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A SIMPLE APPROACH TOWARDS FURTHER ACCURACY IN STRUCTURAL DYNAMIC ANALYSIS

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Abstract. *Accuracy and computational cost are of the most important concerns in structural dynamics and especially time integration analyses. In view of the practical need to control the accuracies of time integration, with tolerable computational cost, a simple approach is in this paper proposed for enhancing the accuracies. The mere pre-requisite, which is also essential in conventional practical analyses, is proper convergence. The adequate performance of the approach is tested via a twenty-storey building in linear and nonlinear regimes.*

1 INTRODUCTION

In the complicated world of science and engineering, the considerable distance between our plans/imaginations and the actual analytical capabilities, leads to the essentiality of approximate computational methods. Considering this, and the significant progresses in electronic engineering, entails more advanced methods, for different practical computations and the accuracies evaluation. Together with accuracy, computational cost is of the most important concerns in numerical computations [1-3]. Accuracy is being studied in view of convergence and order of accuracy [1-6], and computational cost is being measured in terms of the computer memory dedicated and the time spent for the analysis [7-9].

Concentrating on semidiscretized equations of motion, i.e.

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{f}_{\mathbf{int}}(t)=\mathbf{f}(t) \qquad 0 \le t \le t_{end} \\ &\text{Initial conditions:} \left| \begin{array}{l} \mathbf{u}(\mathrm{t}=\mathrm{t}_0)=\mathbf{u}_0 \\ \dot{\mathbf{u}}(\mathrm{t}=\mathrm{t}_0)=\dot{\mathbf{u}}_0 \\ \mathbf{f}_{\mathrm{int}}(\mathrm{t}=\mathrm{t}_0)=\mathbf{f}_{\mathrm{int}_0} \end{array} \right. \\ &\text{Additional constraints: } \mathbf{Q} \end{aligned} \tag{1}$$

(t and t_{end} imply the time and the duration of the dynamic behavior; $\mathbf{M}$ is the mass matrix; $\mathbf{f}_{\text{int}}$ and $\mathbf{f}(t)$ stand for the vectors of internal force and excitation; $\mathbf{u}$ (t), $\dot{\mathbf{u}}$ (t), and $\ddot{\mathbf{u}}$ (t), denote the vectors of displacement, velocity, and acceleration; $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$ and $\mathbf{f}_{\mathrm{int}_0}$ define the initial status of the model [10, 11]; and $\mathbf{Q}$ represents some restricting conditions, e.g. additional constraints in problems involved in impact or elastic–plastic behavior [12, 13]), and their analysis, by time integration, the step by step nature of time integration highlights the importance of accuracy and computational cost in real analyses [1, 3]. Especially, in integration along large numbers of integration steps, the computational costs are considerable, and the accuracies might be affected, by numerical instability [1, 3, 14]. The importance of the concerns increases even further, with attention to the fact that, the main algorithmic parameter, of time integration, i.e. integration step size, affects accuracy and computational cost, in adverse manners [1, 3, 8]. Considering these, a comment for step size selection before any control of the accuracy is [14, 15]

$$\Delta t \cong\le \operatorname{Min}\left(\frac{T}{10}, h\right) \tag{2}$$

(T is the smallest governing period in the response, h is the largest step size preventing numerical stability in linear problems [14-16]). For the sake of the reliability of accuracy, and considering proper convergence [17-19], as an essentiality for practical error evaluation (implied in the inequality in Eq. (2)), it is recommended to repeat the analyses with half steps and compare the responses, while considering the difference of the responses as upper-estimations for the errors (not only in time integration analysis of Eqs. (1), but also, in step-by-step analysis of general ordinary differential equations) [15, 20, 21]. In view of this approach and its enhancement proposed recently [2], and considering time integration, its repetition, and the evaluation of the errors, as a sole computational procedure, the objective in this paper is to increase the efficiency of this practical procedure and arrive at a better approach. A main assumption, in this study, also pre-requisite in practical analyses, is proper convergence, based on which, an approach to implement the recent accuracy-controlling method and materializing less errors with trivial change of computational cost is proposed in this paper. In Section 2, the theoretical basis is first stated for linear systems and then extended to nonlinear systems. The claims are then examined in Section 3, via analysis of a twenty-storey shear building, in a simple linear case and in a practical nonlinear case subjected to seismic excitations. Later, a complementary discussion is presented in Section 4, and finally, with a brief set of the achievements, the paper is concluded in Section 5.

2 THEORY

2.1 Basis for linear analyses

The method currently recommended for practical time integration analysis consists of a time integration analysis with steps sizes satisfying Eq. (2), and at least once repetition of the analysis with half steps [15], i.e.

$$\Delta t = T_{ord}, \frac{T_{ord}}{2} \tag{3}$$

(Δt_{ord} is the step size obtained from Eq. (2)). This leads to the total computational cost

$$C_{current} = A\frac{30}{T} \tag{4}$$

(A is a constant scalar, depending on the mathematical model in Eqs. (1), and the integration scheme [1, 3]), for cases with $\frac{T}{10}$, as the governing term in Eq. (2). Since, in view of the essentiality of proper convergence, and the Lax-Richtmyer equivalence theorem [4, 22], the study is implicitly limited to numerically stable analyses, unconditional stability and $h \to \infty$ is a comment emphasized in the literature by times, e.g. see [6], and meanwhile, most of the broadly accepted time integration methods are unconditionally stable, e.g. see [23-27], Eq. (4) can be considered valid for general conventional time integration analyses.

Different from ordinary time integration, in time integration analyses with controls of errors based on a recently proposed method [2, 19], at least, two repetitions of the time integration analysis are essential. However because of the additional accuracy obtained from the Richardson extrapolation [28-31], implemented in the recent method [2], the integration step size in the first analysis might be different from that introduced, in Eq. (2). Once again concentrating on unconditionally stable time integration methods and the domination of $\frac{T}{10}$ in Eq. (2), by selecting the step sizes in the first analysis and its two repetitions, as noted below:

$$\Delta t = \frac{10\,T_{ord}}{4}, \frac{10\,T_{ord}}{8}, \frac{10\,T_{ord}}{16} \tag{5}$$

the resulting total computational cost would be

$$C_{proposed} \cong A\frac{28}{T}\frac{1+3n_d}{3n_d} \tag{6}$$

(n_d is the number of the degrees of freedom and the last ratio represents the additional computational effort needed for the Richardson extrapolation). In view of Eq. (4) and (6),

$$C_{proposed} \cong< C_{current} \tag{7}$$

or in other words, when implementing Eq. (5) (as the analyses steps in the recent accuracy controlling method [2]), in stead of implementing Eq. (3), (in two time integration analyses, of the ordinary current approach), the computational costs remain unchanged, and this is while, provided the responses converge properly, the accuracies increases in view of the convergence plot typically depicted in Fig. 1. The convergence trends of the responses computed ordinarily and in the process of the recently accuracy-controlling methods are both as straight lines inclined with integer-valued slopes, in Fig. 1, such that the latter (the line corresponding to Richardson extrapolation and the recent accuracy-controlling method) is steeper. This leads to:

$$E_C < E_B \cong E_A \tag{8}$$

(E implies the error), and since E_A and E_C imply the errors of the most exact obtained in the current analysis approach (implementing Eq. (3)), $E_{current}$, and those obtained in the approach proposed here (implementing Eq. (5) and Richardson extrapolation), $E_{propsed}$, we can conclude:

$$E_{proposed} < E_{current} \tag{9}$$

The validity of Eqs. (8) and (9) (also addressed in Fig. 1) can be proved rigorously in view of [32], Eq. (5), the conventional orders of accuracy, i.e. the range of slopes of the two lines in Fig. 1 [1, 3, 6], and the fifty percent reduction of integration step size in each new analysis (see Eqs. (3) and (5)). Consequently, in view of Eqs. (7) and (9), by considering the integration step sizes stated in Eq. (5), when the responses converge properly, we can increase the accuracy, without additional computational cost. Since, the computational cost and the inaccuracy because of nonlinearities (and specifically the nonlinearity iterations and residuals [16, 18]) are not considered in the discussion, the above claim, regarding arriving at further accuracies with no additional computational cost, is yet limited to linear analyses.

○ Responses under consideration in the ordinary process of analysis and evaluation of accuracy

◎ Responses under consideration in analysis and evaluation of accuracy based on Eq. (5) and a recently proposed accuracy controlling method

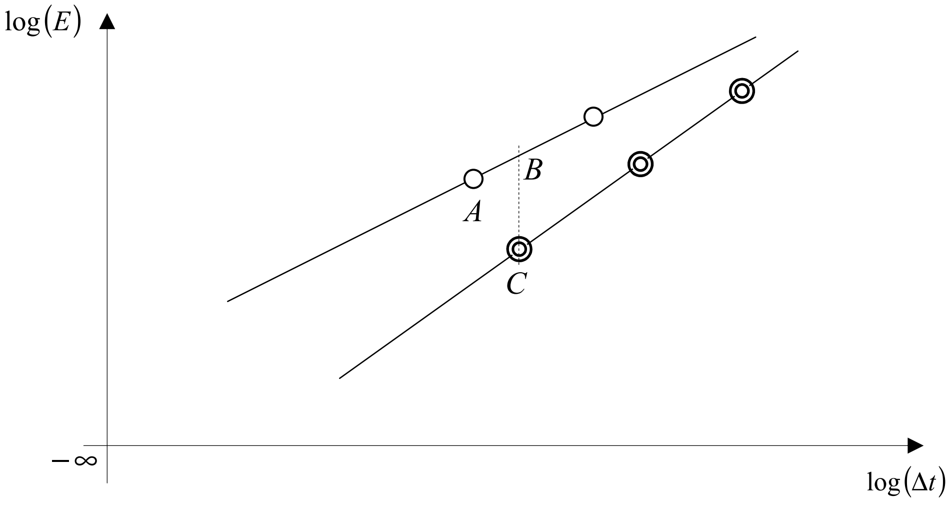

Figure 1: A typical representation of the responses convergence.

2.2 Extension to nonlinear problems

In nonlinear problems, comments regarding integration step size selection, though exist, e.g. see [33] (${}_n\Delta t_{ord} = 0.1\Delta t_{ord}$; ${}_n\Delta t_{ord}$ is the step size recommended for nonlinear problems), are vague; see [18, 34, 35]; special methods are generally essential for maintaining the proper convergence, e.g. see [10, 18, 36, 37]; and proper convergence might not be completely achievable for highly nonlinear problems [18, 38, 39]. Regarding the computational cost, because of the nonlinearity iterations, Eqs. (4) and (6) are not valid, and hence, the validity of Eq. (7) can not guaranteed. Consequently, in presence of nonlinearity, the discussion, presented in Section 2.1, loses its viability. Nevertheless, for low or moderate nonlinear problems (see [38, 39]), where proper convergence is simply achievable [10, 18], and the additional computational cost essential for nonlinearity solutions [40, 41] is small, in view of Eq. (5) and the discussion above, it is reasonable to use the integration step sizes below:

$$\Delta t = \alpha \ {}_n\Delta t_{ord} \ , \frac{\alpha}{2} \ {}_n\Delta t_{ord} \ , \frac{\alpha}{4} \ {}_n\Delta t_{ord} \tag{10}$$

where (see also Eq. (5)),

$$\alpha \cong> 2.5 \ \left(= \frac{0.25}{0.1} \right) \tag{11}$$

and expect trivial additional computational cost, and some additional accuracy. Still there would likely exist a limit of nonlinearity severity, after which, the discussion is not valid.

3 NUMRICAL STUDY

The shear frame introduced, in Fig. 2 and Table 1, and subjected to 0.1 meters initial top lateral displacement, is studied, during a fifty seconds time interval. The exact mid-height dis-

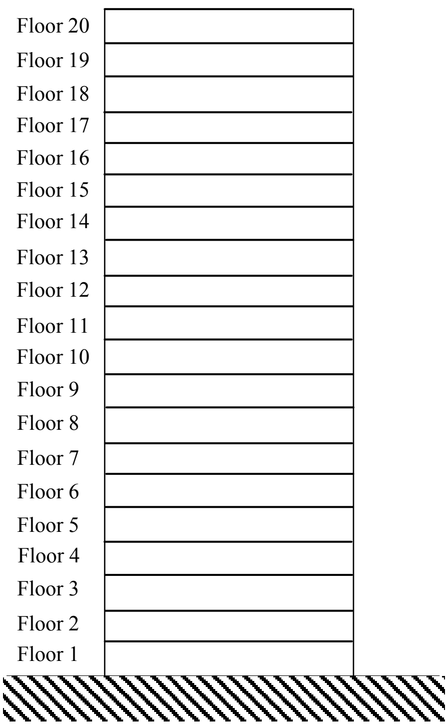

Figure 2: The shear frame under study in the numerical study.

Floor	Mass ($\times 10^{-3}$)	Stiffness ($\times 10^{-3}$)	Damping
1	1060	8600	0
2	1058	8400	
3	1056	8200	
4	1054	7000	
5	1052	6800	
6	1050	6600	
7	1048	6400	
8	1046	6200	
9	1044	6000	
10	1042	5800	
11	1040	5600	
12	1038	5400	
13	1036	5200	
14	1034	5000	
15	1032	4800	
16	1030	4600	
17	1028	4400	
18	1026	4200	
19	1024	4000	
20	1022	3800	

Table 1: Characteristics of the structural system in Fig. 2.

placement is reported in Fig. 3. Considering Fig. 3, and the final displacement therein, an analysis is carried out, with the average acceleration method of Newmark [27] and the integration step size 0.03125, satisfying Eq. (2) and also implying proper convergence; see Fig. 4. Repetition of the analysis with half steps, and then, in the implementation of the new approach, carrying out three analyses with steps satisfying Eq. (5) leads to Table 2. Apparently, Eq. (5), together with conventional time integration, and Richardson extrapolation, has resulted in considerable reduction of errors, i.e. further accuracy, with no additional computational cost (see the third and forth columns of Table 2).

The study is repeated, for a nonlinear seismic case, of the previous problem, by considering linear-elastic/perfect-plastic behaviors, for the columns of the shear frame, in Fig. 2 (the yield limits [42], are all assumed equal to 0.08 meters), and the North-South component of the El Centro strong ground motion, as the excitation [43] (see Fig. 5), the latter affecting Eq. (1) according to the formulation below [15, 43]:

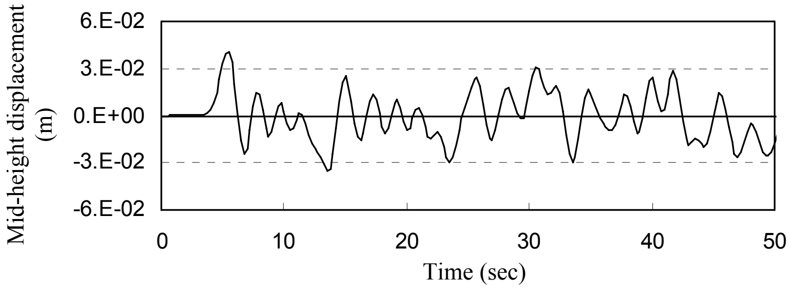

Figure 3: Exact mid-height displacement for the system introduced in Fig. 2 and Table 1 subjected to initial displacement of top floor.

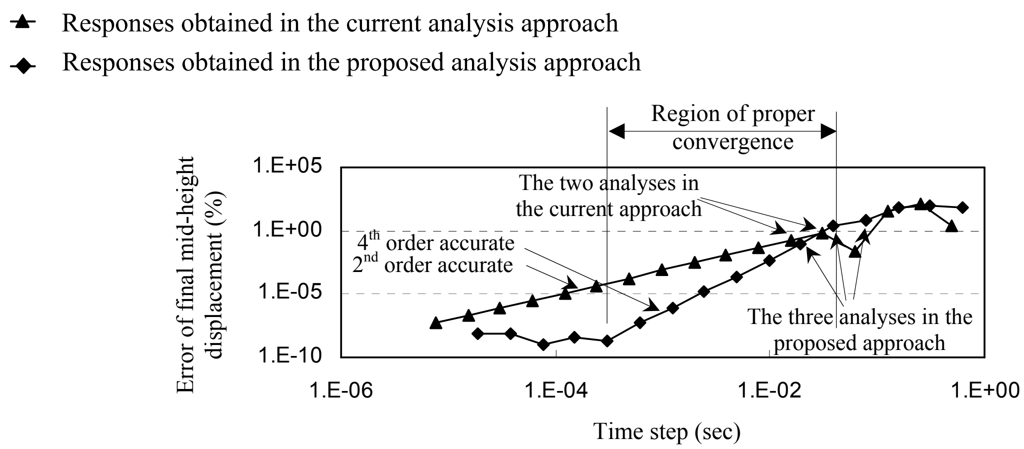

Figure 4: Convergence plots for the final mid-height displacement of the system introduced in Fig. 2 and Table 1 when subjected to initial displacement at top floor and analyzed by the average acceleration.

	Evaluated error (%)	Exact error (%)	Computational cost (Number of steps)
Current approach	0.455	0.1840	4800 (1600+3200)
Proposed approach	0.151	0.0909	4555 (640+1280+2560)*61/60

Table 2: Errors and computational cost for the shear frame introduced in Fig. 2 and Table 1 subjected to top displacement and analyzed by the average acceleration method.

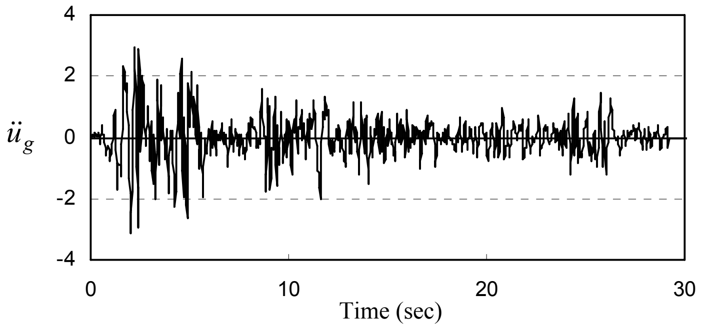

Figure 5: A North-South component of the El Centro strong motion [43].

$$\mathbf{f}(t) = -\mathbf{M}\begin{bmatrix}\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n\end{bmatrix}\ddot{u}_g \tag{12}$$

where, $\ddot{u}_g$ stands for the ground acceleration, and in general,

$$\alpha_1 = \alpha_2 = \cdots \alpha_n = 1 \tag{13}$$

The almost exact mid-height displacement is as displayed in Fig. 6, with attention to which, the smallest dominant period of the response is very small (at most equal to 0.5 sec), and meanwhile, in view of the smallness of the offset of the static equilibrium position, apparent in Fig. 6, the nonlinearity is not severe (see also [43]). The generalized-α method [44, 45], is selected, for time integration (with $r_\infty = 0.8$, as a conventional value for the parameter of the generalized-α method [44]). The nonlinearity iterations are carried out based on conventional fractional time stepping [46, 47], and the method proposed in [18] is implemented to prevent the improper convergence likely in nonlinear analyses. And, finally, in view of Fig. 6, Eqs. (10) and (11), and the fact that in nonlinear problems the "10" in Eq. (2) is recommended to be replaced with "100" [33] (see also [34, 35]), the integration step size, in the first ordinary (current) and proposed analysis approaches, are considered equal to 0.005 sec and 0.0125 sec, respectively, also satisfying the proper convergence pre-requisite (see Fig. 7). The errors and computational costs of the final mid-height displacement, corresponding to the responses obtained in the current and proposed approaches are addressed in Table 3. Similar to the linear case, the performance of the analysis is enhanced from the point of view of accuracy. This implies the fact that the achievements in this paper are not limited to linear problems. In agreement with the discussion presented in Section 2.2, the above mentioned good performance is diminished in repetition of the study, after increasing the severity of the nonlinear behavior, by multiplying the excitation by numbers greater than and sufficiently large, e.g. 100 (see [38, 39]), not reported here for the sake of brevity.

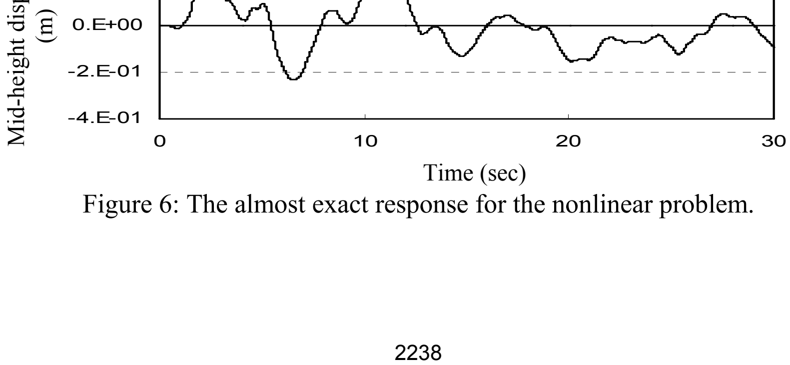

Figure 6: The almost exact response for the nonlinear problem.

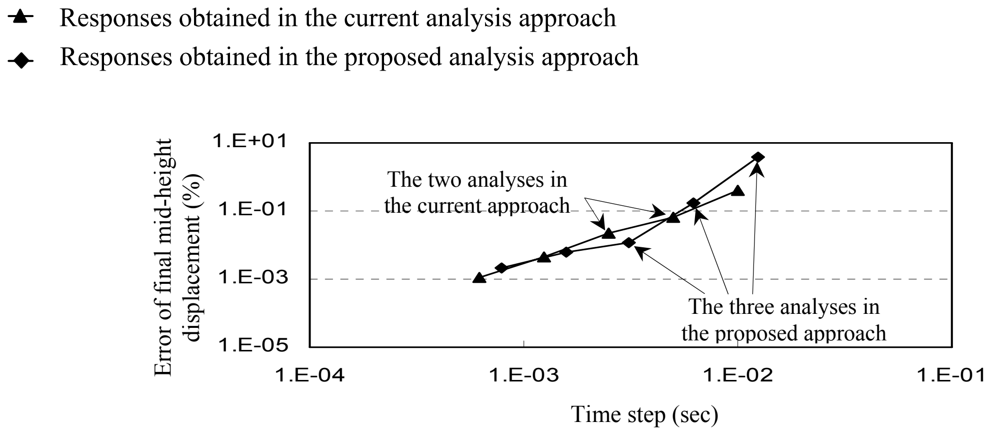

Figure 7: Convergence plots for the final mid-height displacement of the nonlinear problem under consideration when analyzed by the generalized-α ($r_\infty = 0.8$) time integration method.

	Evaluated error (%)	Exact error (%)	Computational cost (Number of steps including the nonlinearity sub-steps)
Current approach	0.0422	0.0224	22879 (8185+14694)
Proposed approach	0.0259	0.0115	24094 (4401+7125+12173)

Table 3: Errors and computational cost for the nonlinear problem under consideration, when analyzed by the generalized-α ($r_\infty = 0.8$) method.

4 DISCUSSION

Based on considering special integration step sizes in the main time integration and repeating the analysis at least twice and extrapolating the computed responses, according to Richardson extrapolation, an analysis approach is proposed for arriving at considerably more accuracy, in price of no or trivial additional computational cost. Proper convergence though is indeed essential even for the current analysis approach (based on at least one repetition of the analysis), it is considered as a key assumption for the approach proposed in this paper. A main question, in this regard, is how can we depict the convergence plot and check the convergence, while, in order to compute the errors,

$$E_{{}_i\mathbf{F}_a} = \left\| {}_i\mathbf{F}_a - \mathbf{F} \right\| \tag{14}$$

($\mathbf{F}$ and ${}_i\mathbf{F}_a$ respectively stand for an approximately computed and the exact values of the arbitrary matrix/vector/scalar under consideration, $E_{{}_i\mathbf{F}_a}$ implies the error and $\| \ \|$ implies an arbitrary norm [48]), the exact response is essential. The answer to this question can be found in the notion of pseudo error and pseudo convergence and their equivalence with error and convergence, from the standpoint of proper convergence [2, 49]. In brief, considering

$$D_i = \left\| {}_i\mathbf{F}_a - {}_{i-1}\mathbf{F}_a \right\| \tag{15}$$

as pseudo error, we can investigate the proper convergence, instead of in the convergence plot, in the log-log plot of pseudo error, D, with respect to integration step size, Δt (being addressed as the pseudo convergence plot), during the analysis repetitions; see also [2].

Another ambiguity in the discussion presented in the previous sections is regarding Eqs. (2), (3), and (5). In fact, in these equations, it is implicitly assumed that the integration step size can be selected continuously from the set of real numbers. This is hardly the case when the excitation is available as a digitized record, e.g. earthquake records [50]. In these cases, conventionally, Eqs. (2) need to be replaced with

$$\Delta t \cong\leq \mathrm{Min}\left(\frac{T}{10}, h, \Delta t_f\right) \quad (16)$$

(Δt_f is the step size by which the excitation record is available). In view of the typology of the proper convergence region in convergence plot, Eq. (16) affects the proposed approach mainly when instead of the analysis with steps sized Δt_{ord}, in the current analysis approach, we need to carry out an analysis with steps sized $2.5\Delta t_{ord}$, in the proposed approach (see Eqs. (3) and (5)). To explain better, when the integration steps in the current approach equal to Δt_f, how, in the proposed approach, can we consider the integration step size $\Delta t = 2.5\Delta t_f$, while there are not excitation stations at all integration stations. A solution is to replace the excitation with an excitation digitized at steps larger than the steps of the original excitation, such that addressing the steps in the two excitations with ${}_1\Delta t_f$ and ${}_2\Delta t_f$,

$$\frac{{}_1\Delta t_f}{{}_2\Delta t_f} = \frac{z_1}{z_2} \quad , \quad z_1, z_2 \in Z^+ \quad (17)$$

Two formulations are proposed for the replacement; see [8, 51, 52], from which, in view of the convergence based nature of the discussion presented in this paper, the first [8, 51] is already implemented in the analyses of the proposed approach in the second example studied in the previous section.

5 CONCLUSION

Provided the integration step sizes, selected for time integration, correspond to points on the convergence plot, with integer-valued slope (i.e. proper convergence is maintained), the accuracy of time integration analyses can be increased considerably, with trivial change in the total computational costs, including the analyses and their repetitions (for controlling the accuracies). This can be materialized, by replacing Eq. (2) with

$$\Delta t \cong\leq \mathrm{Min}\left(\frac{T}{4}, h\right) \quad (18)$$

for the main analysis, while conventionally assuming unconditional stability in linear problems and $h \to \infty$, considering at least two repetitions of the first (main) analysis (each with half steps compared to the previous analysis), and implementing the Richardson extrapolation. Regarding the performance of the proposed approach (based on the presented theoretical and numerical studies):

- For linear problems, the consequence is guaranteed significant increase of accuracy (not proved in this paper for the sake of brevity), in the price of no additional computational cost.
- For nonlinear problems, when the nonlinearity is not severe, the performance can be similar to linear problems. (In presence of considerably severe nonlinearities, the expectation is to observe better performance, when the nonlinearity solution methods and the methods implemented for maintaining proper convergence are more robust.)
- The performance is independent of the integration scheme.

It is meanwhile worth noting that, though, as implied above, proper convergence is an assumption and pre-requisite for the success of the proposed approach, considering the fact that proper convergence is also essential when implementing the current approach, the assumption on proper convergence is not a weak point for the proposed approach. And, finally, regarding practical implementation of the proposed approach,

- Proper convergence need to be investigated in the pseudo convergence plot instead of the convergence plot (the validity of this replacement is recently demonstrated).
- Some considerations regarding digitized excitations may be essential.

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