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REDUCTION METHOD APPLIED TO VIBRATIONS OF VISCOELASTIC SANDWICH SHELLS

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Abstract. *This work deals with the vibrations of sandwich visco-elastic structures. The central visco-elastic layer included between two metallic layers leads to solve a problem of complex and non-linear eigenvalues. Indeed the core material depends on the frequency. Several methods exist to solve this type of problem. But the dimension of matrices to be manipulated can become very large and generate substantial computational times. A reduction technique is proposed herein. It is applied and compared to the high order Newton method.*

INTRODUCTION

To reduce vibrations and noise, the industrialists propose damping sandwich structures, in which a visco-elastic layer is included between two layers of metal material such as aluminum, for instance. The analysis of free vibrations of these structures leads to solve a problem of complex and non-linear eigenvalues. The non-linearity is due to the frequency dependence of the visco-elastic central layer. A lot of models or methods have been developed to solve this non-linear problem. For example, Daya and Potier-Ferry [1] proposed a continuation method. From the linear eigenmode and using the perturbation technique, they determine the frequencies and loss-factors. Duigou and al. [2] developed two iterative methods based on homotopy technique, asymptotic numerical method and Padé approximants. One of them is a sort of high order Newton method. Other authors, like Damanpack and al. [3] or Banerjee and al. [4] used the dynamical stiffness method. Nevertheless, whatever the method used, the matrix sizes to be manipulated in a problem of non-linear vibrations can become quickly very large and generate substantial computational times. In order to reduce this cost, different methods exist like the condensation method proposed by Park and al. [5] or Lima and al. [6], the Lanczos's algorithm used by Chen [7].

In this work, we consider the high order Newton method proposed by Duigou and al [2]. This method is very efficient but it needs substantial computational time because of the matrix sizes to be inverted. That is why we propose to use a reduction method to decrease computational time.

1 PROBLEM OF VIBRATIONS

We consider a sandwich structure including a central layer made up of a visco-elastic material. The latter one generates a complex Young modulus. In order to determine the eigenvalues of this type of sandwich structures, the equation to be solved is written in the following discrete form :

$$\mathbf{R}(\omega, \mathbf{U}) = [\mathbf{K}(0) + E(\omega)\mathbf{K_v} - \omega^2\mathbf{M}][\mathbf{U}] = 0 \tag{1}$$

where **M** is the mass matrix, **K**(0) is the elastic stiffness matrix, E(ω) is the Young modulus depending on the pulsation ω and **U** represents the vector of generalized nodal displacements. The matrix K_υ does not depend on ω. Duigou [8] proposed a numerical method to solve the non-linear eigenvalue problem (1). This method is based on a homotopy technique and a perturbation method. The principle is the following one :

1- A starting point ($\mathbf{U_0}$, p_0) with (p = iω) is chosen and an incremental solution is introduced in the following way :

$$\begin{Bmatrix} \Delta U \\ \Delta p \end{Bmatrix} = \begin{Bmatrix} U - U_0 \\ p - p_0 \end{Bmatrix} \tag{2}$$

2- After inserting (2) into (1) and after several manipulations, the homotopy technique is applied. This one consists in introducing a parameter ε such as for $\varepsilon = 0$, the correction (Δ**U**, Δp) is equal to zero and for $\varepsilon = 1$, the exact non-linear problem (1) is solved.

For example, this modified problem can be defined by :

$$L.\Delta U+G.U_0\Delta p+2p_0\Delta pM\Delta U+\varepsilon(\Delta E(p)-E'(p_0)\Delta p)K_v.U_0$$
$$+\Delta E(p)K_v.\Delta U+(\Delta p)^2M.U_0+(\Delta p)^2M.\Delta U+\varepsilon R_0=0 \quad (3)$$

with $L=K(0)+E(p_0)K_v+p_{0^2}M$ linear operator and $R_0=(K(0)+E(p_0)K_v+p_{0^2}M).U_0$, initial residue.

3- Finally, the perturbation technique is applied, that is to say, the increment ($\Delta\mathbf{U}$,Δp) is sought in the truncated series form depending on the perturbation parameter ε :

$$\Delta U=\sum_{i=1}^{N}\varepsilon^i U_i \quad et \quad \Delta p=\sum_{i=1}^{N}\varepsilon^i p_i \quad (4)$$

4- After inserting these series into the modified problem(3), a series of linear problems for each order is obtained and solved.

As the number of unknowns is greater than the number of equations, an equation to normalize the mode is introduced :

$$\Delta\mathbf{U}^t\,\mathbf{G}\,\mathbf{U_0}=0 \quad (5)$$

Then, the linear problems can be written at the ith truncature order, in the following matrix form :

$$[L_t]\begin{bmatrix}U_i\\p_i\end{bmatrix}=\begin{bmatrix}L & GU_0\\GU_0^t & 0\end{bmatrix}\begin{bmatrix}U_i\\p_i\end{bmatrix}=\begin{bmatrix}F_i^{nl}\\0\end{bmatrix} \quad (6)$$
$$avec \quad L=K(0)+E(p_0)K_v-p_0^2M$$

where $\mathbf{L_t}$ refers to the complex tangent operator and F^{nl}_i is a second member vector changing at each 'i' order but depending only on the variables calculated in previous orders and $(\,)^t$ indicates the transposed operator.

The characteristic parameters of this method are the truncature order of (N) series and a small tolerance parameter enabling to check if the method has converged towards the solution. This resolution technique, named high order Newton method gives good results compared to experimental results. The disadvantage of this method is computational time. Indeed, the problems to solve need triangulating large-size matrices. However, one must take into consideration that the high order Newton method needs one matrix triangulation only and the resolution of “N” linear systems (6) at each iteration for an initial solution ($\mathbf{U_0}$, p_0). In order to reduce computational times, the authors propose – in the reference [2] – to use real matrices instead of consistent complex matrices, corresponding to the $\mathbf{L_t}$ Operator. This method is faster but it diverges in some cases.

The objective of this work is to apply a reduction model to the high order Newton algorithm in order to reduce computational times while preserving the convergence properties of the algorithm.

2 REDUCTION METHOD

The reduction model is applied to the equation (1) : the displacement vector is projected on a small-sized base :

$$U=\Re\ u \quad (7)$$

where **u** is the reduced vector and $\mathcal{R}$, the projection matrix. This matrix is built using the eigenvectors Φ of linear problems and the resulting vectors considering the visco-elastic part of the problem (1), that is to say, according to the reference [9] :

$$\Re=[\Phi\ ,\Psi] \tag{8}$$

With the vectors Ψ , solutions of following linear problems :

$$\Psi=K(0)^{-1}K_v\Phi \tag{9}$$

Introducing the relation(7) in the equation(1) and left-multiplying by $\mathcal{R}^T$, the equation (1) can be written in the reduced form :

$$\mathbf{r}(\omega,\mathbf{u})=[\mathbf{k}(0)+E(\omega)\mathbf{k}_v-\omega^2\mathbf{m}][\mathbf{u}]=0 \tag{10}$$

Where $\mathbf{m}=\mathfrak{R}^T\mathbf{M}\mathfrak{R}$, $\mathbf{k}=\mathfrak{R}^T\mathbf{K}\mathfrak{R}$, $\mathbf{k_v}=\mathfrak{R}^T\mathbf{K_v}\mathfrak{R}$

Then, this reduced equation is solved using the high order Newton method similarly to the real-size problem. A parameter of perturbation is introduced in equation (10) and the unknowns of the reduced problem are sought in the form of truncated series at "N" order, for example.

3 NUMERICAL EXAMPLES

To compare the high order Newton algorithms with or without reduction method, the finite element method is applied. The finite element used is the sandwich shell element with eight nodes and eight degrees of freedom (D.o.F.) developed by Duigou [8] and based on the element of Büchter and al. [9]. In the first part, the Young modulus of the beam is considered constant complex and in the second part, it is considered depending on the temperature. In order to check the convergence of results, a criterion based on the residue calculus is carried out at each step in the following way for the high order Newton method :

- without reduction : $RES=\dfrac{\|R(U,p)\|}{\|K(0).U\|}\leq\delta$
- with reduction : $RES=\dfrac{\|r(u,p)\|}{\|k(0).u\|}\leq\delta$

The tolerance, δ, has to be fixed for each method.

3.1 Example with a constant complex Young modulus

The considered example is a cantilever sandwich beam including a thin central visco-elastic layer and the Young modulus is constant and defined by:

$$E=E_0(1+i\eta c)$$

In this study, the parameter ηc is chosen equal to 1,5, representing quite high damping. The beam has the mechanical properties and the geometrical characteristics given in Table 1. This one is discretized with 424 D.o.F., like in reference [2]. The used parameters by the high order Newton method are : a truncature order N=20, a tolerance equal to 1.10^{-7}. In order to apply the reduction method, 20 vectors are used. Then, the basis (8) is composed of 10 vectors Φ (from a linear vibration computation) and of 10 vectors Ψ solutions of the linear problem (9). The results obtained by the high order Newton method with – referred to as R-HONA) and without reduction – referred to as HONA – are given in Table 2 for the first two bending modes of the beam defined in Table 1.

Mechanical properties	Elastic layer – Aluminum	Visco-elastic layer
Young modulus	Ea = 6,9 1010 N/m2	E0 = 1794 103 N/m2
Poisson ratio	υa = 0,3	υp = 0,3
Density	ρa = 2766 kg/m3	ρp = 968,1 kg/m3
Thickness	ha = 1,524 mm	hp = 0,172 mm

Table 1. Mechanical and geometrical properties of Beam 1.

Mode		HONA [8]	R-HONA
64.32	Iteration	2	3
	Damped frequency	70.19647	70.19646
	Damping coefficient	0.228979	0.228978
	η_m / η_c	0.152653	0.152652
297.85	Iteration	2	3
	Damped frequency	310.4359	310.436
	Damping coefficient	0.295438	0.295435
	η_m / η_c	0.196959	0.196957

Table 2. First two bending modes of Beam 1. $E=E_0(1+i\ \eta_c)$= cst, η_c=1.5 and 20 vectors in the reduction base.

Table 2 shows that the reduction technique proposed enables to obtain the same values as the high order Newton method.

3.2 Example with a Young modulus depending on the temperature

A second cantilever sandwich beam with dimension (178*10 mm2) is considered. But this time, the Young modulus of the central visco-elastic layer is not constant. It is represented by the generalized Maxwell model :

$$E(\omega)=k_0+\eta_0 i\omega+\sum_{j=1}^{N_{\max}}\frac{i\omega}{\left(\frac{i\omega}{k_j}+\frac{1}{\eta_j}\right)} \tag{4}$$

The Maxwell number N_{max} is equal to 129. The other characteristics of the layer are shown in Table 3.

Here, for the finite element simulation, the temperature used is 20°C and the number of nodes used for discretization is equal to 53, that is to say, 424 D.o.F.. The parameters used by the high-order Newton algorithm with and without reduction method are the followings : a truncature order N=20, a tolerance parameter equal to 1.10^{-7} and 20 vectors in the projection basis. The first three bending modes are studied.

In the Table 4, we can note that the computational results carried out with the high order Newton with or without reduction diverge when the truncature order is equal to 20 in the series. That is why, in Table5, the computation of frequencies and dampings is carried out using a truncature order equal to 5. In this case, the two algorithms give good results.

Mechanical properties	Elastic layer – steel	Visco-elastic layer
Young modulus	$E_a = 2{,}1\ \ 10^{11}$ N/m^2	$E_p = 27{,}216\ \ 10^6$ N/m^2
Poisson ratio	$\upsilon_a = 0{,}3$	$\upsilon_p = 0{,}44$
Density	$\rho_a = 7800$ kg/m^3	$\rho_p = 1200$ kg/m^3
Thickness	$h_a = 0{,}6$ mm	$h_p = 0{,}045$ mm

Table 3. Properties of the beam 2.

Modes		HONA	R-HONA
29.96	Iteration	2	5
	Damped frequency	32.7725859	32.7725747
	Damping coefficient	0.033747927	0.033751093
	η_m / η_c	43.2200279	43.2240829
	RES	3.72228416E-11	1.14015107E-08
143.49	Iteration	3	4
	Damped frequency	197.870522	197.864831
	Damping coefficient	0.0981405456	0.0995990691
	η_m / η_c	125.68586	127.553749
	RES	2.53576098E-12	3.25600381E-09
349.78	Iteration	**Diverge**	**Diverge**

Table 4. First three bending modes of Beam 2. N=20.

Mode		HONA	R-HONA
29.99	Iteration	2	4
	Damped frequency	32.7842474	32.785868
	Damping coefficient	0.0335581189	0.0335753581
	η_m / η_c	42.9769478	42.9990273
143.75	Iteration	2	3
	Damped frequency	198.384908	198.635266
	Damping coefficient	0.0928710797	0.0931472569
	η_m / η_c	118.937401	119.291092
349.78	Iteration	3	4
	Damped frequency	548.090895	548.325729
	Damping coefficient	0.0922039246	0.117323568
	η_m / η_c	118.082993	150.253021

Table 5. First benfing modes of the beam 2. N=5.

4 CONCLUSION

These works deal with the vibrations of sandwich visco-elastic structures. The central layer leads to solve a non-linear eigenvalue problem. To do so, two techniques are compared herein : the high order Newton algorithms with and without reduction. Two examples of cantilever sandwich visco-elastic beams are studied. The first one has a constant complex Young modulus ($E= E_0(1+i\eta c)$) and the second one a Young modulus depending on frequency (Model of Maxwell for 20°C). In the two cases, the proposed methods give good results. Numerical tests of these works show that the use of a truncature order close to 5 leads to better convergence than a high order. Currently, numerical tests on more complex geometrical examples needing more final spatial discretization are being carried out in order to quantify computational time gains.

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