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OPTIMAL DESIGN OF VIBRATING RING-STIFFENED CYLINDRICAL SHELL

Sergei B. Filippov and Natalia V. Naumova

Saint Petersburg State University
address
e-mail: s_b_filippov@mail.ru

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Abstract. *Free vibrations of a thin elastic cylindrical shell stiffened by rings of rectangular cross-sections are considered. The parameters of the shell of the minimal weight, having a given fundamental frequency, are found. For the evaluating of the optimal parameters the asymptotic approach is used. The validation of the obtained asymptotic results are fulfilled by finite element analysis.*

1 INTRODUCTION

Ring-stiffened cylindrical shells are widely applied in engineering. Various analytical and numerical methods for analysis of stiffened shells have been developed. In [1] and [2] Fourier series for the solution of static and dynamic problems are used. In the early studies on vibrations and buckling of stiffened shells, the Rayleigh–Ritz method was applied. Nowadays this technique is still popular. For example, in [3] the Ritz method was used for buckling analysis of ring-stiffened cylindrical shells under general pressure loading. The study [4] has shown that the finite element method is quite suitable to analyze the vibration characteristics of ring-stiffened cylindrical shells under external pressure.

The approximate values of the natural frequencies for thin ring-stiffened shells may be obtained by solving eigenvalue problems for linear differential equations. The equations describing the vibration of thin shells contain the dimensionless shell thickness as a small parameter. Therefore, these boundary value problems lend themselves to be solved by asymptotic methods. By means of the asymptotic approach complex eigenvalue problems of the stiffened shells theory can be transformed into problems which have simple analytical solutions.

In [5] asymptotic approaches have been used for calculation of optimal parameters of ring-stiffened cylindrical shell with given mass, for which the fundamental vibration frequency have the largest value. In the presented paper some kind of an inverse problem is studied. We assume that the fundamental vibration frequency is given and seek optimal parameters corresponding to the ring-stiffened shell of the minimal mass.

To get simple approximate formulas for the lower frequencies a combination of asymptotic method is used. First we seek the solutions as a sum of slowly varying functions and edge effect integrals. Thus the initial singularly perturbed system of differential equations is reduced to an approximate system of the smaller order. Then the solution of the approximate eigenvalue problem is obtained by means of the homogenization procedure.

2 BASIC EQUATIONS

Consider the low-frequency free vibrations of a thin circular cylindrical shell stiffened by n_r identical rings at the parallels $s = s_j$, where s is the coordinate in the longitudinal directions, $j = 1, 2, \ldots n_r$ (see Figure 1).

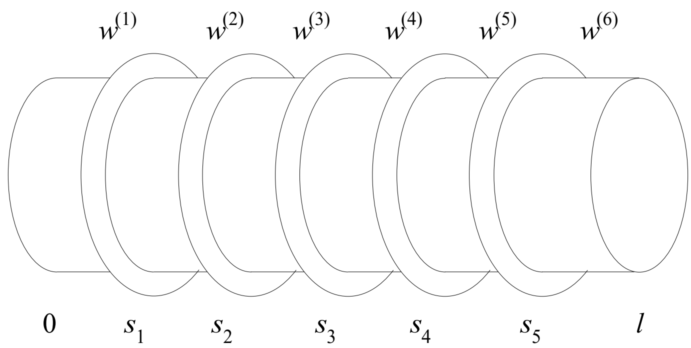

Figure 1: Ring-stiffened cylindrical shell.

If we take the radius R of the cylindrical shell as the characteristic size, then after the separation of variables the dimensionless equations describing vibrations of the shell may be written as

$$\varepsilon^8 \Delta\Delta w^{(j)} - \frac{d^2\Phi^{(j)}}{ds^2} - \lambda w^{(j)} = 0, \qquad \Delta\Delta\Phi^{(j)} + \frac{d^2 w^{(j)}}{ds^2} = 0,$$
$$j = 1, 2, \ldots n, \quad n = n_r + 1, \tag{1}$$

where

$$\Delta w = \frac{d^2 w}{ds^2} - m^2 w, \quad \varepsilon^8 = \frac{h^2}{12\sigma}, \quad \sigma = 1 - \nu^2, \quad \lambda = \frac{\rho R^2 \omega^2}{E}, \tag{2}$$

m is the circumferential wave number, $w^{(j)}$ is the normal deflection, $\Phi^{(j)}$ is the force function, $\varepsilon > 0$ is a small parameter, h is the dimensionless shell thickness, ν is Poisson's ratio, ω is the frequency, ρ is the mass density, and E is Young's modulus. The solutions of equations (1) satisfy four boundary conditions on each shell edges $s = 0$ and $s = l$, where l is non-dimension shell length, and $8n_r$ continuity conditions on the parallels $s = s_j$, $j = 1, 2, \ldots n_r$.

3 First approximation

It is shown in [5] that for sufficiently small ε the lowest eigenvalues, $\lambda \sim \varepsilon^4$, correspond to the eigenfunction with large circumferential wave number $m \sim \varepsilon^{-1}$. We represent the solutions of equations (1) as a sum of the slowly varying function and edge effect integrals. In the first approximation we get the following equations

$$\frac{d^4 w^{(j)}}{ds^4} - \alpha^4 w^{(j)} = 0, \quad \alpha^4 = m^4\lambda - \varepsilon^8 m^8, \quad j = 1, 2, \ldots n. \tag{3}$$

The eigenvalue problem for equations (1) is singularly perturbed, because the equations (1) of order eight transform to differential equations (2) of order four. Therefore, solutions of equations (3) can not satisfy all boundary conditions of initial eigenvalue problem. The problem of extracting two boundary conditions for equations (3) out of four boundary conditions on the shell edges is discussed in detail in [6]. The boundary conditions for equations (3) in the case of freely supported shell edges have the form

$$w = \frac{d^2 w}{ds^2} = 0 \quad \text{for} \quad s = 0, \quad s = l. \tag{4}$$

If characteristic size of the ring cross sections $a \sim \varepsilon^3$, then the boundary conditions for equations (3) at the parallels $s = s_j$ are written as (see [5])

$$w^{(j)} = w^{(j+1)}, \quad \frac{dw^{(j)}}{ds} = \frac{dw^{(j+1)}}{ds}, \quad \frac{d^2 w^{(j)}}{ds^2} = \frac{d^2 w^{(j+1)}}{ds^2}, \quad \frac{d^3 w^{(j)}}{ds^3} - \frac{d^3 w^{(j+1)}}{ds^3} = -c w^{(j+1)}, \tag{5}$$

In case the centers of gravity of the ring cross sections lie on the shell neutral surface $c = m^8 I/h$, where I is the dimensionless moment of inertia of the ring cross-section with respect to the generatrix of the cylinder.

It follows from the second formula (3) that approximate value of frequency parameter is

$$\lambda_k(m) = \alpha_k^4/m^4 + \varepsilon^8 m^4, \tag{6}$$

where α_k is the eigenvalue of problem (3–5). The eigenvalue problem (3–5) also describe the flexural vibrations of a simply supported beam, stiffened by n_r identical springs of stiffness c at the points $s = s_j$.

4 Homogenization

In this section we consider the uniform arrangement $s_j = jl/n$ of the springs on a simply supported beam. If the number n_r of springs is large and the stiffness of each spring c is small, one can use the homogenization method [7] for the approximate evaluation of the eigenvalues α_k. Instead of the problem (3–5) we will solve the equivalent problem for the equation

$$\frac{d^4w}{ds^4} + cw\sum_{j=1}^{n-1}\delta(s-s_j) = \alpha^4 w \tag{7}$$

with the boundary conditions (4). Here $\delta(z)$ is Dirac's delta function. In the new variables $s = xl$ and $w = \hat{w}l$ equation (7) is

$$\frac{d^4\hat{w}}{ds^4} + \hat{c}n\hat{w}\sum_{j=1}^{n-1}\delta(\xi-j) = \kappa\hat{w}, \tag{8}$$

where $\hat{c} = cl^3$, $\kappa = (\alpha l)^4$, and $\xi = nx$. The boundary conditions (4) take the form

$$\hat{w} = \frac{d^2\hat{w}}{dx^2} = 0 \quad \text{for} \quad x = 0, \quad x = 1. \tag{9}$$

Assuming that $n \gg 1$ and $\hat{c}n \sim 1$, we write the solution of equation (8) as

$$\hat{w}(x,\xi) = w_0(x,\xi) + n^{-4}w_4(x,\xi) + \cdots, \quad \kappa = \kappa_0 + n^{-4}\kappa_4 + \cdots, \tag{10}$$

where $w_i(x,\xi) = w_i(x,\xi+1)$ and, consequently,

$$< \frac{\partial^k w_i}{\partial\xi^k} >= \int_{\xi}^{\xi+1}\frac{\partial^k w_i}{\partial\xi^k}d\xi = 0, \quad i = 0,4,\dots, \quad k = 1,2,\dots \tag{11}$$

The operator $< \cdot >$ is called the homogenization operator. The application of this operator to both part of an equation is called the homogenization of the equation.

If we substitute (10) into (8) and (9), then we obtain the equations

$$\frac{\partial^4 w_0}{\partial\xi^4} = 0, \quad \frac{\partial^4 w_4}{\partial\xi^4} + \frac{\partial^4 w_0}{\partial x^4} + \hat{c}n\sum_{i=1}^{n}\delta(\xi-i)w_0 = \kappa_0 w_0 \tag{12}$$

and the boundary conditions

$$w_0 = \frac{\partial^2 w_0}{\partial x^2} = 0, \quad w_4 = \frac{\partial^2 w_4}{\partial x^2} = 0 \quad \text{for} \quad x = 0, \quad x = 1 \tag{13}$$

as a result of equating the coefficients of n^4 and n^0. From the first of equations (12) and (11) it follows that

$$\frac{\partial^3 w_0}{\partial\xi^3} = v_3(x), \quad v_3(x) =< v_3(x) >=< \frac{\partial^3 w_0}{\partial\xi^3} >= 0.$$

A further integration followed by a homogenization gives

$$\frac{\partial^2 w_0}{\partial\xi^2} = v_2(x) = 0, \quad \frac{\partial w_0}{\partial\xi} = v_1(x) = 0, \quad w_0(x,\xi) = v_0(x).$$

After the homogenization of the second of equations (11) we get

$$\frac{d^4 v_0}{dx^4} + \hat{c} n v_0 = \kappa_0 v_0, \tag{14}$$

The first of the boundary conditions (13) takes the form

$$v_0 = \frac{d^2 v_0}{dx^2} = 0 \quad \text{for} \quad x = 0, \quad x = 1. \tag{15}$$

Eigenvalue problem (14), (15) describes the vibrations of a simply supported beam on an elastic base and has the solutions $v_{0k} = \sin(k\pi x)$,

$$\kappa_{0k} = (k\pi)^4 + \hat{c} n, \quad k = 1, 2, \ldots \tag{16}$$

In [8] it has been shown that although formula (16) is derived for $n \gg 1$ and $\hat{c} \sim 1/n \ll 1$, this formula provide good approximations for the exact values of κ_1 even for $n = 2$ (for one ring) and for a sufficiently large stiffness $\hat{c}$.

It follows from (16) that

$$\alpha_k^4 = \beta_k^4 + \eta(\varepsilon m)^8, \tag{17}$$

where

$$\beta_k = \frac{k\pi}{l}, \quad \eta = \frac{cn}{m^8 \varepsilon^8 l} = \frac{nI}{\varepsilon^8 h l}. \tag{18}$$

The dimensionless ring stiffness η is proportional to the ratio D_r/D, where $D_r = EIR^4$ is the bending stiffness of the ring and $D = Eh\varepsilon^8 R^3$ is the bending stiffness of the shell. Substituting (17) into (6) give the following approximate formula

$$\lambda_k(m) = \frac{\beta_k^4}{m^4} + (1+\eta)\varepsilon^8 m^4. \tag{19}$$

To find the frequency parameters

$$\lambda_k^* = \min_m \lambda_k(m)$$

we calculate the partial derivative of the function $\lambda_k(m)$ m and set it equal to zero. The solution of the equation $\partial\lambda_k/\partial m = 0$ has the form

$$m = m_* = \frac{\sqrt{\beta_k}}{\varepsilon(1+\eta)^{1/8}}.$$

The function $\lambda_k(m)$ attains its minimum,

$$\lambda_k^* = 2\varepsilon^4 \beta_k^2 \sqrt{1+\eta} \tag{20}$$

for $m = m_*$. If m_* is an integer, then formula (20) gives the exact result. Replacing m_* by one of the integers closest to m_* we introduce an error whose absolute value is less than 1. Therefore, the relative error of formula (20) is small because $m_* \gg 1$.

It follows from (20) that

$$\lambda_k^*(\eta) = \lambda_k^*(0)\sqrt{1+\eta}, \quad k = 1, 2, \ldots \tag{21}$$

where λ_k^* and $\lambda_k^*(0) = 2\varepsilon^4\beta_k^2$ are the frequency parameters for ring-stiffened and non-stiffened cylindrical shells.

5 Effective stiffness

The lowest frequency parameter λ_1, corresponding to fundamental frequency ω_1, is the important characteristics of a shell. For a non-stiffened freely supported cylindrical shell the approximate value of λ_1 can be found from the formula

$$\lambda_1 \simeq \lambda_1^*(0) = 2\varepsilon^4\beta_1^2. \tag{22}$$

For the ring-stiffened shell with the uniform arrangement of rings the formula

$$\lambda_1^*(\eta) = \lambda_1^*(0)\sqrt{1+\eta}, \tag{23}$$

following from (21) gives the approximate value of $\lambda_1(\eta)$ only if the dimensionless ring stiffness η is not to large. The reason of it is that eigenvalue problem (3–5) in case $s_j = jl/n$ has solutions, which are independent of η. The minimal stiffness-independent eigenvalue $\alpha_n(0) = \pi n/l$ correspond to vibration mode $w = \sin(n\pi s/l)$ which satisfy equation (3) and boundary condition (4) and (5). The minimal stiffness-independent eigenvalue

$$\lambda_n^*(0) \simeq 2\varepsilon^4\beta_n^2 = n^2\lambda_1^*(0)$$

The root of equation $\lambda_1^*(\eta) = \lambda_n^*(0)$ is

$$\eta^* = n^4 - 1.$$

We call η^* effective stiffness. If $\eta \le \eta^*$ then $\lambda_1^*(\eta) \le \lambda_n^*(0)$ else $\lambda_1^*(\eta) > \lambda_n^*(0)$. Therefore for ring-stiffened shell

$$\lambda_1(\eta) \simeq \begin{cases} \lambda_1^*(0)\sqrt{1+\eta}), & 0 \le \eta \le \eta^*, \\ n^2\lambda_1^*(0), & \eta \ge \eta^*. \end{cases} \tag{24}$$

The asymptotic results, obtained from formula (24), are in good agreement with the numerical ones. In particular, for $n = 2$ numerical and asymptotic values of η^* are 14.6 and 15.

6 Optimal design of stiffened shells

We suppose that the fundamental vibration frequency of a ring-stiffened cylindrical shell is given and seek the optimal parameters, for which the mass of the stiffened shell, M_s, has the minimum value. We will compare M_s with the mass of the unstiffened shell, M_0, assuming that both shells have the identical fundamental frequencies, dimensionless length l, radius R and are made of the same materials of the density ρ.

Consider rings with rectangular cross-sections of dimensionless thickness a and width $b = ka$. In this case

$$M_0 = 2\pi R^3\rho l h_0, \quad M_s = 2\pi R^3\rho(lh + n_r a^2 k) \tag{25}$$

where h_0 and h are the dimensionless thicknesses of the unstiffened and the stiffened shells correspondingly.

If the parameters h_0, l, n_r and k are given then ratio M_s/M_0 depends only on a and $d = h/h_0$:

$$F(a,d) = \frac{M_s}{M_0} = d + Aa^2, \quad A = \frac{n_r k}{lh_0}. \tag{26}$$

The fundamental vibration frequency ω_0 of the unstiffened cylindrical shell of the thickness h_0 can be found by the approximate formula, following (2) and (22):

$$\omega_0^2 = \frac{E\lambda_1}{\rho R^2} = \frac{h_0 E\beta_1^2}{\sqrt{3\sigma}\rho R^2} \tag{27}$$

The approximate formula for the fundamental frequency ω_1 of the stiffened shell of the thickness h follows from (2) and (24):

$$\omega_1(\eta) = \frac{hE\beta_1^2}{\sqrt{3\sigma}\rho R^2}\begin{cases} \sqrt{1+\eta)}, & \eta \le \eta^*, \\ n^2, & \eta \ge \eta^*. \end{cases} \tag{28}$$

For the ring with rectangular cross-sections of thickness a and width $b = ka$ the dimensionless moment of inertia of the ring cross-section is $I = ab^3/12$. Substituting I into second formula (18) we get

$$\eta = \frac{Ba^4}{d^3}, \quad B = \frac{\sigma n k^3}{h_0^3 l}. \tag{29}$$

We seek the minimum value of function $F(a, d)$ under the condition $\omega_0 = \omega_1$. Taking into account formulae (27–29) we can represent this condition in the form

$$d = d_* = 1/n^2 \quad \text{for} \quad \eta \ge \eta^*, \quad d^2(1+\eta) = 1 \quad \text{for} \quad \eta \le \eta^*. \tag{30}$$

In case $\eta = \eta_*$ it follows from condition (30) that

$$d = d_*, \quad a = a_* = (\eta_* d_*^3/B)^{1/4}, \quad F = F_* = d_* + Aa_*^2.$$

If $\eta \ge \eta_*$ then

$$d = d_*, \quad \eta = \frac{Ba^4}{d_*^3}, \quad \eta_* = \frac{Ba_*^4}{d_*^3}.$$

Since

$$a > a_*, \quad F(a, d) = d_* + Aa^2 > F_*$$

and F_* is the minimum of function F for $\eta \ge \eta_*$.

Assume that $\eta \le \eta_*$. Then

$$d \le 1, \quad a = \left(\frac{d - d^3}{B}\right)^{1/4}, \quad F(a, d) = f(d) = d + \gamma\sqrt{d - d^3}, \quad \eta = \frac{1}{d^2} - 1,$$

where $\gamma = A/\sqrt{B}$. Taking into account that $\eta_* = 1/d_*^2 - 1$ we obtain inequality $d \ge d_*$. Let us prove that the function $f(d)$ given in the interval $[d_*, 1]$ attains its minimum value at $d = d_*$.

The first and second derivatives of $f(d)$ are

$$f'(d) = 1 + \frac{\gamma(1 - 3d^2)}{2\sqrt{d - d^3}}, \quad f''(d) = -\gamma\left[\frac{(1 - 3d^2)^2}{4(d - d^3)^{3/2}} + \frac{3d}{\sqrt{d - d^3}}\right].$$

It follows from the inequality $n \ge 2$ that $d_* = 1/n^2 \le 1/4$ and $f'(d_*) > 0$. In view of

$$\lim_{d \to 1} f'(d) = -\infty$$

the function $f'(d)$ has the root $d = d_1$ in the interval $[d_*, 1]$. This root is unique because $f''(d) < 0$ and the function $f'(d)$ decreas for $d \in [d_*, 1]$. Hence the function $f(d)$ attains its maximum value at the point $d = d_1$ and

$$\min_{d\in[d_*,1]} f(d) = \min[f(d_*), 1].$$

because $f(1) = 1$. It is clear that one should choose the parameters of the stiffened shell so that the inequality

$$F(a_*, d_*) = f(d_*) = d_* + \gamma\sqrt{d_* - d_*^3} < 1$$

was satisfied. Otherwise the mass of the stiffened shell will be larger than the mass of the unstiffened one. If $f(d_*) < 1$ then the function $f(d)$ attains its minimum value at the point $d = d_*$. For all considered further examples the inequality $f(d_*) < 1$ holds.

Thus for all values of η

$$\min_{a,d} F(a, d) = F(a_*, d_*) = d_* + Aa_*^2,$$

where

$$d_* = \frac{1}{n^2}, \quad a_* = \left(\frac{\eta_* d_*^3}{B}\right)^{1/4} = \left(\frac{d_* - d_*^3}{B}\right)^{1/4} \tag{31}$$

are the optimal parameters.

The optimal value of the parameter η is η_*. Hence, the effective stiffness η_v^* is at the same time the optimal stiffness ensuring the minimal value of the ratio $F = M_s/M_0$.

7 NUMERICAL EXAMPLE

Consider the freely supported unstiffened shell of the thickness $h_0 = 0.01$ and the freely supported shell of the thickness h stiffened by $n_r = n - 1$ rings with square cross-sections for which $k = 1$. The masses of the stiffened and unstiffened shells are M_s and M_0 correspondingly. Both shells have the same length $l = 4$ and Poisson's ratio $\nu = 0.3$.

The values of the optimal parameters d_*, a_* and the ratio M_s/M_0 for various n_r are given in Table 1.

n_r	d_*	a_*	M_s/M_0
1	0,250	0.0268	0.268
2	0,111	0.0200	0.131
4	0,040	0.0137	0.059
6	0,020	0.0106	0.037
8	0,012	0.0088	0.028

Table 1: Optimal values of the parameters vs. n_r.

For the shells under consideration, the ratio M_s/M_0 decreases with the number of the rings, n_r. The mass of the optimal shell stiffened by eight rings is about 35 times less than the mass of the unstiffened shell which has the same fundamental vibration frequency as the stiffened shell. The ratio M_s/M_0, as a function of n_r and k is shown in Table 2.

The ratio M_s/M_0 decreases when k increases, but the suggested approximate approach is applicable only in the case $ka \ll 1$, since we use the beam model of the ring. For large values

	The ratio M_s/M_0.		
n_r	$k=1$	$k=3$	$k=5$
1	0.268	0.260	0.258
2	0.131	0.123	0.120
4	0.059	0.051	0.048
6	0.037	0.030	0.028
8	0.028	0.021	0.019

Table 2: The ratio M_s/M_0 vs. n_r and k.

of k, the ring is wide and it must be treated as an annular thin plate. From the other side if value of k is large the optimal thickness of the rings, a_*, is very small therefore this case is not interesting for applications.

For an estimation of accuracy of asymptotic results the FEM program package ANSYS was utilized. Geometric dimensions and material properties of the unstiffened shell are:

$$R = 0.2 \text{ m},\ l = 4,\ h_0 = 0.01,\ E = 206 \text{ GPa},\ \rho = 7860 \text{ kg/m}^3,\ \nu = 0.3.$$

According formula (27) the approximate value of the fundamental frequency of this shell is 249 Hz. The FEM program gives the fundamental frequency 283 Hz.

In case $n_r = 1$, $k = 3$ the optimal parameters of stiffened shell calculated by means of (31) are $d_* = 0.25$, $a_* = 0.0117$, and the ratio $M_s/M_0 = 0.26$. The approximate value of the fundamental frequency is 249 Hz.

The fundamental frequency of such stiffened shell computed by FEM is equal 264 Hz. The numerical analysis with use of the program package ANSYS has shown, that for $d = 0.292$, $a = 0.0117$ and $b = 6a$ the fundamental frequency of the stiffened shell, 283 Hz, coincides with frequency of the unstiffened shell. The ratio $M_s/M_0 = 0.312$ corresponding to these parameters on 17% is more than its asymptotic approximation, $M_s/M_0 = 0.26$.

8 CONCLUSIONS

The linear differential equations describing free vibrations of thin shells contain the dimensionless shell thickness as a small parameter. The asymptotic technique presented in this paper allows to obtain the simple approximate formulae for the lowest natural vibration frequencies of ring-stiffened shells. These formulae have been used for the evaluation of optimal parameters corresponding to the shell of the minimal weight, having a given fundamental frequency. The solution of the problem of the optimization is obtained in form of closed-form expressions (31). Calculations shows that the mass of the optimal stiffened shell can be about 35 times less than the mass of the unstiffened shell which has the same fundamental vibration frequency.

The numerical evaluation of the optimal parameters of the stiffened shells is time-consuming. Formulae (31) can be used to find first approximation of optimal shell design. Asymptotic results are in satisfactory agreement with the ones obtained by FEM.

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