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COMPUTATIONAL MODELING OF THE WAVES PROPAGATION IN A BLOCK MEDIUM WITH VISCOELASTIC INTERLAYERS

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Abstract. *In the framework of mathematical model of a block medium with elastic blocks interacting through compliant viscoelastic interlayers and its approximation on the basis of equations of the Cosserat continuum, the problems of periodic perturbation of a layer and of a half-space under the action of distributed and localized surface loads are solved numerically. The simple formulas are suggested to determine the elasticity coefficients of the moment continuum by given characteristics of the materials of blocks and interlayers, which provide a good correspondence of the wave fields received by means of the exact and approximate models. Parallel computational algorithms, based on models of the inhomogeneous elasticity theory and the Cosserat elasticity theory, are applied to the analysis of propagation of elastic waves in geomaterials with layered and block microstructure. These algorithms are realized as parallel program systems for GPUs (using CUDA technology) and for multiprocessor computers of the cluster type (using MPI library). Monotone grid-characteristic schemes with a balanced number of time steps in elastic layers or blocks and in viscoelastic interlayers are used. By the analysis of numerical solutions it is shown that the multiblock medium has a resonant frequency of rotational motion of blocks, and this frequency does not depend on the size of a massif and on the boundary conditions at its surface.*

1 INTRODUCTION

Many natural materials, such as rocks and soils, are characterized by inhomogeneous block-hierarchical structure. The block structure is observed at different levels of scale: from the size of crystal grains to large blocks of a rock body, separated by faults. The blocks are connected to each other by means of interlayers of a rock with substantially more compliant mechanical properties [1]. One of the most important technological problems of coal mining is to forecast the sudden collapse of the roof of coal mines. This process is preceded by the weakening of mechanical contact between blocks: the rock becomes weakened microstructure. This state of a material can be found by exciting elastic waves of small amplitude and recording the response to these perturbations, which can be used in the development of special technical devices for timely prediction and prevention of emergencies.

The theory of inhomogeneous media with layered and block microstructure is the field of mechanics, intensively developing during more than half a century. By present time the efficient analytical methods in this field are developed [2, 3, 4], numerical algorithms for the solution of quasi-static and dynamic problems are worked out by means of the finite element and finite difference approximation of continuous models. However such methods are practically unsuitable for the problems on propagation of high-frequency waves in block media, lengths of which are comparable with the size of blocks and layers. This is because of the methodological errors, caused by the approximation viscosity, which are comparable with the physical viscosity. It is necessary to perform computations on fine grids, the mesh size of which is consistent with the characteristic size of the blocks. For these computations it is appropriate to use modern multiprocessor computer systems.

The works [5, 6] are devoted to experimental and theoretical study of low-frequency oscillations of the blocks caused by the propagation of short single impulses of pressure in block media (the so-called pendulum waves). One-dimensional computations of the longitudinal waves of pendulum type on the basis of the equations of a layered medium are carried out in [7]. The purpose of this paper is to work out two-dimensional and three-dimensional models for numerical analysis of the propagation of waves. Computations, performed by means of these models, allow to analyze resonant motion of a block medium due to accounting rotational degrees of freedom of the blocks.

2 EQUATIONS OF A BLOCK MEDIUM

Let us consider the state of plane strain of a block medium, formed by square elastic blocks with sides of the length h, parallel to the coordinate axes x_1, x_2 of a Cartesian coordinate system, and thin interlayers of the thickness δ. Blocks are numbered by pairs of indices k_1 and k_2, taking the values from 1 to N_1 and N_2, respectively. The system of equations of a homogeneous isotropic elastic medium

$$\begin{aligned}
&\rho\,\dot v_1 = \sigma_{11,1} + \sigma_{12,2},\\
&\rho\,\dot v_2 = \sigma_{12,1} + \sigma_{22,2},\\
&\dot\sigma_{11} = \rho\,c_1^2(v_{1,1} + v_{2,2}) - 2\rho\,c_2^2\,v_{2,2},\\
&\dot\sigma_{22} = \rho\,c_1^2(v_{1,1} + v_{2,2}) - 2\rho\,c_2^2\,v_{1,1},\\
&\dot\sigma_{12} = \rho\,c_2^2(v_{2,1} + v_{1,2})
\end{aligned} \tag{1}$$

is fulfilled inside of each block. Here v is the velocity vector, σ is the symmetric stress tensor, ρ is the density of a material, c_1 and c_2 are the velocities of longitudinal and transverse elastic waves, the dot over the symbol and indices after the comma denote the derivatives with respect to time and spatial variables.

Elastic interlayer between neighboring blocks with the numbers (k_1, k_2) and $(k_1 + 1, k_2)$ in the horizontal direction is described by ordinary differential equations, taking into account its mass and the longitudinal and transverse rigidities:

$$\begin{aligned} &\rho' \, \frac{\dot v_1^+ + \dot v_1^-}{2} = \frac{\sigma_{11}^+ - \sigma_{11}^-}{\delta}, \quad \frac{\dot\sigma_{11}^+ + \dot\sigma_{11}^-}{2} = \rho' c_1'^2 \, \frac{v_1^+ - v_1^-}{\delta}, \\ &\rho' \, \frac{\dot v_2^+ + \dot v_2^-}{2} = \frac{\sigma_{12}^+ - \sigma_{12}^-}{\delta}, \quad \frac{\dot\sigma_{12}^+ + \dot\sigma_{12}^-}{2} = \rho' c_2'^2 \, \frac{v_2^+ - v_2^-}{\delta}. \end{aligned} \tag{2}$$

Elastic interlayer between blocks with the numbers (k_1, k_2) and $(k_1, k_2 + 1)$ in the vertical direction is described by the equations:

$$\begin{aligned} &\rho' \, \frac{\dot v_2^+ + \dot v_2^-}{2} = \frac{\sigma_{22}^+ - \sigma_{22}^-}{\delta}, \quad \frac{\dot\sigma_{22}^+ + \dot\sigma_{22}^-}{2} = \rho' c_1'^2 \, \frac{v_2^+ - v_2^-}{\delta}, \\ &\rho' \, \frac{\dot v_1^+ + \dot v_1^-}{2} = \frac{\sigma_{12}^+ - \sigma_{12}^-}{\delta}, \quad \frac{\dot\sigma_{12}^+ + \dot\sigma_{12}^-}{2} = \rho' c_2'^2 \, \frac{v_1^+ - v_1^-}{\delta}. \end{aligned} \tag{3}$$

Here the quantities with superscripts relate to the boundary meshes of interacting blocks. Similar equations were used in [7] for the one-dimensional model of a layered medium. It can be shown that these equations are thermodynamically consistent with the system of equations (1), i.e. for a regular block structure the energy conservation law is fulfilled, in which the sum of kinetic and potential energy is the sum of kinetic and potential energies of blocks and interlayers separately.

When taking into account the viscoelastic strains in interlayers, the equations (2), (3) are replaced by more general equations depending on the rheological scheme of a material [7]. For numerical solution of the equations (1) – (3) and the system, which takes into account the viscoelastic properties of interlayers on the basis of rheological schemes by Maxwell, Kelvin–Voigt and Poynting–Thomson, the parallel computational algorithm is worked out. In this algorithm the Godunov discontinuity decay method is realized for equations of the elasticity theory on a uniform grid with a choice of the maximum permissible time step according to the Courant–Friedrichs–Levy condition. At smaller time step a piecewise-linear ENO–reconstruction of the second order of accuracy is used [8]. The method of two-cyclic splitting with respect to spatial variables is applied for the solution of two-dimensional problems. Splitting method leads to a series of one-dimensional problems, solved by means of a monotone ENO–scheme. Parallelization of the algorithm is performed for computer systems on graphics accelerators using the CUDA technology. Study of the effectiveness of parallel implementation of the algorithm showed acceleration of the program up to 50 times compared to the sequential version.

Created programs are used to solve a series of one-dimensional and two-dimensional problems on the propagation of waves, caused by the influence of short and long (periodic) localized loads at the boundary of a layered (block) medium. Numerical solutions, obtained on the basis of the equations (1) – (3) and in the framework of complete formulation, are compared for Lamb's problem on the action of concentrated force on a surface of massif of a block medium with elastic interlayers. In the complete formulation the interlayers are modeled using the equations of plane theory of elasticity with the pasting together conditions at interfaces. Computations in the complete formulation are performed by means of the 2Dyn_Granular program [9] on the MVS-100K cluster of the Joint Supercomputer Center of RAS. Numerical results for both models are in good correspondence. However, because of a large dimension of the finite-difference grid in interlayers, the computations on the cluster using MPI require on the order more time than the computations on the GPUs using CUDA by simplified model (1) – (3).

In Figs. 1 and 2 one can see the graphs of velocity on the spatial coordinate for the one-dimensional problem on the action of Λ–shaped impulse of a pressure on boundary of a layered medium composed of 512 layers of a rock with microfractures elastic interlayers of a ground [7]. Computations were carried out after reduction of the system of equations to dimensionless variables with the following parameters: $\delta/h = 0.027$, $\rho'/\rho = 0.76$, $a'/a = 7.17$ ($a = 1/(\rho\, c^2)$ and $a' = 1/(\rho' c'^2)$ are the elastic compliances of materials). A uniform finite-difference grid in layers consists of 16 meshes. One mesh is used within each interlayer. Choice of the integer parameters in the form of powers of two is related to the specific features of programming and memory allocation in CUDA. Computations were performed on the eight-core computer with graphics card Tesla C2050.

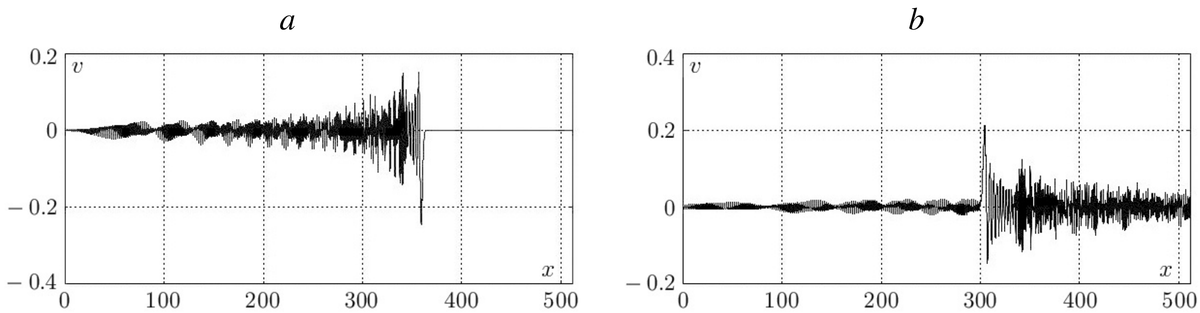


Figure 1: Velocity distribution behind the front of the incident wave (*a*) and the reflected wave (*b*), caused in a layered medium by the influence of a short impulse.

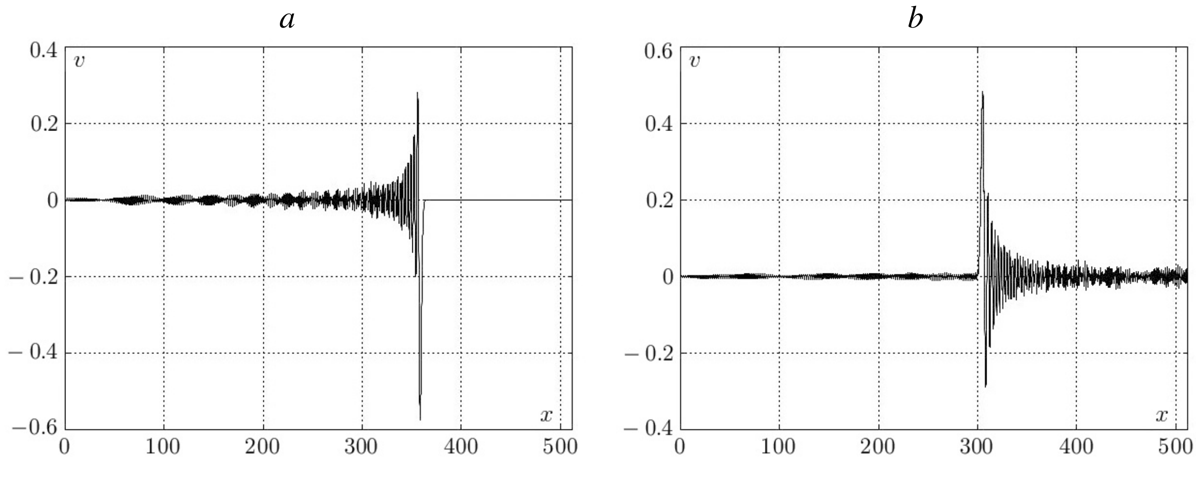


Figure 2: Velocity distribution behind the front of the incident wave (*a*) and the reflected wave (*b*), caused in a layered medium by the influence of a long impulse.

Fig. 1 corresponds to the impulse duration which is equal to the time of passage of elastic wave through one layer, Fig. 2 corresponds to the duration which is two and a half times greater. The impulse of unit amplitude acts on the left boundary of computational domain, the right boundary is fixed. In Figs. 1 *a* and 2 *a* the velocity profiles are shown at the moment when the incident wave goes about 370 layers (6000-th time step of the basic scheme). In Figs. 1 *b* and 2 *b* the reflected wave goes in the opposite direction about 200 layers (12000-th time step of the basic scheme). These results demonstrate a qualitative difference of the wave pattern in layered media as compared with a homogeneous medium. At the initial stage this difference is the appearance of waves reflected from the interlayers – the characteristic oscillations behind the front of loading wave as it passes through the interface. With time, after multiple reflections behind the front of a head wave appears stationary wave pattern, the so-called pendulum wave, whose existence was predicted in [5, 6].

A comparison of Figs. 1 and 2 show that with an increase in impulse duration the amplitude of a head wave increases up to unity, and the amplitude of oscillations behind the front decreases and tends to zero. This is due to the fact that waves, which lengths are considerably greater than the thickness of the interlayer, are practically not reflected from the interlayers. Thus, it is possible to detect a weakened microstructure of layered or block medium only with the help of sufficiently short waves.

Fig. 3 presents the results of two-dimensional computations within the framework of the model in complete formulation. Computations for block media consisting of 6 blocks (Figs. 3 *a* and 3 *b*) and 15 blocks (Figs. 3 *c* and 3 *d*) were performed on a cluster system. The thickness of interlayers is 5 times smaller than the thickness of blocks. The following parameters of materials are used: $\rho = 3700$, $\rho' = 1200\,\text{kg/m}^3$, $c_1 = 3500$, $c_2 = 2100$, $c_1' = 1500$, $c_2' = 360\,\text{m/c}$. In Fig. 3 *a* and 3 *c* the level curves of the stress σ_{11} for Lamb's problem on the action of a normal force, concentrated in the central point of upper boundary of a half-plane, are shown. In Fig. 3 *b* and 3 *d* the level curves of the stress σ_{12} for similar problem on the action of a concentrated tangential force are shown (the x_1 axis is directed downwards, deep into the massif, and the x_2 axis is in the horizontal direction).

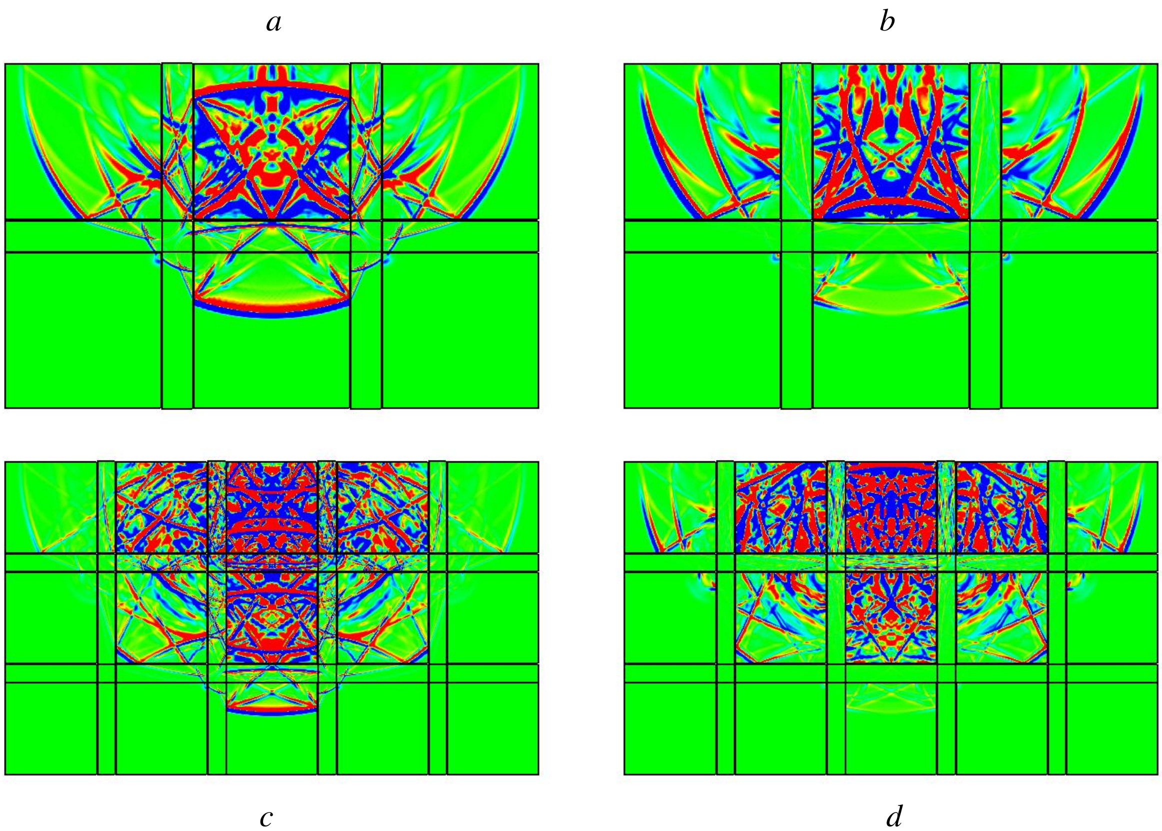
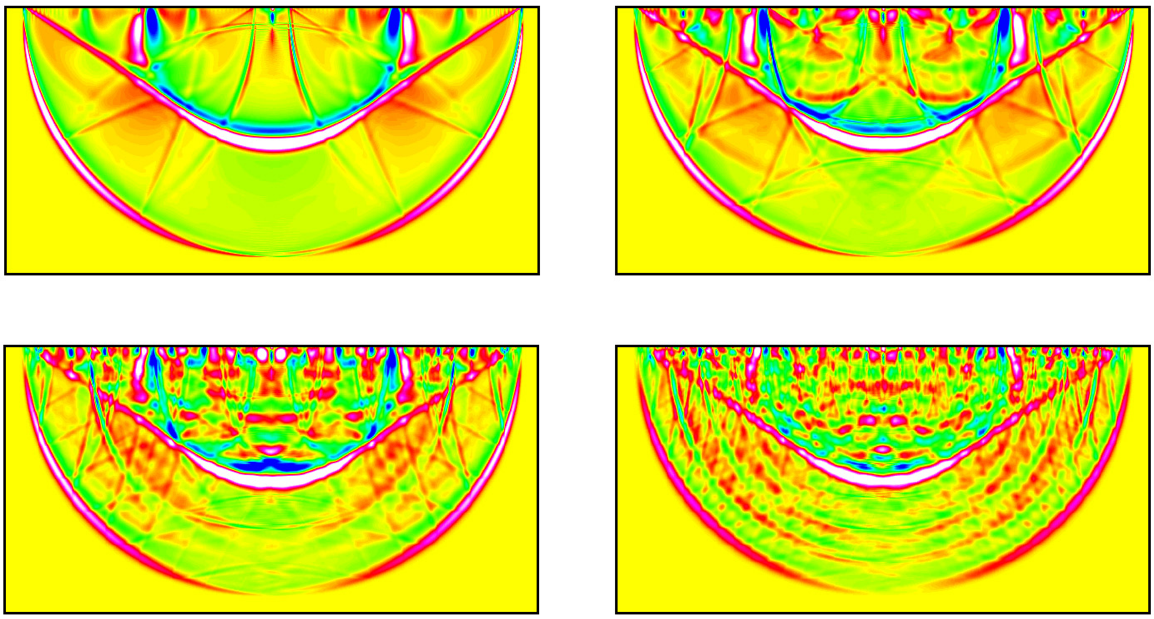


Figure 3: Lamb's problem for a block medium with thick elastic interlayers: level curves of normal stress (*a*, *c*) and tangential stress (*b*, *d*).

Under the action of a impulse load there are formed incident longitudinal and transverse waves, conical transverse waves, propagating deep into the massif with time, and also the Rayleigh surface waves, rapidly damped with depth. Moreover, a series of waves, reflected from interfaces, and a series of surface waves, generated under the arrival of reflected waves on the boundary of a half-plane, appear because of the presence of compliant interlayers.

In Fig. 4 one can see computational results for a multiblock medium (Lamb's problem on the action of a concentrated tangential force), performed on the basis of a simplified model (1) – (3). Computational domains consist of 4×2 blocks of a rock with 10 interlayers of a ground in Fig. 4 *a*, of 8×4 blocks with 52 interlayers in Fig. 4 *b*, of 16×8 blocks with 232 interlayers in Fig. 4 *c*, and of 32×16 blocks with 976 interlayers in Fig. 4 *d*. Computations showed that the increase of the number of blocks and a proportional decrease of the parameters h and δ lead to the appearance of a diffuse wave front, caused by the rotational motion of blocks, behind the front of an incident transverse wave. It is a confirmation of intuitive representation that a multiblock medium with compliant interlayers can be approximated by a generalized continuum. The asymmetry of stress tensor, caused by the rotational motion of blocks, and the couple stresses, associated with the curvature of a regular block structure due to the inhomogeneity of rotations, are taken into account in the equations of a generalized continuum.

a *b*



c *d*

Figure 4: Lamb's problem for a block medium with thin elastic interlayers: level curves of tangential stress; 4×2 blocks of a rock with 10 interlayers of a ground (*a*), 8×4 blocks with 52 interlayers (*b*), 16×8 blocks with 232 interlayers (*c*), 32×16 blocks with 976 interlayers (*d*).

3 A BLOCK MEDIUM AS THE COSSERAT CONTINUUM

Known property of layered and block media, tested many times experimentally and by means of numerical computations (see, for example, [6, 7]), is that the waves of small amplitude, the length of which is larger than the size of layer or block, move in such a medium almost as in a homogeneous medium with some effective elastic moduli. Strictly speaking, this applies only to the waves of translational motion. Perturbation of rotational motion of the blocks leads to the appearance of new types of waves. Let us describe the propagation of long waves in a multiblock medium by means of the system of equations of the Cosserat continuum, which is widely used in the modeling of structurally inhomogeneous materials at different scale levels [10, 11, 12]. To do this, assuming that the block size is small compared with the size of all massif, let us calculate the phenomenological parameters of the moment continuum via the elasticity parameters of materials of blocks and interlayers.

One of the simplest ways of recalculation is connected with the requirement of correspondence between models in the special schemes of quasistatic or dynamic deformation [13, 14, 15, 16]. The scheme of deformation of a block medium, in which blocks rotate by a small angle φ, and their centers of mass remain fixed, is shown in Fig. 5 *a*. The deformation of blocks can be neglected taking into account the compliance of interlayers. In this approximation, the interlayers are in a state of pure shear with the angle of shear $\gamma = h\,\varphi/\delta$. The corresponding tangential stress $\tau = \mu'\gamma$ ($\mu' = \rho' c_2'^2$ is the shear modulus for a material of interlayer) is proportional to the rotation angle. The proportionality coefficient $f_\alpha = \tau/\varphi = \mu' h/\delta$ is one of the phenomenological parameters of the Cosserat continuum.

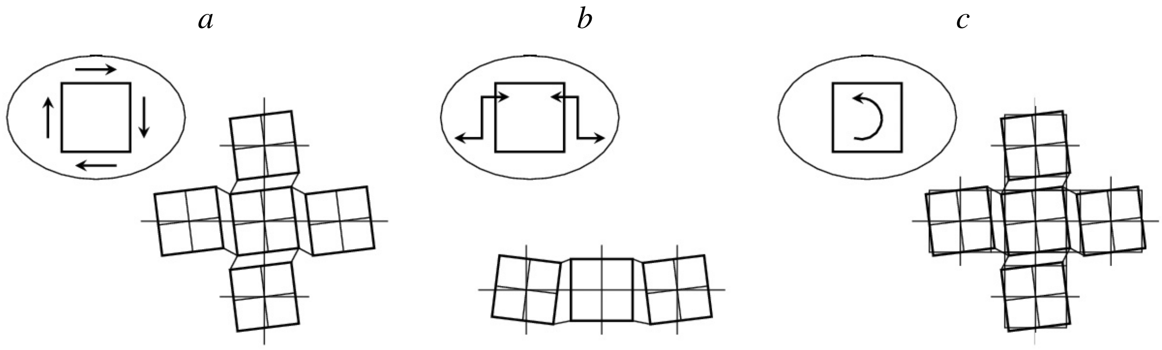


Figure 5: Schemes of deformation of a block medium: uniform rotation of blocks (*a*), nonuniform rotation with formation of curvature (*b*), uniform torsion (*c*).

In the scheme, represented in Fig. 5 *b*, the displacements of centers of mass are also absent. The curvature $æ = \partial\varphi/\partial x$ is formed because of the counter rotation of neighboring blocks. It can be estimated by the formula: $æ = \varphi/(h+\delta)$. Linear distribution of the strain $\varepsilon = -\varphi\, y/\delta$ and the normal stress $\sigma = E'\varepsilon$ appears in the interlayer. Here x and y are the horizontal and vertical coordinates, $E' = \rho' c_2'^2 (3c_1'^2 - 4c_2'^2)/(c_1'^2 - c_2'^2)$ is the Young modulus of a material of interlayer. Action of the stress is reduced to the action of the bending moment $m = E'\varphi\, h^3/(12\delta)$. Consequently, one more parameter of the Cosserat continuum can be defined by the formula: $f_\beta = E'(h+\delta)h^3/(12\delta)$.

In Fig. 5 *c* one can see the scheme of uniform torsion of a block medium around the axes passing through the fixed centers of mass of blocks perpendicular to the plane of the figure. In this state the stresses in interlayers are determined by solving the problem of elastic torsion. Projections of the displacement vector in interlayer onto horizontal and vertical axes are expressed by formulas: $u_x = -y\,\varphi\, z/\delta$ and $u_y = x\,\varphi\, z/\delta$. According to Hooke's law, the tangential stresses $\tau_x = \mu'\partial u_x/\partial z$ and $\tau_y = \mu'\partial u_y/\partial z$ are found. The torsional moment is defined as the integral

$$M = \int\limits_{-h/2}^{h/2}\int\limits_{-h/2}^{h/2} (x\,\tau_y - y\,\tau_x)\,dx\,dy = \frac{\mu'\varphi}{\delta}\int\limits_{-h/2}^{h/2}\int\limits_{-h/2}^{h/2} (x^2 + y^2)\,dx\,dy = \frac{\mu'\varphi\,h^4}{6\delta}.$$

The torsion $æ_0 = \partial\varphi\partial z$ is estimated using the above formula for the curvature. Therefore the phenomenological parameter $f_\kappa = M/æ_0$, which equals the ratio of torsional moment to torsion, can be calculated by the formula: $f_\kappa = \mu'(h+\delta)h^4/(6\delta)$.

In the spatial state the elastic properties of an isotropic moment continuum are characterized by six independent parameters. Recalculation of three of them is performed via f_α, f_β and f_κ. One more parameter can be found from the condition of realizability of a generalized plane stress state. It turns out that in general case the constitutive equations of the Cosserat elasticity theory do not allow simultaneous equality to zero of the projections of the angular velocity vector and the components of the couple stress tensor, which are absent in this state.

The complete system of equations for description of the dynamics of the Cosserat continuum in terms of components of the linear velocity vector v_p, the angular velocity vector ω_p, the asymmetric stress tensor σ_{pq} and the couple stress tensor m_{pq} can be represented in the next form [17, 18]:

$$\begin{aligned}
&\rho_0\,\dot v_p = \sigma_{1p,1} + \sigma_{2p,2} + \sigma_{3p,3}\,,\\
&\dot\sigma_{pp} = \Big(k + \frac{4\mu}{3}\Big)v_{p,p} + \Big(k - \frac{2\mu}{3}\Big)(v_{q,q} + v_{r,r})\,,\\
&\dot\sigma_{pq} = (\mu+\alpha)\,v_{q,p} + (\mu-\alpha)\,v_{p,q} - 2\alpha\,\omega_r\,,\\
&\dot\sigma_{qp} = (\mu-\alpha)\,v_{q,p} + (\mu+\alpha)\,v_{p,q} + 2\alpha\,\omega_r\,,\\
&j_0\,\dot\omega_p = m_{1p,1} + m_{2p,2} + m_{3p,3} + \sigma_{qr} - \sigma_{rq}\,,\\
&\dot m_{pp} = \Big(\kappa + \frac{4\eta}{3}\Big)\omega_{p,p} + \Big(\kappa - \frac{2\eta}{3}\Big)(\omega_{q,q} + \omega_{r,r})\,,\\
&\dot m_{pq} = (\eta+\beta)\,\omega_{q,p} + (\eta-\beta)\,\omega_{p,q}\,,\\
&\dot m_{qp} = (\eta-\beta)\,\omega_{q,p} + (\eta+\beta)\,\omega_{p,q}\,.
\end{aligned} \tag{4}$$

Here the indices p, q and r take the values $1, 2, 3$ and besides $q = p+1 \bmod 3$, $r = q+1 \bmod 3$. The symbol j_0 denotes the inertial characteristic of a material, which is equal to the product of the moment of inertia of a block and the number of blocks in a unit volume:

$$j_0 = jN, \quad j = \rho\,\frac{h^5}{6}, \quad N = \frac{1}{(h+\delta)^3}.$$

Comparing the equations of the system (4) and the equations for stresses and couple stresses, coefficients of which are the parameters f_α, f_β and f_κ, one can obtain the following conversion formulas:

$$\alpha = \frac{f_\alpha}{2} = \frac{\mu' h}{2\delta}, \quad \eta + \beta = \frac{f_\beta}{h} = \frac{E'(h+\delta)h^2}{12\delta}, \quad \kappa + \frac{4\eta}{3} = \frac{f_\kappa}{h^2} = \frac{\mu'(h+\delta)h^2}{6\delta}. \tag{5}$$

In a generalized plane stress state, which is realized in the layer of the thickness h loaded in the plane with lateral surfaces $x_3 = 0$ and $x_3 = h$ free from stresses, the quantities $v_{1,3}$, $v_{2,3}$, $v_{3,1}$, $v_{3,2}$, ω_1, ω_2, $\omega_{3,3}$ and $\sigma_{13} = \sigma_{31}$, $\sigma_{23} = \sigma_{32}$, σ_{33}, m_{31}, m_{32}, m_{33} are identically zero. Such variant is allowed by the system (4) only if the parameters η and β are equal between themselves. Otherwise, $\omega_{3,1} = \omega_{3,2}$ follows from the equations for the couple stresses m_{31} and m_{32}, i.e. nonuniform rotation of the blocks in a generalized plane stress state is impossible. Thus, the parameters conversion formulas (5) are closed by the equality $\eta = \beta$.

Parameters, characterizing the compliance of a material in tension–compression and in shear, are determined from the condition of coincidence of velocities of plane longitudinal and transverse elastic waves for a block medium and for an effective homogeneous material. In the directions of coordinate axes the waves in a block medium move with the average velocities:

$$\bar{c}_1 = c_1\, c_1'\, \frac{h+\delta}{h\, c_1' + \delta\, c_1}, \qquad \bar{c}_2 = c_2\, c_2'\, \frac{h+\delta}{h\, c_2' + \delta\, c_2}.$$

Taking into account the expressions for velocities of longitudinal and transverse waves in the Cosserat medium, the phenomenological parameters k and μ are calculated as follows:

$$\mu = \rho_0\, \bar{c}_2^2 - \alpha, \qquad k = \rho_0\, \bar{c}_1^2 - \frac{4\mu}{3}, \qquad \rho_0 = \frac{\rho\, h + 3\rho'\delta}{(h+\delta)^3}. \tag{6}$$

For example, let us consider a block medium with parameters of materials, presented in section 1, and with thicknesses of blocks and interlayers, characteristic for a brick masonry: $h = 0.05$ and $\delta = 0.01$ m. Calculation of the parameters of the Cosserat continuum for this medium by formulas (5), (6) gives:

$$k = 16.9,\ \ \mu = 3.07,\ \ \alpha = 0.389\ \text{GPa}, \qquad \kappa = 7.92,\ \ \eta = \beta = 286\ \text{kN}.$$

The material density is $\rho_0 = 2560$ kg/m^3, the inertial inertial characteristic of rotation is $j_0 = 0.892$ kg/m. Velocities of longitudinal, transverse, rotational and torsional waves for these parameters are:

$$\bar{c}_1 = \sqrt{\frac{k + 4\mu/3}{\rho_0}} = 2864, \ \ \bar{c}_2 = \sqrt{\frac{\mu + \alpha}{\rho_0}} = 1163,$$

$$\bar{c}_3 = \sqrt{\frac{\eta + \beta}{j_0}} = 800, \ \ \bar{c}_4 = \sqrt{\frac{\kappa + 4\eta/3}{j_0}} = 660\ \text{m/s}.$$

The natural frequency of the rotational motion, which is calculated by the period of natural oscillations $T = \pi\sqrt{\alpha/j_0} = 15.05$ ms, is equal to $\nu_* = 1/T = 6.645$ kHz.

It should be noted that since in the model of the Cosserat continuum the particles of a material microstructure are considered as undeformable, the approximation of a block medium by means of a moment continuum is possible only if the elastic compliance of interlayers is significantly more compared with the compliance of blocks, and if, in addition, the thickness of interlayers is not too small compared with the size of blocks. Incorrectness of the approximation manifests, ultimately, in violation of the inequalities

$$k,\ \mu,\ \alpha > 0, \qquad \kappa,\ \eta,\ \beta > 0.$$

If these inequalities are fulfilled, then potential energy of the Cosserat continuum is a positive definite quadratic form [18], and it provides the hyperbolicity of the system of equations (4).

4 RESONANCES IN A BLOCK MEDIUM

Resonant excitation of a medium due to the perturbation of rotational motion of blocks is modeled on the basis of equations of the Cosserat continuum. In Fig. 6 the graphs of dependence of the modulus of amplitude of tangential stress on the frequency are shown, which are obtained from the solution of a problem on uniform cyclic shear of viscoelastic layer of the thickness H. Phenomenological parameters of a material are pointed at the end of previous section. The tangential stress belongs to the rigidly fixed lower side of the layer. At the upper side the linear velocity varies on periodic law with the frequency ν. the layer thickness is varied. The similar graphs for ideal inviscid media have a system of resonant peaks with infinite amplitudes. Viscosity is used as a smoothing mechanism. The process of shear is described by equations (4), in which, according to the Boltzmann viscoelasticity theory, the products of parameters of a medium on the characteristics of the strain state are replaced by the convolutions of relaxation kernel corresponding to these parameters on the same characteristics.

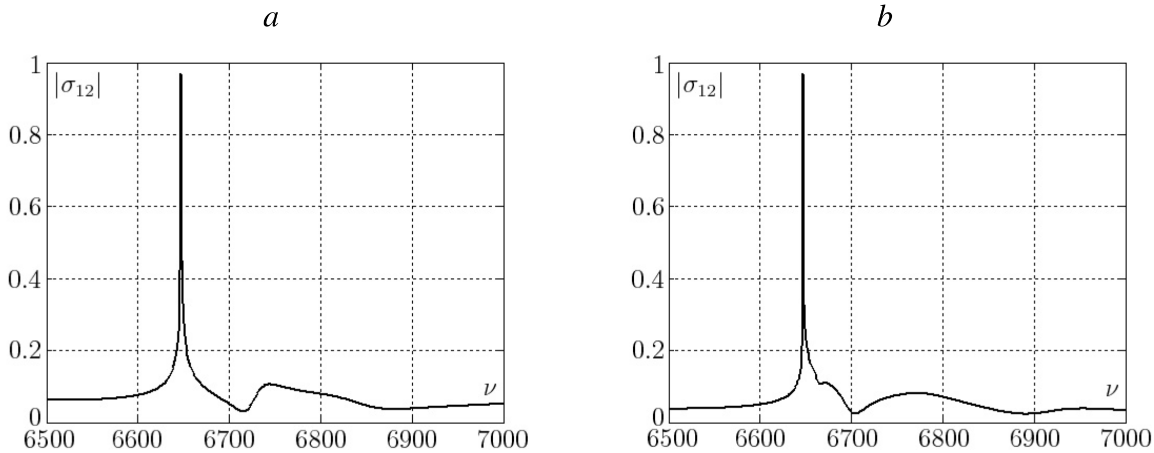


Figure 6: Amplitude-frequency characteristics of a viscoelastic layer: $H = 0.5\,\mathrm{m}$ (*a*) and $H = 1\,\mathrm{m}$ (*b*).

Assume that the axis x_1 of a Cartesian coordinate system is directed deep into the layer, and the axis x_2 is directed horizontally. The system of equations of plane shear waves with rotation of the particles, moving in a moment continuum in the direction x_1, has the form:

$$
\begin{aligned}
&\rho_0\,\dot v_2 = \sigma_{12,1}\,, && j_0\,\dot\omega_3 = m_{13,1} + \sigma_{12} - \sigma_{21}\,,\\
&\dot\sigma_{12} = (\mu+\alpha)*v_{2,1} - 2\,\alpha*\omega_3\,, && \dot\sigma_{21} = (\mu-\alpha)*v_{2,1} + 2\,\alpha*\omega_3\,,\\
&\dot m_{31} = (\eta-\beta)*\omega_{3,1}\,, && \dot m_{13} = (\eta+\beta)*\omega_{3,1}\,.
\end{aligned}
\qquad (7)
$$

From the system (7) one can obtain the amplitude equations:

$$
\begin{aligned}
&2\pi i\,\nu\rho_0\,\hat v_2 = \hat\sigma_{12,1}\,, && 2\pi i\,\nu j_0\,\hat\omega_3 = \hat m_{13,1} + \hat\sigma_{12} - \hat\sigma_{21}\,,\\
&2\pi i\,\nu\,\hat\sigma_{12} = (\hat\mu+\hat\alpha)\,\hat v_{2,1} - 2\hat\alpha\,\hat\omega_3\,, && 2\pi i\,\nu\,\hat\sigma_{21} = (\hat\mu-\hat\alpha)\,\hat v_{2,1} + 2\hat\alpha\,\hat\omega_3\,,\\
&2\pi i\,\nu\,\hat m_{31} = (\hat\eta-\hat\beta)\,\hat\omega_{3,1}\,, && 2\pi i\,\nu\,\hat m_{13} = (\hat\eta+\hat\beta)\,\hat\omega_{3,1}\,.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&4\,\pi^2\nu^2\rho_0\,\hat v_2 + (\hat\mu+\hat\alpha)\,\hat v_{2,11} - 2\,\hat\alpha\,\hat\omega_{3,1} = 0\,,\\
&4\,(\pi^2\nu^2 j_0 - \hat\alpha)\,\hat\omega_3 + (\hat\eta+\hat\beta)\,\hat\omega_{3,11} + 2\,\hat\alpha\,\hat v_{2,1} = 0\,.
\end{aligned}
$$

Hence, $A\,\hat{v}_{2,1111} + 4\,B\,\hat{v}_{2,11} + 16\,C\,\hat{v}_2 = 0$, where

$A = (\hat{\mu}+\hat{\alpha})(\hat{\eta}+\hat{\beta}),\;\; B = \hat{\alpha}^2+\pi^2\nu^2\rho_0(\hat{\eta}+\hat{\beta})+(\pi^2\nu^2 j_0-\hat{\alpha})(\hat{\mu}+\hat{\alpha}),\;\; C = \pi^2\nu^2\rho_0(\pi^2\nu^2 j_0-\hat{\alpha}).$

Let λ_1 and λ_2 be the roots of biquadratic characteristic equation

$$\frac{\lambda_{1;2}^2}{2} = \frac{-B \pm \sqrt{B^2-4AC}}{A}.$$

Then general solution for the amplitude of linear velocity depends on four constants:

$$\hat{v}_2 = A_1\,e^{\lambda_1 x_1} + A_2\,e^{-\lambda_1 x_1} + B_1\,e^{\lambda_2 x_1} + B_2\,e^{-\lambda_2 x_1}.$$

The Fourier transform of angular velocity takes the form:

$$2\,\hat{\alpha}\,\hat{\omega}_3 = -\,\frac{æ_1}{\lambda_1}\Big(A_1\,e^{\lambda_1 x_1} - A_2\,e^{-\lambda_1 x_1}\Big) - \frac{æ_2}{\lambda_2}\Big(B_1\,e^{\lambda_2 x_1} - B_2\,e^{-\lambda_2 x_1}\Big),$$

where $æ_{1;2} = 4\,\pi^2\nu^2\rho_0 + \lambda_{1;2}^2\,(\hat{\mu}+\hat{\alpha})$.

Boundary conditions for the layer are formulated in terms of the velocities v_2 and ω_3:

$$v_2\Big|_{x_1=0} = v_0\,e^{2\,\pi i\,\nu\,t}, \quad \omega_3\Big|_{x_1=0} = 0, \quad v_2\Big|_{x_1=H} = \omega_3\Big|_{x_1=H} = 0.$$

They lead to a system of four linear algebraic equations for finding the constants of integration:

$$\begin{aligned}
&A_1 + A_2 + B_1 + B_2 = v_0, \quad æ_1(A_1 - A_2) + æ_2(B_1 - B_2) = 0,\\
&A_1\,e^{\lambda_1 H} + A_2\,e^{-\lambda_1 H} + B_1\,e^{\lambda_2 H} + B_2\,e^{-\lambda_2 H} = 0,\\
&æ_1\Big(A_1\,e^{\lambda_1 H} - A_2\,e^{-\lambda_1 H}\Big) + æ_2\Big(B_1\,e^{\lambda_2 H} - B_2\,e^{-\lambda_2 H}\Big) = 0.
\end{aligned}$$

Solution of the system can be written in explicit form, but it is not included in this paper because of the cumbersome expression. The amplitude of tangential stress at the lower boundary of the layer is calculated by formula:

$$2\,\pi i\,\nu\,\hat{\sigma}_{12} = \frac{(\hat{\mu}+\hat{\alpha})\lambda_1^2 - æ_1}{\lambda_1}\Big(A_1\,e^{\lambda_1 H} - A_2\,e^{-\lambda_1 H}\Big) + \frac{(\hat{\mu}+\hat{\alpha})\lambda_2^2 - æ_2}{\lambda_2}\Big(B_1\,e^{\lambda_2 H} - B_2\,e^{-\lambda_2 H}\Big).$$

The Kelvin–Voigt viscoelasticity theory is used in calculations, according to which the complex moduli are linear functions of the frequency, in particular $\hat{\mu} = \mu + 2\pi i\,\nu\tilde{\mu}$. Imaginary parts of the complex moduli are chosen so as to achieve the necessary smoothing of solutions.

Comparing the graphs in Fig. 6, obtained for the thicknesses $H = 0.5$ and $H = 1$ m, one can see that the resonance peak at the frequency 6.645 kHz, equal to the natural frequency of rotational motion of the particles, is present regardless of the layer thickness.

5 LOCALIZED PERTURBATIONS

In the computations of impulsive action on a block elastic medium on the basis of two-dimensional and three-dimensional equations of the Cosserat continuum the specific low-frequency pendulum waves, caused not by translational motion (as in the original paper [5]) but the rotation of blocks, were found. The method of solving the problem on high-performance cluster systems and some results of computations are presented in the monograph [18].

Numerical results for 2D Lamb's problem on the normal action of a concentrated impulsive load on the boundary of an elastic block are represented in Fig. 7. Fig. 7 *a* shows the level curves of normal stress for a homogeneous elastic medium (in the framework of the classical theory of elasticity), and Fig. 7 *b* shows the level curves of normal stress for the Cosserat medium. Computations were performed on 15 processors of a cluster, the uniform difference grid consists of 1000×500 meshes. In both figures one can see the incident longitudinal and transverse waves, conical transverse waves and the Raleigh surface waves, moving from the loading point inside the computational domain. The essential difference between Figs. 7 *a* and 7 *b* is that in a moment medium behind the front of the transverse wave an additional system of low-frequency waves, caused by rotational motion of the particles, is observed. Oscillations behind the front of the transverse wave have the period T. The corresponding wavelength is evaluated by the velocity of transverse waves as $\bar{c}_2\, T$.

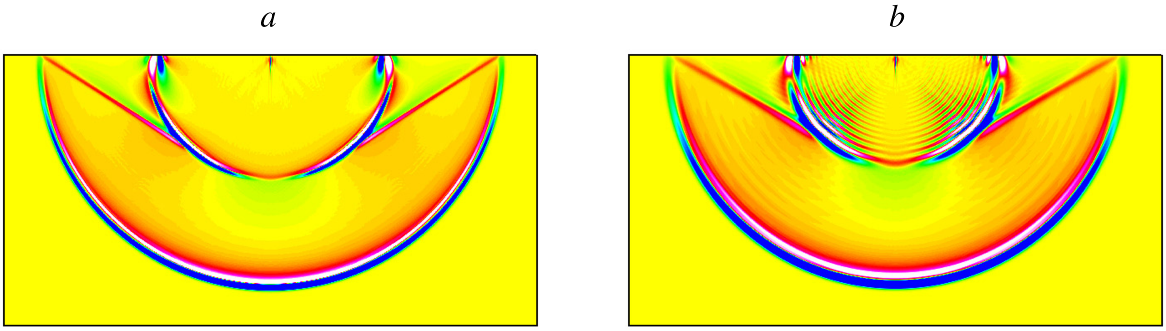


Figure 7: 2D Lamb's problem for a homogeneous elastic medium (*a*) and for the Cosserat elastic medium (*b*): level curves of normal stress.

More accurate expression for the phase velocity c of the pendulum wave of rotational motion can be obtained using the dispersion equation of plane shear waves with rotation of the particles [17, 18]:

$$\Big(1-\frac{\bar{c}_2^2}{c^2}\Big)\Big(\pi^2\nu^2\Big(1-\frac{\bar{c}_3^2}{c^2}\Big)-\frac{\alpha}{j_0}\Big)=\frac{\alpha^2}{\rho_0\, j_0\, c^2}.$$

For $\nu=\nu_*$ this equation gives

$$c=\bar{c}_2\Big(1+\frac{\alpha}{\rho_0\,\bar{c}_3^2}\Big)^{-1}.$$

It turns out that $c\approx\bar{c}_2$ only for small values of the parameter α and for relatively high velocities $\bar{c}_3$. If the velocity $\bar{c}_3$ is low, then c is also low. Thus, in the model of a reduced Cosserat continuum, where the couple stress tensor is zero, the waves of this type are standing waves.

Computations of 3D problems have confirmed the main qualitative difference of the wave field in the Cosserat continuum as compared with the classical elasticity theory, which consists in the appearance of oscillations of the rotational motion of particles on the wave fronts. Comparative calculations with different values of scale of the microstructure of a material were performed in [18]. A direct proportional dependence of the period of natural oscillations from this scale was found numerically.

In Fig. 8 one can see results for the problem on the action of a concentrated rotational moment, periodic in time, obtained by means of the 3Dyn_Cosserat program [19]. Computations were performed for a material with the above parameters on 64 processors of the MVS-100K cluster of the Joint Supercomputer Center of RAS. The loading scheme (Fig. 8 *a*), and level surfaces of the angular velocity ω_2 for the resonance frequency $\nu = \nu_*$ (Fig. 8 *b*) and for the nonresonance frequency $\nu = 1.75\,\nu_*$ (Fig. 8 *c*) at different moments of time are represented. The maximum amplitude of oscillations of the angular velocity is achieved at the point of load application, and the wavelength depends essentially on the frequency. Comparison of the graphs shows that for the frequency of external action, equal to the natural frequency ν_* of rotational motion of the particles, the growth of amplitude with time occurs and a more smooth decay of oscillations with increasing the distance from the point of load application, characteristic of the acoustic resonance, takes place (Fig. 8 *b*). It should be noted that if normal stress or tangential stress, linear velocity or angular velocity, concentrated in space, acts at the boundary point of a computational domain, then the resonance excitation of a medium does not take place.

a *b* *c*

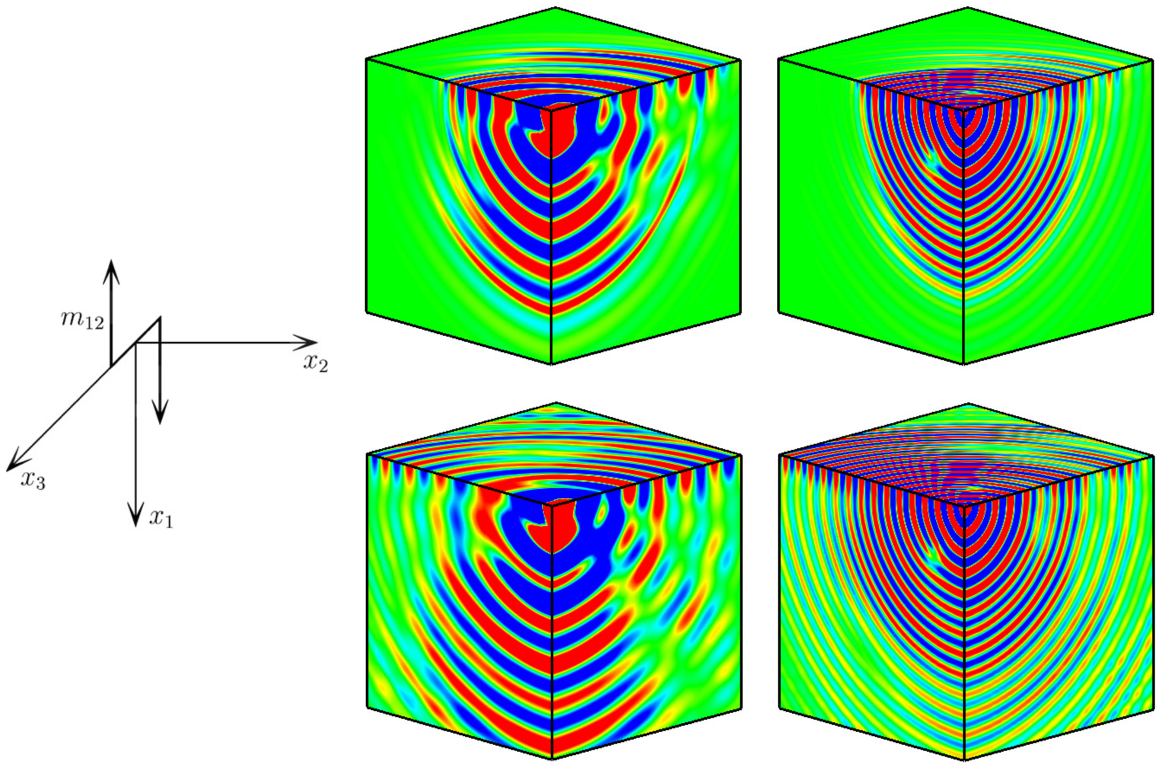


Figure 8: 3D problem on the periodic action of concentrated rotational moment: loading scheme (*a*), level surfaces of angular velocity for the resonance frequency (*b*) and for the nonresonance frequency (*c*); $t = 1.5\,\text{ms}$ (*above*), $t = 2.10\,\text{ms}$ (*below*).

The existence of resonance frequency in the Cosserat continuum model, which does not depend on the size of a sample and is essentially a phenomenological parameter of a material, was found in [17, 18]. It was shown in [20] that the resonance frequency, connected with rotational motion of particles in the microstructure of a material, is also present in the models of micropolar elastic thin shells.

6 CONCLUSIONS

- To describe the wave processes in a multiblock medium with elastic blocks interacting through compliant viscoelastic interlayers, the simplified mathematical model, received by averaging the equations of deformation of the interlayers in thickness, and the model of the Cosserat continuum, in which the independent rotations of the blocks are taken into account along with the translational degrees of freedom, were applied.

- From a comparison of the velocities of elastic waves and the coefficients of elastic resistance of a medium to rotation and torsion of the blocks, the formulas for recalculation of phenomenological parameters of the Cosserat continuum by given characteristics of materials of the blocks and the interlayers were obtained.

- Algorithms and parallel programs are worked for numerical realization of considered models on multiprocessor computer systems of the cluster type and on the systems with GPUs.

- Parallel computational algorithms, based on models of the inhomogeneous elasticity theory and the Cosserat elasticity theory, are applied to the analysis of propagation of elastic waves in geomaterials with layered and block microstructure.

- The results of analysis of the oscillation processes show that the Cosserat medium possesses the eigenfrequency of acoustic resonance, which appears under certain conditions of perturbation and depends only on the inertial properties of the microstructure particles and the elasticity parameters of a material.

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