pmatrix}\dot{u}^1\\ \dot{u}^2\\ \dot{u}^3\end{pmatrix}=-\begin{bmatrix}0 & -n^3 & n^2\\ n^3 & 0 & -n^1\\ -n^2 & n^1 & 0\end{bmatrix}\begin{pmatrix}u^1\\ u^2\\ u^3\end{pmatrix}\quad\text{or}\quad\dot{\underline{u}}=-\tilde{n}\underline{u}\,. \tag{41}$$

Simple inspection verifies that Eq. (40), together with condition (36), can be fulfilled simultaneously, indeed, provided that the non-zero affinities take the following constant values

$$\Lambda^1_{23}=-\Lambda^1_{32}=\Lambda^2_{31}=-\Lambda^2_{13}=\Lambda^3_{12}=-\Lambda^3_{21}=1\,, \tag{42}$$

on all of $M(3)$. Therefore, there appears the need for the selection of an appropriate non-natural basis. Since $M(3)$ forms a Lie group, this basis can conveniently be obtained by extending a basis $\{\underline{e}_i(e)\}$, defined in $m(3)\equiv T_eM(3)$, on all of $M(3)$ through a right or left translation. Taking into account Eq. (1), the latter choice is preferred, leading to

$$\underline{e}_i(p(t))=L_{p(t)*}\underline{e}_i(e)\,,\qquad i=1,2,3\,, \tag{43}$$

so that each basis vector $\underline{e}_i(p(t))$ is part of a left invariant vector field on $M(3)$. In essence, this is identical to Eq. (18), since for a given point p these vectors are fixed and their dependence on t is only implicit. To stress this, the basis vector $\underline{e}_i(p(t))$ will next be denoted simply by $\underline{e}_i(p)$. An immediate consequence of this basis choice is that if $\underline{v}(t)$ is any left invariant vector field on $M(3)$, then its representative vector at point p can be expressed in the form

$$\underline{v}_p(t) = v_p^i(t)\underline{e}_i(p) .$$

Moreover, application of Eq. (20) yields

$$v_p^i(t) = v_e^i(t) \equiv v^i(t) . \tag{44}$$

Then, Eq. (39) can be rewritten in the form $n_p^i(t) = n_e^i(t) \equiv n^i(t)$, illustrating that the tangent vector to an autoparallel curve of $M(3)$ is part of a left invariant vector field on $M(3)$. Therefore, taking into account Eqs. (36) and (42), it is convenient to choose the left invariant canonical connection, expressed by Eq. (25a), as most appropriate for $M(3)$. An immediate consequence of this, with enormous significance, is that the autoparallels of $M(3)$ will coincide with its one parameter Lie subgroups and their left translations (see end of Section 3.2). In this respect, Eq. (41) can be seen as a means of determining the components of any vector $\underline{u}_p(t)$ of $T_pM(3)$, obtained by parallel translation of a vector $\underline{u}_e(t)$ of $T_eM(3)$ along the autoparallel curve of $M(3)$ connecting point $p(t)$ to the identity e. In fact, since the coefficient matrix $\tilde{n}$ in Eq. (41) is constant, this solution can be expressed in the form

$$\underline{u}_p(t) = B(t)\underline{u}_e(t) \quad \Rightarrow \quad \underline{u}_e(t) = A(t)\underline{u}_p(t) , \tag{45}$$

with

$$A(t) = \exp(t\tilde{n}) \tag{46}$$

and

$$B(t) = A^{-1}(t) = \exp(-t\tilde{n}) . \tag{47}$$

Among the infinity of available choices, the specific selection of the affinities expressed by Eq. (42) leads to a 3×3 matrix $A(t)$, given by Eq. (46). Here, this matrix appears in Eq. (45) as a linear transformation from $T_pM(3)$ to $T_eM(3)$.

The ideas presented above can provide a clear interpretation of rigid body kinematics. For instance, the basis $\{\underline{e}_i(p)\}$ obtained by left translation of a basis $\{\underline{e}_i(e)\}$ of $T_eM(3)$ on all of $M(3)$, as specified by (43), corresponds to a basis which remains fixed on the rigid body during its motion (body frame). In addition, the components of the tangent vector $\underline{w}(t)$ to the path $\gamma(t)$ of the motion, are components of the angular velocity of the body in the local basis $\{\underline{e}_i(p)\}$. That is

$$\underline{w}(t) = w^i(t)\underline{e}_i(p) . \tag{48}$$

This vector belongs to $T_pM(3)$ at point p and can be viewed as the outcome of the left translation of a vector of $T_eM(3)$

$$\underline{\Omega}(t) = \Omega^i(t)\underline{e}_i(e) , \tag{49}$$

known as the convective angular velocity [3]. Based on (44), the following choice is made for its components

$$w^i(t) = \Omega^i(t) . \tag{50}$$

Alternatively, vector $\underline{w}(t)$ can also be obtained by a parallel transfer of another vector of $T_eM(3)$, through the autoparallel curve of $M(3)$ joining the identity to the current point p. If

$$\underline{\omega}(t) = \omega^i(t)\underline{e}_i(e) , \tag{51}$$

then, according to Eq. (49), its components satisfy

$$w^i(t) = B^i_j(t)\omega^j(t) \quad \text{and} \quad \omega^i(t) = A^i_j(t)w^j(t) .$$

Next, the results obtained in this section are illustrated by Fig. 1. The actual motion of the body is described by a path $\gamma(t)$ on $M(3)$, starting from the identity element at $t = 0$. Then, for a fixed time t, points on the autoparallel curve of $M(3)$ connecting the current point $p(t)$ to the origin e, say $\eta_p(s)$, are located by another parameter, say s, related to the length of this path. In this case, in order to distinguish the true path $\gamma(t)$ from the autoparallel $\gamma_p(s)$, it is more appropriate to replace matrix $A(t)$ in Eq. (46) by

$$Q(s, \underline{n}(t)) = \exp(s\tilde{n}(t)) , \tag{52}$$

for $0 \le s \le t$ and $Q(t, \underline{n}(t)) = A(t) = R(t)$. This means that at any given time t, matrices A and Q coincide with matrix R. This is a manifestation of Euler's theorem, stating that one can move a rigid body, rotating about a fixed point, from an initial to any final position, through a pure rotation about a fixed axis [13]. In addition, the tangent vector $\underline{w}(t)$ to the actual path $\gamma(t)$ represents the angular velocity of the body and creates two images, $\underline{\Omega}(t)$ and $\underline{\omega}(t)$, in $T_eM(3)$ at any time t.

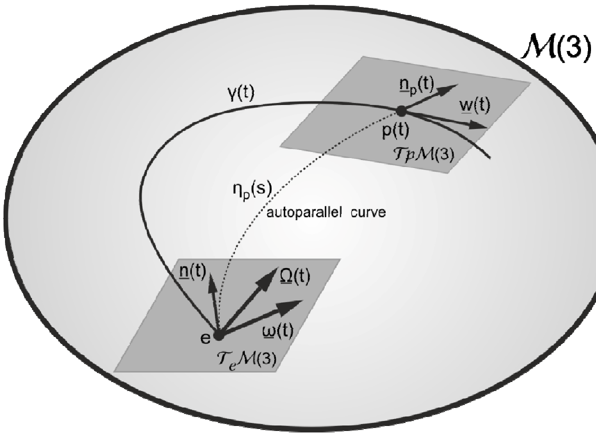

Fig. 1. Geometrical interpretation of spherical motion in $M(3)$

Finally, note that there exist several closed form expressions for matrix $A(t)$ in terms of rotation parameters. Among them, the most commonly known is probably the Rodrigues formula

$$A(t) = \exp[\tilde{\Psi}(t)] = I + \frac{\sin\|\underline{\Psi}\|}{\|\underline{\Psi}\|}\tilde{\Psi} + \frac{1}{2}\frac{\sin^2(\frac{1}{2}\|\underline{\Psi}\|)}{(\frac{1}{2}\|\underline{\Psi}\|)^2}\tilde{\Psi}\tilde{\Psi} .$$

Direct comparison with Eq. (50) shows that the quantity $\underline{\Psi}$, known as the Cartesian rotation vector [9], is defined by

$$\underline{\Psi}(t) = t\underline{n}(t)\,. \tag{53}$$

4.2 Group representation of *M(3)* on *SO(3)*

The group properties of $M(3)$ can be obtained by a group representation on $SO(3)$, through an appropriate map [16]. This map, say Φ from $M(3)$ to $SO(3)$, is selected to be an isomorphism [14]. In order to derive its explicit form, appropriate coordinates systems on the manifolds and bases on the tangent spaces should first be chosen.

In many occasions, it is beneficial to employ a special set of local coordinates, known as canonical [2]. Specifically, let $\eta(s)$ be the autoparallel curve of a manifold M^n emanating from a point p of the manifold at $s=0$, with a tangent vector $\underline{n}$ on the tangent space T_pM^n. If the coordinates of the origin (or pole) p are selected as

$$p^i = 0\,, \quad i = 1,\dots,n$$

and the tangent vector $\underline{n}$ is expressed over a basis $\{\underline{e}_i\}$ of T_pM^n in the form $\underline{n} = n^i\underline{e}_i$, then the canonical coordinates of any point q on the curve $\eta(s)$ are uniquely specified to be

$$q^i = sn^i\,. \tag{54}$$

Canonical coordinates specify uniquely any point q of M^n in the vicinity of the pole p. In the special case of a Lie group, it was shown at the end of Sect. 3 that the affinities Λ^i_{jk} can be obtained from the structure constants c^i_{jk}. In the particular case of $M(3)$, it was demonstrated in Sect. 4.1 that the left invariant canonical connection is the most natural choice for it, with affinities given by (42). Then, its autoparallel curves are identified by (39), which leads to (54), with $n=3$. Therefore, (54) defines a canonical coordinate system on $M(3)$ with origin at its identity element e. Moreover, direct comparison of (54) with (53) reveals that the canonical coordinates coincide with the components of the so called Cartesian rotation vector.

Based on the above, a canonical coordinate system is first placed on $M(3)$ with origin at e for locating its points. Next, a basis $\{\underline{e}_i(e)\}$ is selected at $m(3)$, according to conditions that will be stated more explicitly in Sect. 5. This basis is then extended to all points q of $M(3)$ by left translation

$$\underline{e}_i(q) = q\underline{e}_i(e)\,, \tag{55}$$

creating a non-natural basis [11].

According to material presented in Sections 3 and 4.1, an autoparallel curve emanating from the identity element e of $M(3)$ coincides with an one parameter Lie subgroup, which corresponds to pure rigid body rotation about an axis determined by the tangent vector to the curve at the identity. Conversely, given any point q on the manifold, one can find a vector on the tangent space at the identity element of $M(3)$, representing an axis of rotation of the body, which is tangent to the unique one parameter subgroup emanating from e and passing from q. Moreover, these special curves are captured by the corresponding exponential map of the group. In fact, the exponential map of a group G is a local diffeomorphism from a neighborhood of zero in T_eG onto a neighborhood of e in G [10]. In addition, this map can be extended over all of G through a left translation. Therefore, guided by Eq. (46), the mapping Φ

from $M(3)$ to $SO(3)$ is selected in the form

$$\Phi(q) = \exp(s\tilde{n}) , \tag{56}$$

where $\tilde{n} = spin(\underline{n})$ and $\underline{n}$ is the vector of $m(3)$ which is tangent to the autoparallel curve starting from the identity e and passing from point q of $M(3)$. Based on (54), the components of q depend on the parameters s and $\underline{n}$. Therefore, the quantity

$$Q(s,\underline{n}) = \Phi(q(s,\tilde{n})) \tag{57}$$

represents a 3×3 exponential matrix. Since matrix Q is orthogonal, it is an element of $SO(3)$, indeed. In view of Eq. (54), this matrix represents a map from the canonical coordinates of point q to $\mathbb{R}^9$. Moreover, since the identity element e of $M(3)$ has canonical coordinates ($0,0,0$), it turns out from Eqs. (56) and (57) that

$$\Phi(e) = Q(0,\underline{n}) = I , \tag{58}$$

which verifies that Φ maps the identity element e of $M(3)$ to the identity element I of $SO(3)$.

This information is enough to determine the tangent mapping of Φ at the identity point e of $M(3)$ by

$$\Phi(e)_*(\cdot) = spin(\cdot) . \tag{59}$$

Then, the Lie bracket of the Lie algebra $m(3)$ is defined by

$$[\underline{X}_1(e), \underline{X}_2(e)] = vect([Y_1(I), Y_2(I)]) , \tag{60}$$

while the tangent mapping of Φ at an arbitrary point q of $M(3)$ is obtained in the form

$$\Phi(q)_* = \Phi(q)\Phi(e)_* = Q\Phi(e)_* .$$

This completes the geometrical picture of the mapping between $M(3)$ and $SO(3)$, shown in Fig. 2.

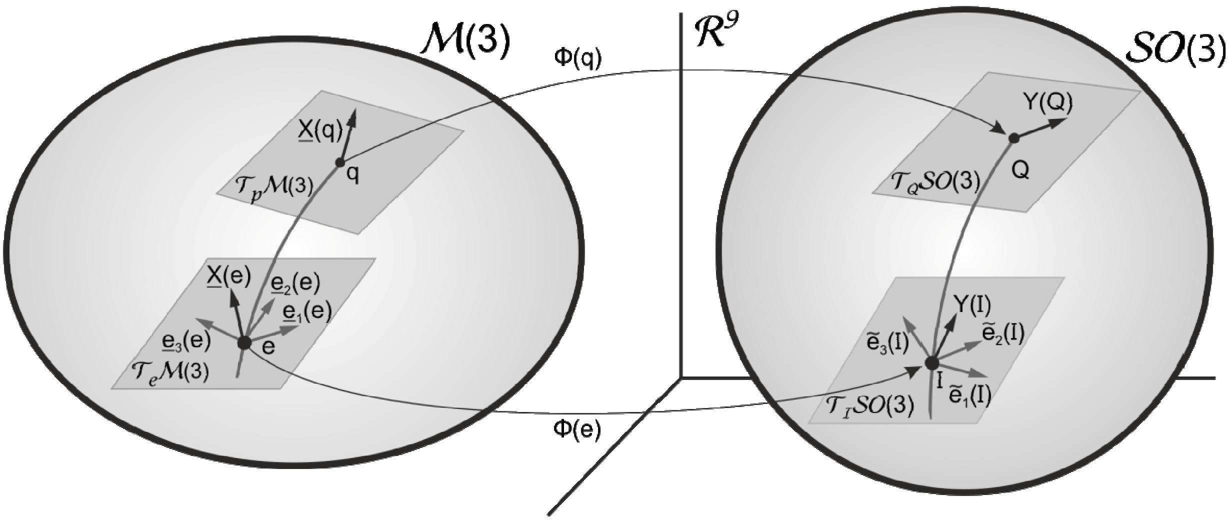

Fig. 2. Definition of mapping Φ for a group representation of $M(3)$ on $SO(3)$.

Finally, direct computation yields

$$c_{ij}^{k} = \tilde{c}_{ij}^{k}, \tag{61}$$

which determines the structure constants c_{ij}^{k} in $m(3)$ and consequently completes the definition of its Lie bracket in terms of the structure constants $\tilde{c}_{ij}^{k}$ of the basis in $so(3)$. At the same time, this result establishes the affinities of the connection on all of $M(3)$, through Eqs. (27), (34) and (42).

5 APPLICATION TO RIGID BODY DYNAMICS

Study of rigid body dynamics necessitates consideration of the dual space $T_pM(3)^*$ to $T_pM(3)$. This space includes elements known as covectors, which are linear functionals on $T_pM(3)$ [12]. More specifically, if $\underline{v}$ is a vector of $T_pM(3)$, then there exists a covector $\underset{\sim}{v}^*$ of $T_pM(3)^*$, such that

$$\underset{\sim}{v}^*(\underline{u}) \equiv \langle \underline{v}, \underline{u} \rangle, \quad \forall \underline{u} \in T_pM(3), \tag{62}$$

where $\langle \cdot,\cdot \rangle$ denotes the inner product of $T_pM(3)$. In addition, a basis $\{\underset{\sim}{e}^i\}$ (with $i = 1,2,3$) of $T_pM(3)^*$ is called dual basis to the basis $\{\underline{e}_i\}$ of $T_pM(3)$, provided that

$$\underset{\sim}{e}^i(\underline{e}_j) = \delta_j^i, \tag{63}$$

where δ_j^i is a Kronecker delta symbol. Then,

$$\underset{\sim}{v}^* = v_i \underset{\sim}{e}^i,$$

with components v_i determined by

$$v_i = g_{ij} v^j, \tag{64}$$

where the scalars

$$g_{ij} = \langle \underline{e}_i, \underline{e}_j \rangle \tag{65}$$

are components of the metric tensor in basis $\{\underline{e}_i\}$ of $T_pM(3)$.

The choice of the body frame can now be performed. First, it turns out successively that

$$\underset{\sim}{w}^*(\underline{w}) = \langle \underline{w}, \underline{w} \rangle = w^i w^j g_{ij} = \Omega^i g_{ij} \Omega^j .$$

Therefore, direct comparison of the last result with Eq. (14) shows that the kinetic energy of the body can be expressed by

$$T = \tfrac{1}{2} \langle \underline{w}, \underline{w} \rangle = \tfrac{1}{2} \Omega^i g_{ij} \Omega^j, \tag{66}$$

provided that

$$g_{ij} = J_{ij}, \tag{67}$$

with $J_O = [J_{ij}]$. Next, a basis $\{\underline{e}_i(e)\}$ can be selected for the tangent space $m(3)$ through (65), so that it satisfies

$$\langle \underline{e}_i(e), \underline{e}_j(e) \rangle = J_{ij}. \tag{68}$$

This basis is then left translated at any point p of $M(3)$, according to (55). Therefore, if the metric tensor is chosen to be left invariant, then

$$\langle \underline{e}_i(p), \underline{e}_j(p) \rangle = J_{ij}, \qquad \forall p \in M(3). \tag{69}$$

The geometry of manifold $M(3)$ presents some significant differences with the classical rotation group $SO(3)$. The first appears in the connection. Specifically, it was shown in Sect. 4.1 that the most convenient choice for describing spherical motion on $M(3)$ is the left invariant canonical connection, defined by Eq. (25a). This choice leads to a non-Riemannian manifold, with torsion and no curvature, which is in sharp contrast to the geometrical properties of the ordinary $SO(3)$.

Another important deviation between $M(3)$ and $SO(3)$ occurs in the metric tensor. By employing the orthogonality property (4), the components of the metric tensor in $SO(3)$ can be determined in the form

$$\hat{g}_{ij} = 2\delta_{ij}. \tag{70}$$

This metric is symmetric and positive definite and can be shown to be compatible with the connection chosen [11]. This verifies that $SO(3)$ is a Riemannian manifold, since it also possesses curvature and is torsionless. However, the components of the metric tensor of $SO(3)$ are different than those of the inertia tensor in $M(3)$, whose components are defined by (67).

The picture of the dynamics of rigid body spherical motion can now be completed by illuminating the role of $T_pM(3)^*$ and the selection of $\gamma(t)$. Direct comparison of (67) and (13) demonstrates that the covector $\underset{\sim}{w}^*$ corresponding to the angular velocity vector $\underline{w}$, is the angular momentum vector. Namely

$$\underset{\sim}{w}^* = \underset{\sim}{H}_O \quad \Rightarrow \quad H_i = J_{ij}\Omega^j. \tag{71}$$

Also, comparison of Eqs. (62) and (66) shows that the dual product of the angular velocity vector $\underline{w}$ with the angular momentum $\underset{\sim}{H}_O$ yields the kinetic energy of the body. That is,

$$T = \tfrac{1}{2}\underset{\sim}{w}^*(\underline{w}) = \tfrac{1}{2}\underset{\sim}{H}_O(\underline{w}),$$

which is equivalent to Eq. (14). In addition, the selection of the solution curve $\gamma(t)$ on $M(3)$ is performed by Euler's law

$$\dot{\underset{\sim}{H}}_O = \underset{\sim}{M}_O, \tag{72}$$

where $\underset{\sim}{M}_O = M_i \underset{\sim}{e}^i$ is the resultant moment with respect to point O. Therefore, by employing (71) and taking into account that the components J_{ij} are constant, (72) becomes

$$J_{ij}\dot{\Omega}^j + \tilde{\Omega}^j_i J_{jk}\Omega^k = M_i, \tag{73}$$

which matches Eq. (15), with

$$[-\Lambda^j_{ki}\Omega^k] \equiv [\tilde{\Omega}^j_i] = \tilde{\Omega} = spin(\underline{\Omega}). \tag{74}$$

Finally, the affinities selected for $M(3)$ according to (42) match its group and manifold properties but are not compatible with its metric, expressed by (67). This implies that the metric is not preserved under parallel transfer along an arbitrary path. For instance, differentiation of both sides of (66) with respect to time and simultaneous application of the symmetry condition of the inertia tensor $J_{ij} = J_{ji}$, yields eventually

$$\dot{T} = \Omega^i J_{ij} \dot{\Omega}^j \,. \tag{75}$$

Among all the possible paths, only on the real one it is true that $\dot{\Theta}^m = \Omega^m$. Therefore, for $\underset{\sim}{M}_O = \underset{\sim}{0}$, i.e., for torque free motion

$$J_{ij} \dot{\Omega}^j = -\tilde{\Omega}^j_i J_{jk} \Omega^k = -(-\Lambda^j_{mi} \Omega^m) J_{jk} \Omega^k = \Lambda^j_{mi} \Omega^m H_j \,,$$

which after substituting in (75) yields

$$\dot{T} = \Omega^i (\Lambda^j_{mi} \Omega^m H_j) = (\Lambda^j_{mi} \Omega^i \Omega^m) H_j = 0 \,,$$

due to the anti-symmetry property of the affinities selected for $M(3)$. This shows that the kinetic energy is conserved along the path corresponding to the actual motion of the body.

6 SYNOPSIS

Finite rigid body rotation has been treated in this study. Borrowing ideas from Lie group theory provided a solid foundation for a thorough and consistent investigation of rigid body kinematics and dynamics. As a result, the following elegant geometrical picture emerged.

First, the orientation of a rigid body was represented by a single point, while the motion over a finite time interval was described by a curve on a three dimensional manifold. Then, it was demonstrated that, contrary to common belief, the well known special orthogonal group $SO(3)$ is not appropriate for describing either the kinematics or the dynamics of large rigid body rotation. In fact, a new manifold was introduced, named $M(3)$, which is diffeomorphic to $SO(3)$. Specifically, a significant contribution of this work was the selection of a canonical connection for $M(3)$, so that its autoparallel curves, representing pure rotation of the body, coincide with its one parameter Lie subgroups, which are located conveniently by the exponential map. This led to a manifold possessing torsion and no curvature, in contrast to the classical rotational group $SO(3)$, which is a Riemannian manifold with curvature and no torsion. Moreover, the exponential map provided the ground for choosing a holonomic coordinate system in determining uniquely the points on $M(3)$. In particular, the components of the classical Cartesian rotation vector were picked up as canonical coordinates. However, an anholonomic coordinate frame was selected for expressing the vectors of the tangent space at the current point. This frame is fixed on the body (body frame) and was obtained mathematically by a left translation of an appropriate basis at the identity. Moreover, all the important geometrical properties of $M(3)$, were also selected by a suitable representation on the rotation group $SO(3)$.

Next, the emphasis was put on dynamics. The components of the metric tensor were chosen to match the components of the mass moment of inertia tensor of the body. This provided the body frame at the identity. Then, the equations of motion were derived by applying Euler's law. Finally, since the connection and metric of $M(3)$ are not compatible, in contrast to the case of $SO(3)$, the inner product in the tangent space (representing kinetic energy) is not preserved along any arbitrary curve. However, it was proved that the kinetic energy is conserved along the true path, during a torque-free motion.

The theoretical results of this study provide valuable tools for developing efficient techniques for their geometrically exact temporal discretization in all areas of mechanics, involving rigid body rotation. This will be demonstrated in future publications.

REFERENCES

[1] J. Argyris, V.F. Poterasu, Large rotations revisited application of Lie algebra, *Computer*

Methods in Applied Mechanics and Engineering, **103**, 11-42, 1993.

[2] R.M. Murray, Z. Li, S.S. Sastry, *A Mathematical Introduction to Robot Manipulation*, CRC Press, Boca Raton, Florida, 1994.

[3] J.C. Simo, L. Vu-Quoc, A geometrically-exact rod model incorporating shear and torsion-warping deformation, *Int. J. Solids Struct*., **27**, 371-393, 1991.

[4] D. Negrut, E.J. Haug, H.C. German, An implicit Runge-Kutta method for the integration of differential-algebraic equations of multibody dynamics, *Multibody System Dyn.*, **9**, 121-142, 2003.

[5] O. Brüls, A. Cardona, M. Arnold, Lie group generalized-α time integration of constrained flexible multibody systems, *Mechanism and Machine Theory*, **48**, 121-137, 2012.

[6] W. Flugge, *Tensor Analysis and Continuum Mechanics*, Springer-Verlag, New York, 1972.

[7] D.H. Sattinger, O.L. Weaver, *Lie Groups and Algebras with Applications to Physics, Geometry and Mechanics*, Springer-Verlag, New York, 1986.

[8] W. Bertram, Differential Geometry, Lie Groups and Symmetric Spaces over General Base Fields and Rings, *Memoirs of the American Mathematical Society*, **192**, 1-202, 2008.

[9] M. Geradin, A. Cardona, *Flexible Multibody Dynamics*, John Wiley & Sons, New York, 2001.

[10] J.E. Marsden, T.S. Ratiu, *Introduction to Mechanics and Symmetry, second ed.*, Springer Science+ Business Media Inc., New York, 1999.

[11] J.G. Papastavridis, *Tensor Calculus and Analytical Dynamics*, CRC Press, Boca Raton, 1999.

[12] T. Frankel, *The Geometry of Physics: An Introduction*, Cambridge University Press, New York, 1997.

[13] A.A. Shabana, *Dynamics of Multibody Systems, third ed.*, Cambridge University Press, New York, 2005.

[14] E. Paraskevopoulos, S. Natsiavas, A new look into the kinematics and dynamics of finite rigid body rotations using Lie group theory, *Int. J. Solids Struct.*, **50**, 57-72, 2013.

[15] S.V. Shabanov, Constrained systems and analytical mechanics in spaces with torsion, *J. Phys. A: Mathematical and General*, **31**, 5177-5190, 1998.

[16] B.C. Hall, *Lie Groups, Lie Algebras and Representations: An Elementary Introduction*, Springer-Verlag, New York., 2003.