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DEVELOPMENT OF AN EFFICIENT UNSTRUCTURED FINITE VOLUME SOLVER FOR A NEW CONSERVATION LAW IN FAST STRUCTURAL DYNAMICS

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Abstract. *In this paper, the Jameson-Schmidt-Turkel scheme (JST) is implemented to solve a new mixed methodology in fast transient dynamics [1, 2]. The use of this numerical technique leads to a significant cost reduction compared to standard Finite Volume methods, which require the use of linear reconstruction and slope limiters to guarantee second order accuracy. Crucially, the JST algorithm is specially designed to be used in conjunction with unstructured meshes and is particularly suited for problems with discontinuities, since the artificial dissipation term appearing in the scheme includes an implicit shock capturing term. All of the above ingredients enable the efficient simulation of realistic large scale problems within the context of fast structural dynamics. In this paper, the JST scheme has been combined with a two-stage Total Variation Diminishing (TVD) Runge-Kutta time integrator. The capabilities of the resulting numerical technique, including the preservation of angular momentum and the excellent behaviour in bending dominated scenarios, are demonstrated for a series of benchmark problems. The solutions will be compared with solutions obtained using other methodologies, such cell centred upwind Finite Volume, Two Step Taylor Galerkin or Stream Upwind Petrov Galerkin (SUPG)[4][3][6][5]*

1 INTRODUCTION

Recently, a new Lagrangian mixed formulation [1] has been developed for the simulation of fast transient dynamics problems. This methodology is in the form of a system of first order conservation laws, where the linear momentum and the deformation gradient tensor are regarded as the two main conservation variables. The current paper presents a new implementation using the Jameson-Schmidt-Turkel (JST) scheme [3] widely know within the CFD community. The scheme uses a central differences approach, equivalent to a Galerkin Finite Element Discretisation with linear elements plus a blend of a non divided Laplacian and a biharmornic operator in order to add artificial diffusion. The attractiveness of this scheme relies mainly on computational cost aspects. First of all, it is a nodal based finite volume and therefore, the number of evaluations of the stress tensor (constitutive model) will be reduced drastically as compared to a cell centred scheme since the number of elements is from 5 to 7 times the number of nodes in a tetrahedral mesh. Secondly, the computational effort when computing the flux gradients is reduced to a half in a vertex centred mesh since the loops are performed on edges instead of faces. Furthermore, the combination of the artificial dissipation term and the shock capturing switch allows obtaining a second order monotonicity preserving algorithm without the use of linear reconstruction and slope limiters.

The work presented is the spatial discretisation of a set of governing equations (conservation of linear momentum and deformation gradient) by using the JST scheme. In order to adapt the scheme to the specificities of the problem, dissipation will only be added to the first equation, whereas the update of the deformation gradient will be left as a numerical gradient of the velocities with no additional dissipation. This will allow satisfying the compatibility conditions of the deformation mapping. Special care will be taken in the integration of the boundary fluxes, by the use of a weighted average of nodes at the boundary faces. The spatial discretisation will be combined with a two stages Total Variation Diminishing (TVD) Runge-Kutta time integrator in order to advance the solution in time. The displacements are integrated in time using a trapezoidal rule which, combined with a Lagrange multiplier minimisation procedure, ensure the conservation of angular momentum. An additional correction to the numerical dissipation of the linear momentum is as well added in order to ensure the conservation of that variable.

2 GOVERNING EQUATIONS

A mixed system of conservation laws was presented in [1] as

$$\begin{aligned}
\frac{\partial \mathbf{p}}{\partial t} - \nabla_0 \cdot \mathbf{P} &= \rho_0 \mathbf{b} \\
\frac{\partial \mathbf{F}}{\partial t} - \nabla_0 \cdot (\mathbf{v} \otimes \mathbf{I}) &= \mathbf{0} \\
\frac{\partial E_T}{\partial t} - \nabla_0 \cdot \left(\mathbf{P}^T \mathbf{v}\right) &= 0
\end{aligned} \tag{1}$$

where $\mathbf{p} = \rho_0 \mathbf{v}$ is the linear momentum, ρ_0 is the material density, $\mathbf{v}$ is the velocity field, $\mathbf{b}$ is the body force per unit mass, $\mathbf{F}$ is the deformation gradient tensor, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, E_T is the total energy per unit of undeformed volume, $\mathbf{I}$ is the identity tensor and ∇_0 describes the material gradient operator in undeformed space. The above system of equations can be rewritten in a more compact form, describing a first order hyperbolic system as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \boldsymbol{F}_I}{\partial X_I} = \mathbf{S} \tag{2}$$

where

$$\mathbf{U} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \\ E_T \end{pmatrix}, \quad \boldsymbol{F}_I = \begin{pmatrix} -P_{1I}(\mathbf{F}) \\ -P_{2I}(\mathbf{F}) \\ -P_{3I}(\mathbf{F}) \\ -\delta_{I1} v_1 \\ -\delta_{I2} v_1 \\ -\delta_{I3} v_1 \\ -\delta_{I1} v_2 \\ -\delta_{I2} v_2 \\ -\delta_{I3} v_2 \\ -\delta_{I1} v_3 \\ -\delta_{I2} v_3 \\ -\delta_{I3} v_3 \\ -P_{iI} v_i \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \rho_0 b_1 \\ \rho_0 b_2 \\ \rho_0 b_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall I = 1, 2, 3 \tag{3}$$

Additionally, the conservation law (2) has to be supplemented with a constitutive model which satisfies the objectivity requirement as well as the relevant laws of thermodynamics.

3 SPACE DISCRETISATION

The JST is a vertex centred Finite Volume Method and, as such, requires the use of a dual mesh for the definition of control volumes. In this paper, the median dual approach for triangular meshes, as presented in [4] and [5], has been chosen. This approach constructs the dual mesh by connecting edge midpoints with element centroids in two dimensions (see Figure 1) and edge midpoints with face centroids and element centroids in three dimensions. Such a configuration ensures that only one node of the initial mesh exists in each control volume. For a given edge connecting nodes e and α an area vector is then defined as

$$\mathbf{C}^{e\alpha} = \sum_{k \in \Gamma_{e\alpha}} A_k \mathbf{N}_k \tag{4}$$

where $\Gamma_{e\alpha}$ is the set of facets belonging to edge $e\alpha$, A_k is the area of a given facet and $\mathbf{N}_k$ is the normal vector of the facet (see Figure 1).

On account of the definition of the dual mesh, the defined area vector satisfy $\mathbf{C}^{e\alpha} = -\mathbf{C}^{\alpha e}$. This area vector enables a substantial reduction in the computational cost when computing the boundary integral used in the Green Gauss divergence theorem (classical in FVM), since it saves an additional loop on facets. Consider a hyperbolic system of conservation laws generally written as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \boldsymbol{F}_I}{\partial X_I} = \mathbf{0} \tag{5}$$

where $\mathbf{U}$ is the vector of conserved variables and $\boldsymbol{F}_I$ the flux vector. This set of equations can be discretised in space by using the extended JST scheme (see [4] and [**?**]) to give, for a given

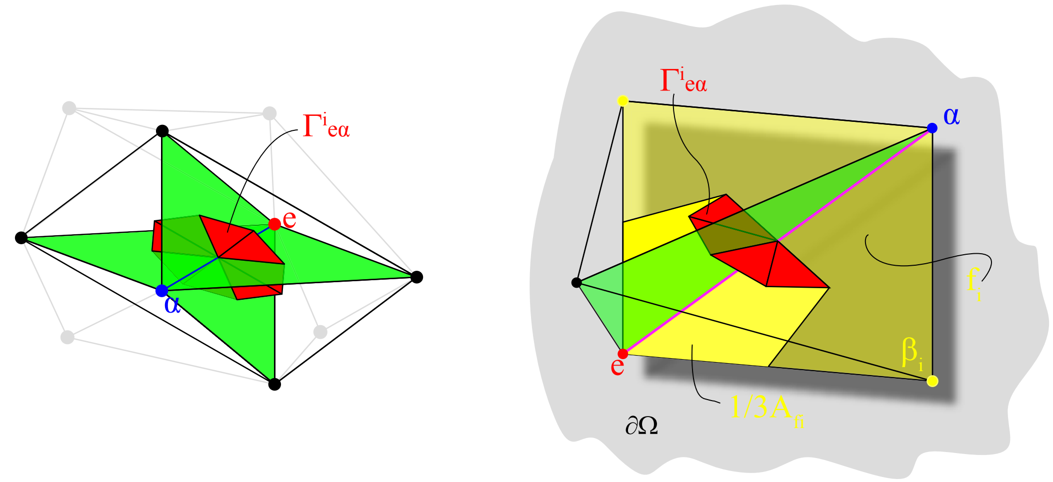

Figure 1: Set of facets related to an interior edge (left) and boundary edge (right) in three dimensions. The green surfaces correspond to the interior faces to which the edge belongs, whereas the dark yellow surfaces correspond to the boundary faces. The red surfaces are the set of interior facets $\Gamma_{e\alpha}$ corresponding to edge $e\alpha$. The bright yellow zone is the contributory area of the face $e\alpha\beta$ to node e.

node e,

$$\frac{d\mathbf{U}_e}{dt} = -\frac{1}{V_e}\left(\sum_{\alpha\in\Lambda_e}\frac{1}{2}(\boldsymbol{F}^e+\boldsymbol{F}^\alpha)\mathbf{C}^{e\alpha} + \sum_{\substack{\alpha\in\Lambda_e^B\\ \beta\in(\Lambda_e^B\bigcap\Lambda_\alpha^B)}}\frac{6\boldsymbol{F}^e+6\boldsymbol{F}^\alpha+6\boldsymbol{F}^\beta}{8}\mathbf{N}^{e\alpha\beta}A^{e\alpha\beta}\right) + \frac{1}{V_e}\mathbf{D}(\mathbf{U}_e) \tag{6}$$

where Λ_e is the set of nodes connected to node e by an edge, Λ_e^B is the set of nodes connected to node e by a boundary edge, $\boldsymbol{F}$ is a matrix gathering the flux vectors as $\boldsymbol{F} = (\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3)$ and $\mathbf{D}(\mathbf{U}_e)$ is a dissipative operator. The first term of the equation is the actual Green Gauss evaluation of the cell boundary fluxes. The dissipative operator reads

$$\mathbf{D}(\mathbf{U}_e) = \sum_{\alpha\in\Lambda_e}\varepsilon_{e\alpha}^{(2)}\Psi_{e\alpha}\theta_{e\alpha}\left(\mathbf{U}_\alpha - \mathbf{U}_e\right) - \varepsilon_{e\alpha}^{(4)}\Psi_{e\alpha}\theta_{e\alpha}\left(\mathbf{L}(\mathbf{U}_\alpha) - \mathbf{L}(\mathbf{U}_e)\right) \tag{7}$$

where $\varepsilon_{e\alpha}^{(2)}$ and $\varepsilon_{e\alpha}^{(4)}$ are discontinuity switches which activate second or fourth order differences operators, $\Psi_{e\alpha}$ is the spectral radius and $\theta_{e\alpha}$ denote geometrical weights. The second order differences operator is defined as

$$\mathbf{L}(\mathbf{U}_e) = \sum_{\alpha\in\Lambda_e}\theta_{e\alpha}(\mathbf{U}_\alpha - \mathbf{U}_e) \tag{8}$$

The first term of the dissipative operator provides the second order differences operator and the second term the fourth order differences operator. The fourth order differences operator avoids the appearance of the odd-even decoupling of the solution (that would result from using averaged fluxes) whilst maintaining the second order accuracy of the scheme. The second order differences operator is introduced to smear out the solution in the vicinity of a shock whilst reducing the solution to first order locally.

4 TIME DISCRETISATION

The time discretisation is performed using a Total Variation Diminishing (TVD) Runge-Kutta time integrator as proposed by Shu and Osher [6]. For a set of equations discretised in space, but left continuous in time (method of lines) at a given node e,

$$\frac{d\mathbf{U}_e}{dt} = -\mathbf{R}_e(\mathbf{U}_e, t) \tag{9}$$

the method computes the solution at time step t^{n+1} from the solution at time step t^n as

$$\begin{aligned} \mathbf{U}_e^{*} &= \mathbf{U}_e^{n} - \Delta t \mathbf{R}_e(\mathbf{U}_e^{n}, t^n) \\ \mathbf{U}_e^{**} &= \mathbf{U}_e^{*} - \Delta t \mathbf{R}_e(\mathbf{U}_e^{*}, t^{n+1}) \\ \mathbf{U}_e^{n+1} &= \frac{1}{2}\left(\mathbf{U}_e^{n} + \mathbf{U}_e^{**}\right) \end{aligned} \tag{10}$$

In addition, the displacements are integrated in time using the trapezoidal rule, which combined with a Lagrange multiplier minimisation procedure, allows for the conservation of angular momentum.

5 NUMERICAL RESULTS

5.1 Punch test

A square flat plate of unite side length is constrained to move tangentially on the east, west and south sides, whereas it is free on the north side. The plate is subjected to an initial uniform velocity $v_{punch} = 100$ m/s on its right half side. The plate is composed of a NeoHookean rubber material with Young's modulus $E = 1.7 \times 10^7 Pa$, density $\rho_0 = 1.1 \times 10^3$ Kg/m^3 and Poissons ratio $\nu = 0.45$. The problem shows the performance of the method in nearly incompressible scenarios, with absence of volumetric locking and spurious modes (checker board) for the pressure. Figure 2 compares results obtained using Mean Dilatation technique and standard Finite Element Method (FEM) for the standard displacement based formulation and Cell Centred Finite Volume, Stream Upwind Petrov Galerkin (SUPG) and JST using the proposed formulation. It can be seen how the FEM solution suffers from volumetric locking, while the Mean Dilatation technique is capable of circumventing it. In addition, both solutions exhibit spurious oscillations in the pressure distribution. The methodologies based in the proposed conservation law formulation (equation (1)) alleviate both the volumetric locking and the appearance of the spurious pressure oscillations.

5.2 Spinning cube (3D)

A unit side cube free at its six faces is released with an initial angular velocity $\omega_0 = \frac{105}{\sqrt{3}}(1,2,3)^T$ rad/s. The cube is composed of a Neohookean nearly incompressible material with Young's modulus $E = 1.7 \times 10^7 Pa$, density $\rho_0 = 1.1 \times 10^3$ Kg/m^3 and Poissons ratio $\nu = 0.45$. Figure 3 shows the evolution in time of the deformed configuration and distribution of pressure. The algorithm is able to preserve the angular and linear momentum, as it is shown in Figure 4.

5.3 Column twist (3D)

This example is shown in order to show the robustness of the problem in highly nonlinear deformation scenarios. A column of rubber material is clamped on its bottom surface and free

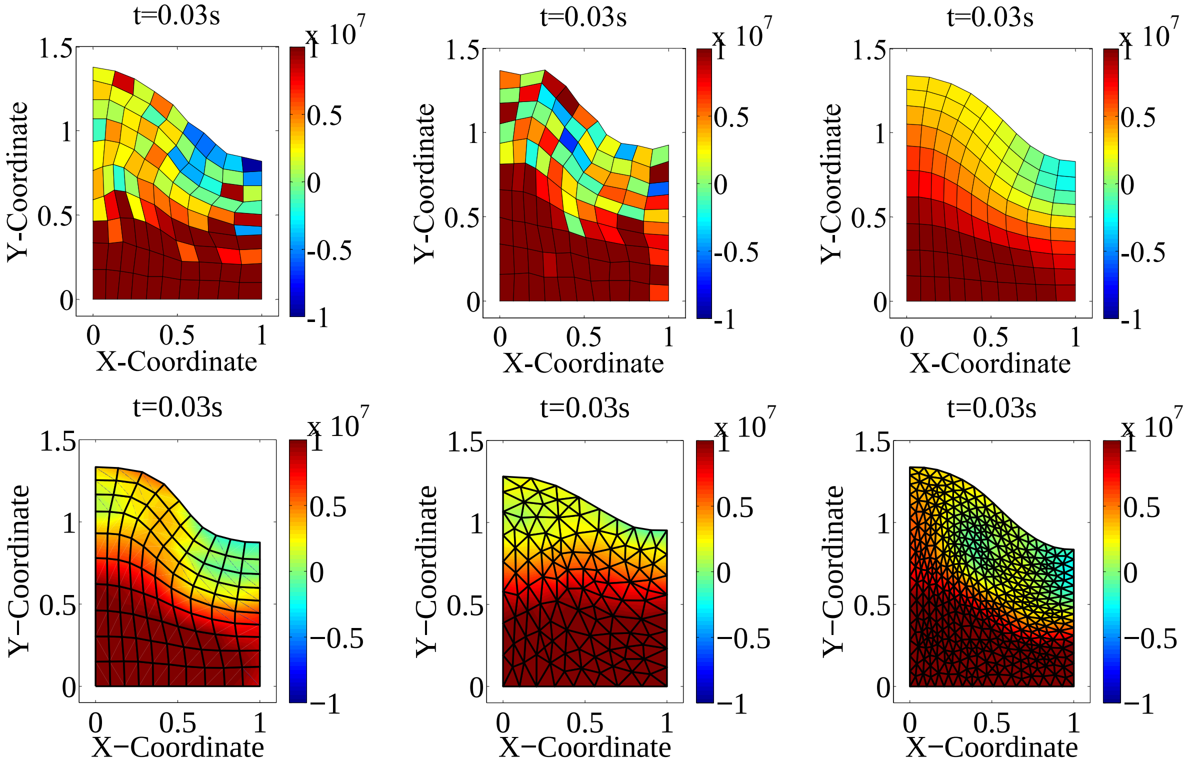

Figure 2: Numerical solution of the punch test case with an initial uniform velocity at the right hand side $v_{punch} =$ 100 m/s. Material properties $E = 1.7 \times 10^7 Pa$, $\rho_0 = 1.1 \times 10^3 Kg/m^3$, $\nu = 0.45$ for a Neohookean material. The solution is shown at time $t = 0.03$s for different discretisations. From left to right and top to bottom: Mean dilatation technique, standard FEM, Cell Centred Finite Volume, SUPG and JST for the last two plots. The solution is obtained using a discretisation of 121 nodes for all the cases except for the last JST solution, where the number of nodes is doubled.

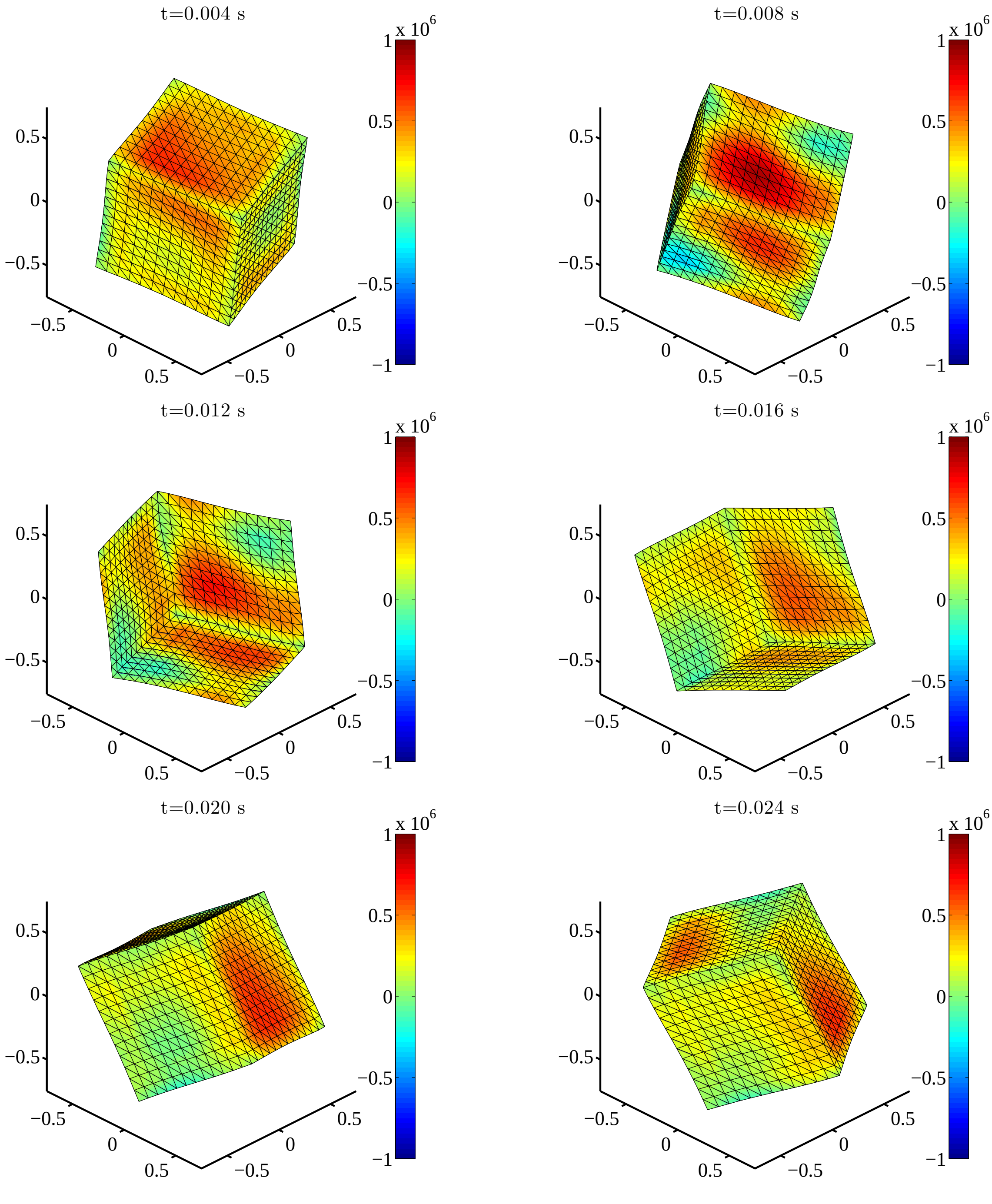

Figure 3: Spinning cube. Evolution in time of the pressure distribution in the deformed configuration. Initial angular velocity $\omega_0 = \frac{105}{\sqrt{3}}(1,2,3)^T$ rad/s. Neohookean nearly incompressible material with Young's modulus $E = 1.7 \times 10^7 Pa$, density $\rho_0 = 1.1 \times 10^3$ Kg/m^3 and Poissons ratio $\nu = 0.45$. JST spatial discretisation with $h = 1/12$ m, $\kappa^{(4)} = 1/128$ and $\alpha_{CFL} = 0.4$.

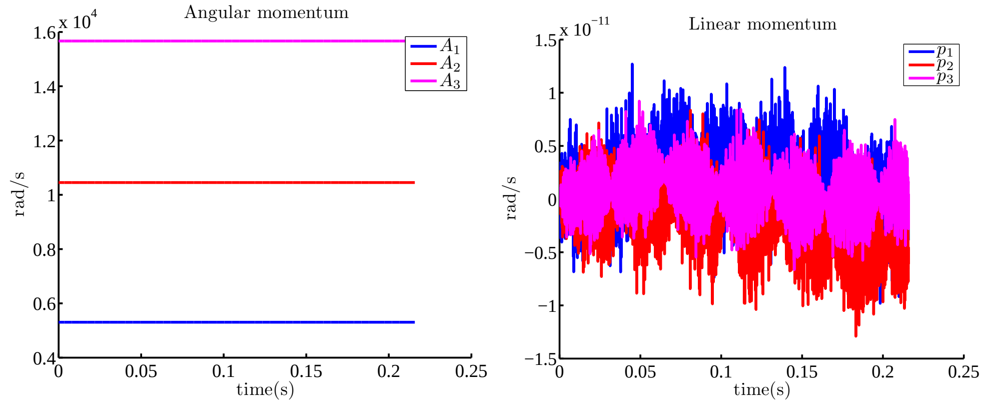

Figure 4: Spinning cube.(a) Conservation of angular momentum. (b) Conservation of linear momentum. Initial angular velocity $\omega_0 = \frac{105}{\sqrt{3}}(1,2,3)^T$ rad/s. Neohookean nearly incompressible material with Young's modulus $E = 1.7 \times 10^7 Pa$, density $\rho_0 = 1.1 \times 10^3$ Kg/m^3 and Poissons ratio $\nu = 0.45$. JST spatial discretisation with $h = 1/12$ m, $\kappa^{(4)} = 1/128$ and $\alpha_{CFL} = 0.4$.

on the rest. An initial angular velocity $\omega = (0, 0, 105)^T$ rad/s is imposed in the whole body except for its clamped bottom. The bar is then left oscillating in time. The problem is set by using a Neohookean nearly incompressible material with Young's modulus $E = 1.7 \times 10^7 Pa$, density $\rho_0 = 1.1 \times 10^3$ Kg/m^3 and Poissons ratio $\nu = 0.45$. Figure 5 shows the evolution in time of the problem. As it can be seen, the deformation of the column is well captured even as the solution advances in time. No numerical oscillations or instabilities are present.

6 CONCLUSIONS

The JST scheme has been implemented for a new mixed conservation law in fast transient dynamics for triangular and tetrahedral meshes. The implementation has been carried out with special care to numerical stability, fulfillment of compatibility conditions and treatment of boundary conditions. This results in an adapted JST scheme, where the numerical dissipation is only added to the first equation (conservation of linear momentum) and the boundary conditions are treated using an external loop on faces, where a weighted average of nodal values ensures accuracy and robustness of the solution. In addition, the numerical algorithm is modified to ensure preservation of linear and angular momentum. A set of numerical results has been presented both for two and three dimensions. These numerical results have proven second order convergence both for stresses and velocities. Furthermore, they circumvent the volumetric locking and spurious pressure modes as they appear in standard Finite Element (displacement based formulation) using linear elements for triangle and tetrahedra. The solutions compare well with other methodologies that discretise the proposed formulation, such as cell centred Finite Volume or Stream Upwind Petrov Galerkin. The proposed methodology allows for further research into more realistic real life problems. In fact, irreversible problems involving shocks can be easily implemented, due to the straightforward implementation of constitutive models and the built-in shock capturing term. Contact problems can as well be investigated by using Riemann-Solvers on the external faces.

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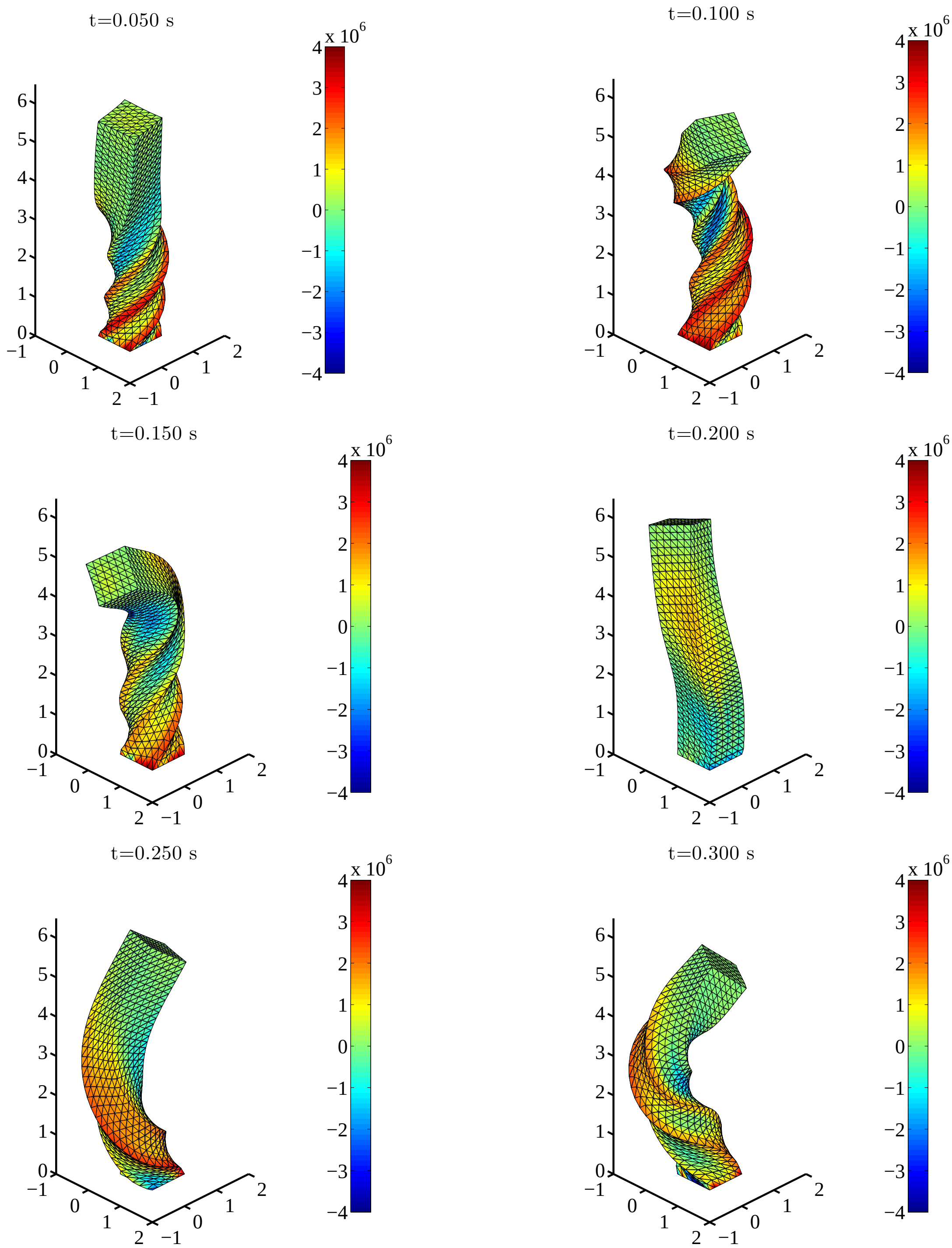

Figure 5: Twisting column. Evolution in time of the pressure distribution in the deformed configuration. Initial angular velocity $\omega_0 = \frac{105}{\sqrt{3}}(1, 2, 3)^T$ rad/s. Neohookean nearly incompressible material with Young's modulus $E = 1.7 \times 10^7 Pa$, density $\rho_0 = 1.1 \times 10^3$ Kg/m^3 and Poissons ratio $\nu = 0.45$. JST spatial discretisation with $h = 1/6$ m, $\kappa^{(4)} = 1/128$ and $\alpha_{CFL} = 0.4$.

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