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LIQUID FREE VIBRATION ANALYSIS IN A CIRCULAR CYLINDRICAL RIGID CAVITY USING THE h-p FINITE ELEMENT METHOD

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Abstract. *A new hierarchical finite element is developed to analyze the free vibration of a fluid in a rigid cylindrical cavity. It is a bi-hierarchical four nodes quadrilateral element with three degrees of freedom (three displacements) by node. The specificity of this element is a double increase of the hierarchical mode number independently according to both axial and radial directions. The first advantage is that the solution is accurate for high ratio dimensions cavities. The second advantage is the possibility of using only one element to idealize the fluid. The third advantage is that the pressure and the velocity potential are not considered as unknowns, only the displacements describe the fluid behavior. Through the comparison results, this element gives good results accuracy with simple idealization for a fluid in a rigid cylindrical cavity.*

1 INTRODUCTION

For coupled systems, obtaining analytical solutions is difficult or impossible however, numerical methods based specifically on the finite element method can easily approach the solution. In general, fluid-structure interaction can be divided into five categories; each category requires a language and unknowns that may not suit others. As shown in Figure 1, the category two is characterized by the non existence of the free surface that reduces the phenomenon only to the fluid vibration. The fluid is the only field that can vibrate and therefore deform.

In the added mass approach, a part of the mass of the fluid is added to the structural model along the interface between the two fields. This approach neglects the compressibility of the fluid. In the Eulerian approach ([1], [2]), the velocity potential, the pressure or velocity describe the behaviour of the fluid, while displacements represent the movements of the structure. The solution of the coupled system can then be obtained by solving the two systems separately but by considering the effects of the interaction in an iterative way ([3]).

In the Lagrangian approach, the fluid behaviour is in general described by a field of displacements ([4], [5]). The motion of the fluid and the structure being both described by only one field of displacement, this approach has the advantage of very easily satisfying the compatibility and the equilibrium conditions along the interface. So a fluid structure system in interaction with a complex geometry can effectively be analyzed with this method.

The liquid in this analysis is regarded as nonviscous, irrotational and incompressible. Such simplification allows displacements, pressures or the velocity potentials to be the variables in the liquid field. But for this study, only displacements will be regarded as variables. This allows a compatibility allowing a facility to incorporate the liquid elements in finished programs of element for the structural analysis.

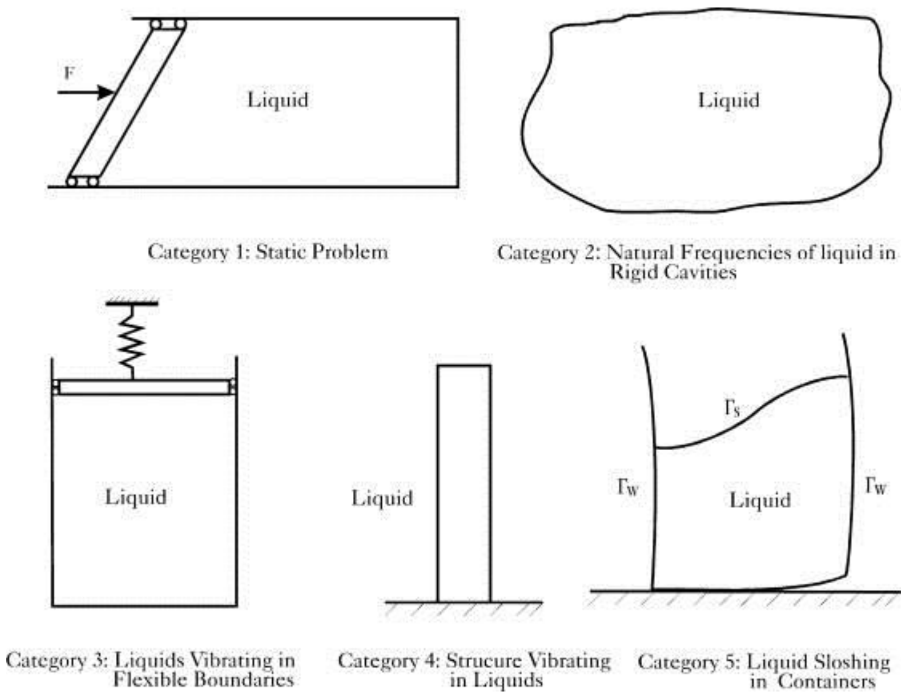


Figure 1: Fluid structure categories

The propose of this work is to present a new displacement based bi-hierarchical finite element to idealize the vibration of a nonviscous, irrotational and incompressible fluid in a cylindrical rigid cavity. This element combined to structural hierarchical finite elements of the same geometrical feathers ([6]) can easily idealize fluid structure interaction. Because of its hierarchical feature, only one element can idealize all the fluid.

2 GOVERNING EQUATION

The fluid motion is governed by the Euler equation:

$$\nabla p = -\rho_f . \dot{v} \tag{1}$$

Where p is the pressure, ∇ is a gradient operator ρ_f is the fluid density and v is the velocity vector of the fluid motion.

In this formulation, velocity and pressure field describe the motion of the fluid. Using the relationship between pressure and volume, one can write:

$$p = -k_f . \nabla^T u \tag{2}$$

Where u is the displacement vector and k_f is the bulk modulus.

Introducing the small amplitude motion assumption expressed by:

$$v = \dot{u} \tag{3}$$

The Euler equation is reduced to an acoustic wave equation:

$$\nabla^2 p = \frac{1}{c^2} . \ddot{p} \tag{4}$$

Where c is the acoustic speed in the fluid given by

$$c = \sqrt{\frac{k_f}{\rho_f}} \tag{5}$$

In the equation (4), only the pressure field describes the fluid motion

The same wave equation can be derived from the following Navier equation for an isotropic, homogeneous and elastic medium:

$$G.\nabla^2 u + (k_f + G).\nabla\nabla^T u = \rho_f . \ddot{u} \tag{6}$$

Where, G is the shear modulus. For the case of inviscid fluid, the shear modulus equals zero. Equation (6) becomes:

$$k_f . \nabla\nabla^T u = \rho_f . \ddot{u} \tag{7}$$

3 HIERARCHICAL FINITE ELEMENT FORMULATION

The hierarchical finite element method known under the name of the p-version of the finite element method is more precise and its convergence is faster than that of the h-method. Indeed, when the exact solution is analytical everywhere the rate of convergence is exponential, whereas that of the h-method is only algebraic. The quality of the solutions is not very sensitive to the distortions of the elements, which allows the use of flattened elements or great ratio on sides without penalizing the precision too much. The hp version of the finite element method has as a characteristic to increase the precision by increasing both the degree of the polynomial of interpolation and the number of finite elements as for the standard finite element method ([7]).

The equation of motion of the fluid admit the representation of the Radial, circumferential and axial displacement components u, v, and w following respectively R, Z and θ in the following form

$$\begin{cases} u(r,z,\theta) = \bar{u}(r,z).\cos n\theta \\ v(r,z,\theta) = \bar{v}(r,z).\cos n\theta \\ w(r,z,\theta) = \bar{w}(r,z).\sin n\theta \end{cases} \tag{8}$$

Dependence on time is removed by assuming that the displacement varies sinusoidal in phase and at the same frequency. The displacement functions, $\bar{u}(r,z)$, $\bar{v}(r,z)$ and $\bar{w}(r,z)$ are expressed in terms of nodal displacements of the finite element by appropriate interpolation functions.

3.1 Shape functions selection

The hierarchical shape functions are generally selected in the Serendipity space. In this paper, the shape functions are built from the shifted Legendre orthogonal polynomials introduced by ([8]). These polynomials are defined in the interval [0,1]. They can be classified in three categories, nodal shape functions, side shape functions and internal shape functions. Their recurring form is:

$$\begin{cases} f_i(x) = 1 - x \\ f_i(x) = x \\ f_{i+2}(x) = \int_0^x P_i(\alpha).d\alpha \end{cases} \tag{9}$$

Where $P_i(\alpha)$ are the shifted Legendre polynomials defined by :

$$\begin{cases} P_0(\alpha) = 1 \\ P_1(\alpha) = 2.\alpha - 1 \\ P_{i+1}(\alpha) = \dfrac{1}{i+1} \cdot [(-2i - 1 + (4i+2)\alpha) \cdot P_i(\alpha) - i \cdot P_{i-1}(\alpha)] \qquad \text{i} = 1,2,3,\ldots \end{cases} \tag{10}$$

The shape functions are given on the basis of one-dimensional hierarchical finite element. The origin of the non-dimensional coordinates is at the left end of the element.

Having the shape functions, the displacement functions $\overline{u}(r,z)$, $\overline{v}(r,z)$ and $\overline{w}(r,z)$ can be expressed:

$$\begin{cases} \overline{u}(r,z) = \sum_{1}^{n} N_i(r,z).u_i \\ \overline{v}(r,z) = \sum_{1}^{n} N_i(r,z).v_i \\ \overline{w}(r,z) = \sum_{1}^{n} N_i(r,z).w_i \end{cases} \tag{11}$$

Where $$N_i(r,z) = f_k(r).g_l(z) \tag{12}$$

With: k = 1,..,p+1 et l = 1,..,q+1. p and q being the hierarchical modes number according respectively to ξ and η, and f and g are the shape functions.

A weak form of the expression (7) for a fluid region Ω subject to natural boundary conditions and critical respectively on the boundaries Γ_p and Γ_f can be expressed:

$$\int_{\Omega} \left\{ \rho_f.\tilde{u}^T.\ddot{u} + k_f.\left(\nabla^T \tilde{u}\right)^T.\left(\nabla^T u\right) \right\} d\Omega + \rho_f.g \int_{\Gamma_f} \left(\tilde{u}^T.n\right).\left(n_g^T.u\right) d\Gamma = -\int_{\Gamma_p} \left(\tilde{u}^T.n\right).\overline{p} d\Gamma \tag{13}$$

Where $\overline{p}$ is the prescribed pression on Γ_p, n_g the gravity direction vector, n the direction vector normal to corresponding boundary, g the gravitational acceleration, Γ_f the free surface, and $\tilde{u}$ is a virtual displacement field.

The second term of the equation left part, result from the free surface boundary condition relatively to the sloshing movement.

Using the same shape functions matrix for the displacement fields u et $\tilde{u}$, one can write :

$$\{u\} = [N].\{q\} \tag{14}$$

$$\{\tilde{u}\} = [N].\{\tilde{q}\} \tag{15}$$

Where $\{q\}$ is the nodal displacement vector, and $\{\tilde{q}\}$ is an arbitrary constant vector.

Basing on the variationnel form of the expression (13) and replacing u and $\tilde{u}$ by their expressions respectively (14) and (15), one can obtain the following matrix equation:

$$\left[M_f\right].\{\ddot{q}\} + \left(\left[K_f\right] + [S]\right).\{q\} = \{f\} \tag{16}$$

Analogically to (13), this equation gives the mass matrix, the stiffness matrix associated to the volumetric deformation, the sloshing stiffness matrix and the charge vector. the mass and stiffness matrices expressions are respectively:

$$\left[M_f\right] = \int_{\Omega} \rho_f.[N]^T.[N].d\Omega \tag{17}$$

$$\left[K_f\right] = \int_{\Omega} k_f . \left[B_f\right]^T . \left[B_f\right] d\Omega \tag{18}$$

3.2 Idealization of the fluid

The fluid is divided into four nodes hierarchical axisymmetric quadrilateral isoparametric finite elements (Fig.2) with only the volumetric deformation stifeness [K_f]. The element size is arbitrary. They may all be of the same size or may all be different. The fluid can also be idealized by only one element if the geometry is varying linearly or if it is constant.

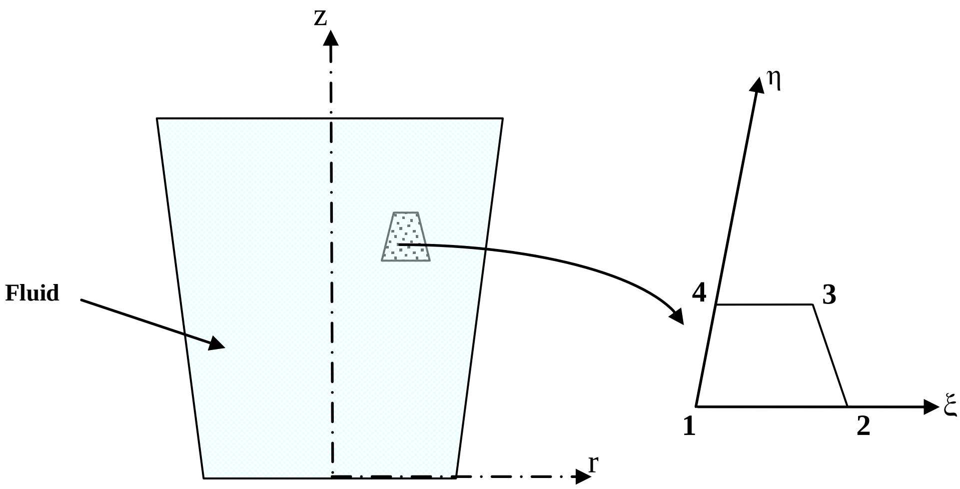


Figure 2: Hierarchical internal axisymmetric fluid finite element

3.2.1 Fluid stiffness matrix

The fluid deformation is given by:

$$\{\nabla\}.\{\delta\} = \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} + \frac{u}{r} + \frac{1}{r}.\frac{\partial w}{\partial \theta} \tag{19}$$

Where $\{\nabla\}$ is a differential operator. According to generalized displacements, the deformation is written:

$$\left\{\varepsilon_f\right\} = \left\{d_f\right\}.\left\{\bar{\delta}\right\} \tag{20}$$

Where $\left\{d_f\right\}$ is a differential operator defined by:

$$\left\{d_f\right\} = \left[\left(\frac{\partial}{\partial r} + \frac{1}{r}\right).\cos n\theta \quad \frac{\partial}{\partial z}.\cos n\theta \quad \frac{n}{r}.\cos n\theta\right] \tag{21}$$

The stiffness matrix of the fluid element is given by the expression (18):

$$\left[K_f\right] = \int_{\Omega} k_f . \left[B_f\right]^T . \left[B_f\right] d\Omega$$

Where: k_f is the fluid bulk modulus, Ω is the volume containing the fluid and $\lfloor B_f \rfloor$ is a matrix defined by:

$$\lfloor B_{f\,i} \rfloor = \{d_f\}.[N_i] \tag{22}$$

The element stiffness matrix can be written, as

$$\left[K_{f\,e}\right] = k_f . k . \pi \int_0^1 \int_0^1 \sum_{i=1}^{p+1} \sum_{j=1}^{q+1} \left[B_{f\,i}\right]^T . \left[B_{f\,j}\right] r . |J| d\xi . d\eta \tag{23}$$

Where k = 2 for n = 0 and k = 1 for n = 1,2,… (n: circumferential wave number)

3.2.2 Fluid mass matrix

The fluid element mass matrix is given by (17):

$$\left[M_f\right] = \int_{\Omega} \rho_f . [N]^T . [N] d\Omega$$

Where: ρ_f is the fluid density, and Ω is the volume containing the fluid.

By replacing the matrix $[N]$ by its submatrices, the fluid element mass matrix can be written:

$$\left[M_{f\,e}\right] = k\pi . \int_0^1 \int_0^1 \rho . \sum_{i=1}^{p+1} \sum_{j=1}^{q+1} [N_i]^T . \left[N_j\right] |J| . r . d\xi . d\eta \tag{24}$$

3.4 Numerical integration

The double integral appearing in the forms of the mass and stiffness matrices results in a numerical integration. For his implementation, one uses the Gauss quadrature expressed by:

$$\int_0^1 f(x) d(x) = \sum_{i=1}^{Np} W_i . f(x_i) \tag{25}$$

Where, Np is the integration points number. In order to optimize calculations, the integration points number increase automatically with the of interpolation polynomial degree.

4 RESULTS AND DISCUSSIONS

The convergence and comparison studies must be carried out to ensure the reliability of the results. The results are given by the frequency parameter Ω wich is expressed in terms of the vibration frequency ω by:

$$\Omega = \frac{\omega}{2.\pi} \tag{26}$$

The effectiveness of the proposed fluid element was examined for the vibration of a fluid in a rigid circular cylindrical cavity. It is assumed that the cavity has a radius equal to 1 and a height equal to 1 as shown in Figure (3). The bulk modulus and the density are equal to 1. To validate the results, an ANSYS program has been elaborated, where axisymmetric fluid elements with four nodes were used. Because spurious modes can not be eliminated, the real patterns will be recognized by plotting each mode.

Table 1 shows the convergence of the first six modes in terms of both of hierarchical mode numbers p and q following respectively the radial and axial directions for one and two elements. Considering the square shape of the cavity, the two hierarchical mode numbers will be equal.

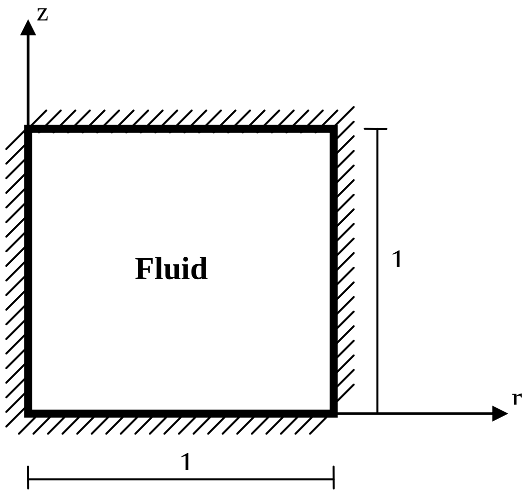


Figure 3: Fluid in a cylindrical rigid cavity

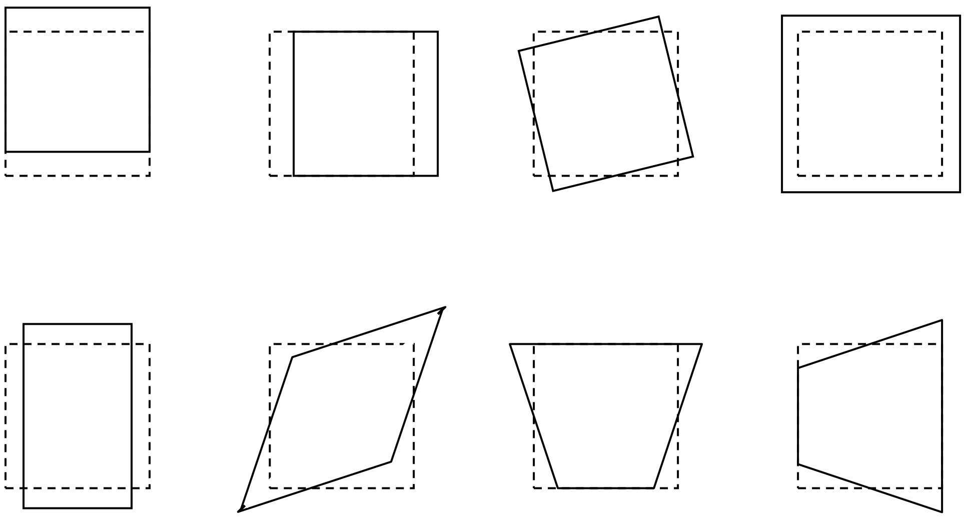


Figure 4: Eight independent displacement modes of a four node fluid element

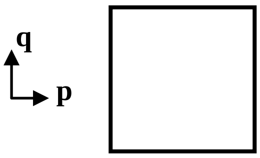
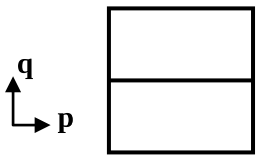
Element number	q	p	Modes							
			1	2	3	4	5	6	7	8
1 	2	2	0.501	0.609	0.789	0.991	1.106	1.152	1.221	1.864
	4	4	0.498	0.605	0.773	0.983	1.091	1.092	1.186	1.483
	6	6	0.496	0.603	0.766	0.970	1.049	1.084	1.157	1.366
	8	8	0.496	0.603	0.766	0.969	1.071	1.079	1.132	1.306
	ANSYS		0.496	0.602	0.757	0.965	1.068	1.077	1.130	1.287
2 	2	2	0.499	0.608	0.785	0.984	1.101	1.136	1.216	1.787
	4	4	0.497	0.604	0.770	0.979	1.082	1.087	1.183	1.411
	6	6	0.496	0.603	0.766	0.969	1.073	1.082	1.145	1.364
	8	8	0.496	0.603	0.766	0.969	1.071	1.079	1.132	1.296
	ANSYS		0.496	0.602	0.757	0.965	1.068	1.077	1.130	1.287

Table 1: Convergence and frequency parameter comparison of a fluid in a circular cylindrical cavity (R =1, H=1).

Table (1) show that for the two idealization one and two elements, an accuracy of two digits after the comma is reached for the first two modes for p=2 and q=2.

For modes 3, 4, 5; the convergence is reached for p=6 and q=6 for a one element model, and for p=4 and q=4 for a two elements model. The sixth mode reaches the convergence for p=8 and q=8 for one element, and p=6 and q=6 for two elements.

It is apparent that by increasing the number of elements, the hierarchical mode numbers p and q necessary to reach convergence are smaller. But the degrees of freedom number becomes more important. The matrices size is smaller if one increases p and q rather than the elements number.

The figure (4) shows the eight independant modes of a four node fluid element. The first six modes are constant-strain modes and the other two are bending modes. One can found these two kind of modes for the bi-hierarchical fluid element in figure (5). The first five modes are constant-strain modes and the other three are bending modes. These modes show the shift of a liquid relative to the walls while being in contact, which reflects the modeling of the interaction that is imposing the same normal displacement to the sides. In this case, the normal displacement is equal to zero because the liquid is in a rigid cavity.

5 CONCLUSION

The bi-hierarchical fluid finite elements presented in this study are able to give accurate frequencies for fluids in axisymmetric cylindrical cavities. The results show clearly that these elements can be easily used for the analysis of the free vibration of a fluid in a closed cavity. With this element one is not constrained any more to have the same number of hierarchical modes in the two main directions (radial and axial) to idealize a fluid in a cavity wich can be slender or lowered. Also, only one element can idealize a fluid in a rigid cavity. Finally these bi-hierarchical finite elements allows triple increase in the accuracy, finite elements number, and radial and axial hierarchical modes numbers.

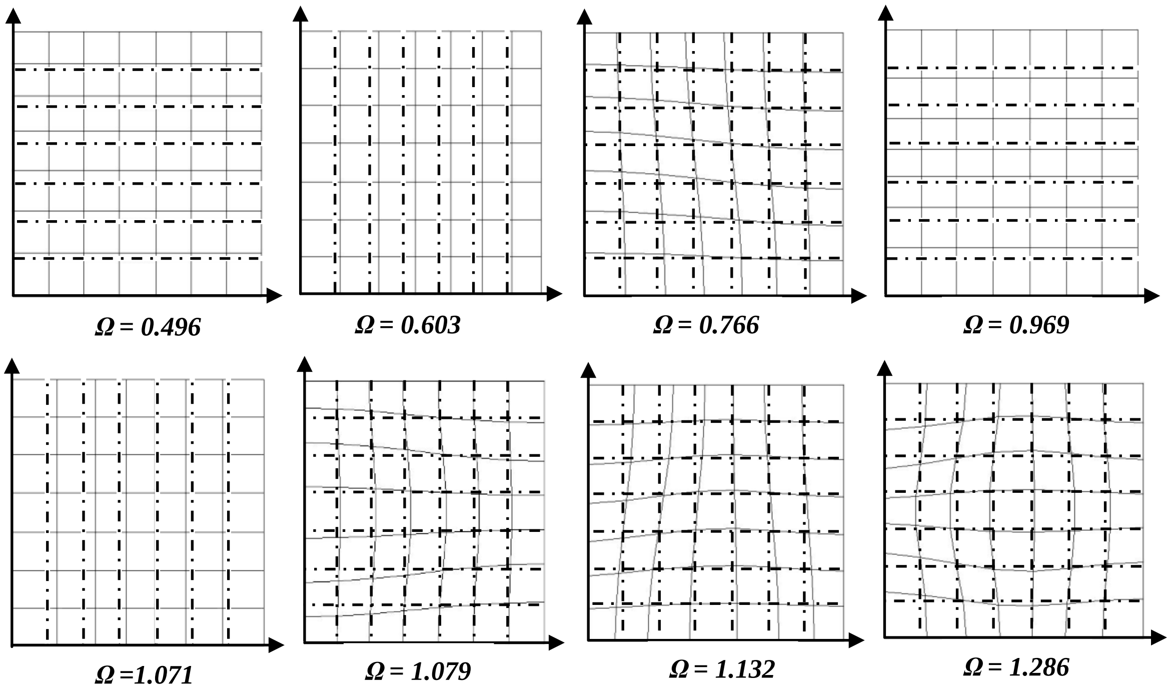


Figure 4: First height acoustic modes of a liquid in a cylindrical rigid cavity

REFERENCES

[1] Haroun, M. A., 1983. Vibration Studies and Tests of Liquid Storage Tanks. J. of Earth. Engng and Struc. Dyn., 11, 179-206.

[2] Veletsos, A.S., Tang, Y. & Tang, H., 1990. Soil-Structure Interaction Effects for Liquid Containing Storage Tanks. Proceedings of the Fourth U.S. National Conference on Earthquake Engineering, California.

[3] Nomura, T. and Hughes, T. J. R., 1992. An arbitrary Lagrangian Eulerian finite element method for interaction of fluid and a rigid body, Comp. Meth . Appl. Mech. Engng., 95, 115-138.

[4] Wilson, E., & Khalvati, M., 1983. Finite Elements For the Dynamic Analysis of Fluid-Solid Systems. International Journal of Numerical Methods in Engineering, 19, 1657-1668.

[5] Chen HC, Taylor RL. 1990. Vibration analysis of fluid-solid systems using a finite element displacement formulation. Int J Numer Methods Eng. 29:683-698.

[6] Ouissi, M.N.,& Houmat, *A. 2009*, Non axisymmetric free vibration analysis of linearly varying thickness shells of revolution by a bi-hierarchical finite element. *Latin American Journal of Solids and Structures*, Vol 6, No 2. 105-129.

[7] Szabo B.A. & Babuska I., Finite element analysis. John Wiley & Sons, 1991.

[8] Houmat A., 2004. Three dimensional hierarchical finite element free vibration analysis of annular sector plates. Journal of Sound and Vibrations. 276, 181-193.