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TRANSIENT RESPONSE OF FUNCTIONALLY GRADED BEAMS UNDER ARBITRARY DYNAMIC LOADING

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Abstract. *In this contribution, functionally graded beams (FGBs) with an arbitrary gradation of the material properties along the beam thickness are considered. Such FGBs are of special interest in civil and mechanical engineering to improve both the thermal and the mechanical behaviours of the beams.*
In order to simplify the coupled governing differential equations for the longitudinal and transversal vibration of FGBs based on the Euler-Bernoulli beam theory, the gradation of the material properties is described by means of simple functions. The free vibration solution (mode shapes and eigenfrequencies) has been derived analytically and can be applied to all boundary conditions of beams.
In this study, vibrations of FGBs due to periodic and non-periodic dynamic loadings are investigated. The dynamic response solution of the beams is derived analytically or numerically by means of the modal analysis with the eigenfrequencies from the free vibration analysis. Comparison is also made with the numerical results obtained by the Finite Element Method. The obtained solutions and their applications will be also discussed.

1 INTRODUCTION

For many applications in civil and mechanical engineering, classical beam elements can be replaced by novel and advanced structures like multi-layered composites or sandwich structures. They offer often better structural properties and can be optimized for different purposes according to the specific requirements. However, the transition between those layers made up of often very different materials is abrupt, and this may lead to delamination or cracks on the interfaces.

To avoid this problem, functionally graded materials (FGMs) can be employed, which possess a smooth property transition inside structural elements. They may consist of different materials which are mixed gradually with different ratios, or they consist of the same materials with varying properties like porosity, fiber content or fiber orientation, as the nature shows us in many cases (e.g. bamboo stem or human and animal bones).

For the engineering application, an appropriate theory is needed to correctly describe the static and dynamic behaviours of FGBs. In Ref. [2] closed-form solutions of the stress distributions, eigenfrequencies and eigenfunctions have been derived for FGBs based on Euler-Bernoulli and Timoshenko theory by means of a single differential equation of motion for the deflection. In Ref. [5] static Green's functions for functionally graded Euler-Bernoulli and Timoshenko beams are presented. In combination with Betti's theorem the Green's functions are applied to calculate internal forces and stress analysis of FGBs. Moreover, approximate eigenfunctions and eigenfrequencies are obtained using Fredholm's integral equation and the static Green's functions of FGBs.

However, these previous considerations did not take into account the coupling between the longitudinal and the transversal displacements and its effects on the eigenfrequencies and mode shapes of FGBs. This approximation is exact only for a symmetrical material gradation and not valid for general cases with an arbitrary material gradation.

In contrast to the work presented in [2], in Ref. [3] the coupling effects of the longitudinal and transverse displacements on the deformation and internal forces of FGBs have been investigated for different boundary conditions. In Ref. [4] free vibrations of Euler-Bernoulli-type FGBs have been considered. The eigenfrequencies and mode shapes of FGBs have been calculated for different boundary conditions by taking into account the coupling effects.

The objective of this paper is to extend the free vibration solution to FGBs subjected to periodic and non-periodic dynamic loadings. The eigenfrequencies and mode shapes obtained from the free vibration analysis are used to calculate the dynamic response of the beams by means of the modal analysis. The analytical results are also validated with the numerical results obtained by the Finite Element Method (FEM) and analyzed concerning their accuracy and their applications.

2 DIFFERENTIAL EQUATIONS OF FUNCTIONALLY GRADED BEAMS

We consider a functionally graded Euler-Bernoulli beam with an arbitrary material gradation in thickness direction as shown in Fig. 1. Since the geometrical principal axis does not correspond to the elastic principal axis for non-symmetric Young's modulus and density functions $E(z)$ and $\rho(z)$, there will be a coupling between the longitudinal and the transverse displacements. Most previous publications [1, 2] neglected this coupling in static and dynamic analysis of FGBs. The coupled partial differential equations (PDEs) describing the beam vibration are given by

$$E_0\frac{\partial^2 u}{\partial x^2} - E_1\frac{\partial^3 w}{\partial x^3} - \rho_0\frac{\partial^2 u}{\partial t^2} + \rho_1\frac{\partial^3 w}{\partial x \partial t^2} = n\left(x,t\right), \tag{1}$$

and

$$E_2\frac{\partial^4 w}{\partial x^4} - E_1\frac{\partial^3 u}{\partial x^3} + \rho_0\frac{\partial^2 w}{\partial t^2} + \rho_1\frac{\partial^3 u}{\partial x \partial t^2} = q\left(x,t\right), \tag{2}$$

with the abbreviations $\rho_i = \int_A z^i \rho(z)\, dA$ and $E_i = \int_A z^i E(z)\, dA$.

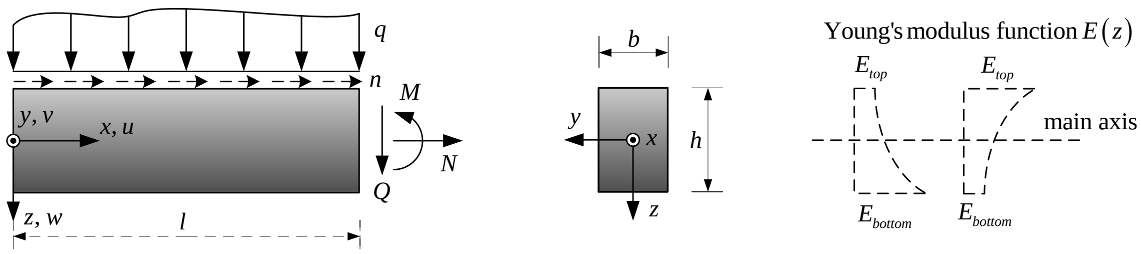

Figure 1: Functionally graded Euler-Bernoulli beam with non-symmetric material properties.

For the free vibration solution (homogeneous PDE), the time-dependent and time-independent components can be separated by using the ansatz

$$\begin{aligned} u(x,t) &= U(x)\cos\left(\omega t - \alpha\right), \\ w(x,t) &= W(x)\cos\left(\omega t - \alpha\right), \end{aligned} \tag{3}$$

with the assumption of the same eigenfrequency ω for both longitudinal and transverse vibrations. It leads to the ordinary differential equations

$$E_0\frac{d^2U}{dx^2} - E_1\frac{d^3W}{dx^3} = \omega^2\left(\rho_1\frac{dW}{dx} - \rho_0 U\right), \tag{4}$$

and

$$E_2\frac{d^4W}{dx^4} - E_1\frac{d^3U}{dx^3} = \omega^2\left(\rho_1\frac{dU}{dx} + \rho_0 W\right). \tag{5}$$

The solution of Eqs. (4) and (5) can be written as

$$\begin{aligned} W(x) = C_1\sinh\left(\kappa_1 x\right) + C_2\cosh\left(\kappa_1 x\right) + C_3\sin\left(\kappa_2 x\right) + C_4\cos\left(\kappa_2 x\right) \\ + \overline{\varepsilon} D_1\kappa_3\cos\left(\kappa_3 x\right) - \overline{\varepsilon} D_2\kappa_3\sin\left(\kappa_3 x\right) \end{aligned} \tag{6}$$

$$\begin{aligned} U(x) = \overline{\gamma} C_1\kappa_1\cosh\left(\kappa_1 x\right) + \overline{\gamma} C_2\kappa_1\sinh\left(\kappa_1 x\right) + \overline{\delta} C_3\kappa_2\cos\left(\kappa_2 x\right) \\ - \overline{\delta} C_4\kappa_2\sin\left(\kappa_2 x\right) + D_1\sin\left(\kappa_3 x\right) + D_2\cos\left(\kappa_3 x\right), \end{aligned} \tag{7}$$

with

$$\overline{\gamma} = \frac{E_1\kappa_1^2 + \omega^2\rho_1}{E_0\kappa_1^2 + \omega^2\rho_0}; \quad \overline{\delta} = \frac{E_1\kappa_2^2 + \omega^2\rho_1}{E_0\kappa_2^2 + \omega^2\rho_0}; \quad \overline{\varepsilon} = \frac{E_1\kappa_3^2 + \omega^2\rho_1}{E_2\kappa_3^4 - \omega^2\rho_0}. \tag{8}$$

The eigenvalues κ_1 and κ_2 define the transversal-dominated and κ_3 the longitudinal-dominated terms of the mode shapes $W(x)$ and $U(x)$.

3 MODAL ANALYSIS OF FUNCTIONALLY GRADED BEAMS

Once the homogeneous solution of Eqs. (4) and (5) is found, we can rewrite the transient response $w(x,t) = w_h(x,t) + w_p(x,t)$ and $u(x,t) = u_h(x,t) + u_p(x,t)$ as a series expansion of the mode shapes multiplied by a time-dependent term $T_k(t) + \bar{T}_k(t)$ as follows

$$w(x,t) = \sum_{k=1}^{\infty} W_k(x)\left(T_k(t) + \bar{T}_k(t)\right), \qquad u(x,t) = \sum_{k=1}^{\infty} U_k(x)\left(T_k(t) + \bar{T}_k(t)\right). \tag{9}$$

In analogy to homogeneous beams, the terms T_k and $\bar{T}_k$ describe the influence of the k^{th} mode shape on the global system response and must fulfill the homogeneous or inhomogeneous differential equations

$$\begin{aligned} \ddot{T}_k(t) + \omega_k^2 T_k(t) &= 0 \qquad \text{and} \\ \ddot{\bar{T}}_k(t) + \omega_k^2 \bar{T}_k(t) &= \frac{q(x,t)}{M_k}, \end{aligned} \tag{10}$$

respectively, with the k^{th} eigenfrequency $\omega_k = \sqrt{\dfrac{K_k}{M_k}}$ and the terms K_k (generalized stiffness) and M_k (generalized mass) given by

$$K_k = E_2 \int_0^l \left(W_k''(x)\right)^2 dx + E_0 \int_0^l \left(U_k'(x)\right)^2 dx - 2E_1 \int_0^l W_k''(x) U_k'(x)\, dx, \tag{11}$$

$$M_k = \rho_0 \int_0^l \left[W_k^2(x) + U_k^2(x)\right] dx - 2\rho_1 \int_0^l W_k'(x) U_k(x)\, dx. \tag{12}$$

The homogeneous solution with the initial values $w(x,0) = w_0$, $u(x,0) = u_0$, $\dot{w}(x,0) = \dot{w}_0$ and $\dot{u}(x,0) = \dot{u}_0$ can be written as

$$w_h(x,t) = \sum_{k=1}^{\infty} W_k(x)\left(G_{1,k}\cos\omega_k t + G_{2,k}\sin\omega_k t\right), \tag{13}$$

$$u_h(x,t) = \sum_{k=1}^{\infty} U_k(x)\left(G_{1,k}\cos\omega_k t + G_{2,k}\sin\omega_k t\right), \tag{14}$$

where the constants of integration can be determined by

$$G_{1,k} = \frac{\int_0^l \left[w_0 W_k(x) + u_0 U_k(x)\right] dx}{\int_0^l \left[W_k^2(x) + U_k^2(x)\right] dx}; \quad G_{2,k} = \frac{\int_0^l \left[\dot{w}_0 W_k(x) + \dot{u}_0 U_k(x)\right] dx}{\omega_k \int_0^l \left[W_k^2(x) + U_k^2(x)\right] dx}. \tag{15}$$

We can describe the transversal and longitudinal dynamic loading with a product ansatz as $q(x,t) = q^*(x)f(t)$ and $n(x,t) = n^*(x)g(t)$, and obtain for the particular solution of the transient response of the system

$$w_p(x,t) = \sum_{k=1}^{\infty} W_k(x)\frac{\omega_k}{K_k}\int_0^l W_k(\xi)q^*(\xi)\, d\xi \int_{-\infty}^{t} f(\tau)\sin\left[\omega_k(t-\tau)\right] d\tau, \tag{16}$$

$$u_p(x,t) = \sum_{k=1}^{\infty} U_k(x)\frac{\omega_k}{K_k}\int_0^l U_k(\xi)n^*(\xi)\, d\xi \int_{-\infty}^{t} g(\tau)\sin\left[\omega_k(t-\tau)\right] d\tau, \tag{17}$$

where the last term is a Duhamel's integral [6].

4 CANTILEVER FGB UNDER A VERTICAL LOAD AT THE FREE BEAM END

For the first example we consider a cantilever FGB with non-symmetric material gradation as shown in Figure 1 with the Young's modulus $E(z) = E_{bottom} + (E_{top} - E_{bottom})\left(\frac{1}{2} - \frac{z}{h}\right)^2$ and the density distribution $\rho(z) = \rho_{bottom} + (\rho_{top} - \rho_{bottom})\left(\frac{1}{2} - \frac{z}{h}\right)^2$, with the ratios $\frac{E_{bottom}}{E_{top}} = 0.2$ and $\frac{\rho_{bottom}}{\rho_{top}} = 0.2$. The longitudinal boundary conditions may be CC (constrained at both ends) or CF (constrained-free). The geometrical data and material properties as well as the first 10 eigenfrequencies are listed in Table 1. These eigenfrequencies are calculated with the present coupled beam theory (first two columns) and compared with the eigenfrequencies (last two columns) given by the uncoupled beam theory of Li [2]. For the CC boundary conditions, this example presents a difference of about $7.5\ \%$ for the first eigenfrequency and shows that the coupling between the longitudinal and the transversal beam vibrations should not be neglected.

Geometrical data
Length $l = 8.0\,[m]$
Height $h = 40\,[cm]$
Width $b = 20\,[cm]$
Material properties
$E_{top} = 20.000\,[N/mm^2]$
$\rho_{top} = 2.500\,[kg/m^3]$

k	$\omega_{k,CC}$	$\omega_{k,CF}$	$\omega_{k,CC}$ [2]	$\omega_{k,CF}$ [2]
1	17.986387	16.729478	16.729845	16.729845
2	107.43758	104.87972	104.84415	104.84415
3	296.17575	293.91101	293.56665	293.56665
4	576.91977	555.36037	575.27346	555.36037
5	942.38759	576.73058	950.96825	575.27346
6	1100.5816	955.15818	1110.7207	950.96825
7	1441.8223	1430.2464	1420.5819	1420.5819
8	1983.0178	1666.0811	1984.1185	1666.0811
9	2215.4974	2003.4353	2221.4415	1984.1185
10	2683.8256	2676.4925	2641.5779	2641.5779

Table 1: Material properties and first 10 eigenfrequencies $\omega_k\ [1/s]$ of the cantilever FGB

4.1 Harmonic loading $F(t) = F_0 \sin(\Omega t)$

The first considered case is a sinusoidal excitation. Figure 2 shows the vertical and horizontal displacements at the beam end for $F_0 = 50\ N$, $\Omega = 20\ s^{-1}$ and $0 \leq t \leq 4\ s$.
All FEM results have been obtained using 2D plain stress elements (PLANE82) in ANSYS 13.0 to model the functionally graded beam as a layered beam with 32 constant layers. In all cases $a)$, $b)$, $c)$ and $d)$, we find a very good agreement between the results of this study and the numerical results obtained with the FEM software. The results of Li [2] did not take into account the coupling between the longitudinal and transversal vibrationsand are therefore only valid for the vertical deflection if the beam is not constrained longitudinally. However, the model of Li [2] gives wrong results for the longitudinal displacements and the transversal vibration of longitudinally constrained beams, as shown in $a)$ and $b)$.

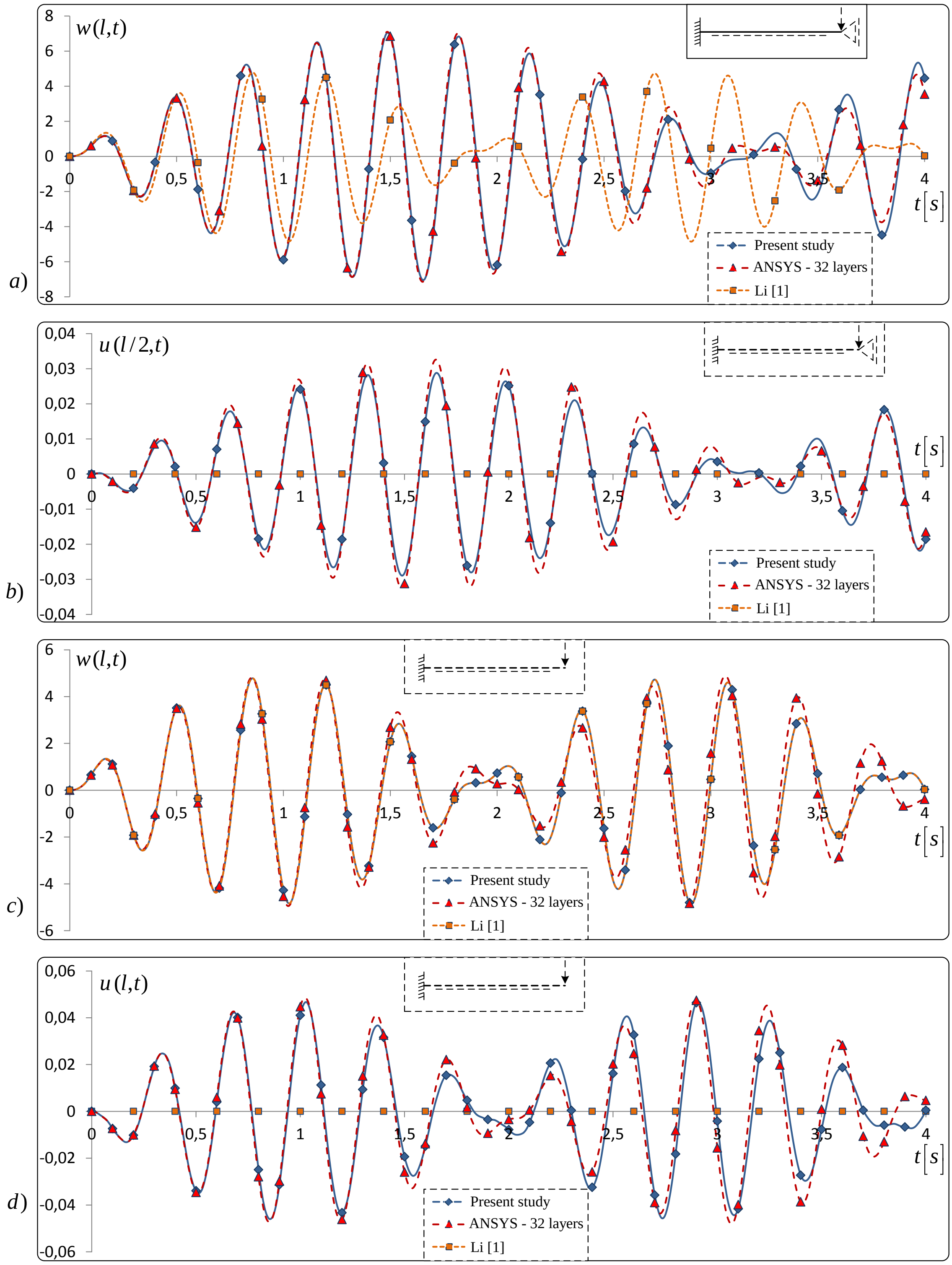

Figure 2: Displacements of the FGB for the first 4 s: a) Vertical displacement at the beam end and b) Horizontal displacement in the middle of the beam with CC boundary conditions, c) Vertical and d) Horizontal displacements at the beam end with CF boundary conditions.

4.2 Triangular impulse loading

The second loading for this example is a non-periodic triangular impulse loading which can be described by

$$F(t) = F_0 \left(1 - \frac{t}{t_0}\right), \tag{18}$$

with an impulse duration of $t_0 = 0.3\ s$ and the intensity $F_0 = 50\ N$. In Fig. 3, the vertical displacements of the beam due to this impulse loading are given. For the vertical vibration as shown in $a)$ and $b)$ we find again a good agreement between the results of this study and the numerical results. The time of calculation for $b)$ took $t_{calc} = 57.947\ s$ for the present study with MATLAB, and $t_{calc} = 257.667\ s$ for the FEM results performed with ANSYS. In $c)$ the vertical displacement of the beam for the time steps $t = 0.05\ s$, $t = 0.10\ s$, $t = 0.15\ s$, $t = 0.20\ s$, $t = 0.25\ s$ and $t = 0.30\ s$ is shown.

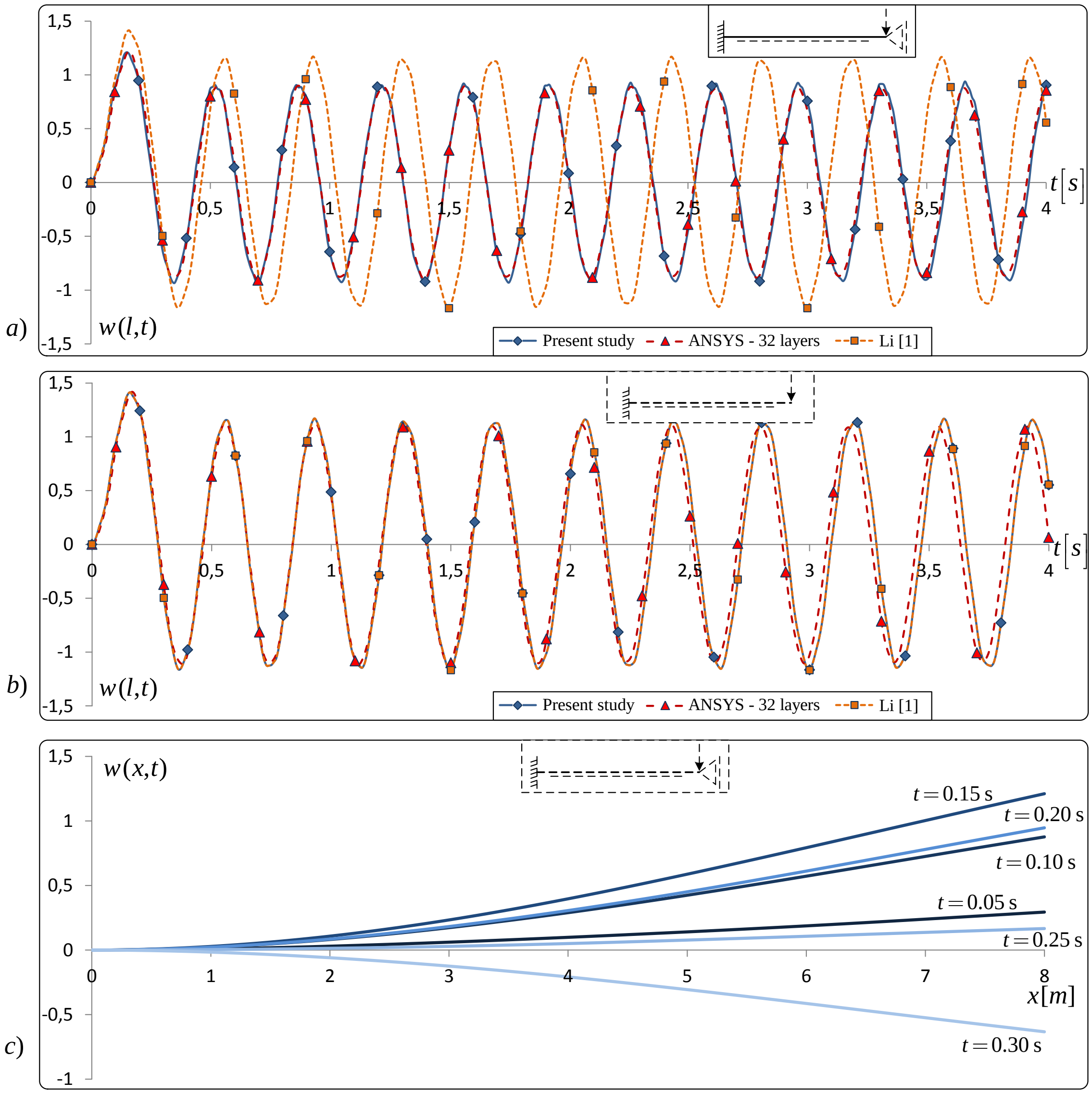

Figure 3: Vertical displacements of the FGB for the first $4\ s$: $a)$ With CC boundary conditions, $b)$ With CF boundary conditions, and $c)$ Beam deflection at 6 different time steps.

5 SIMPLY SUPPORTED FGB UNDER A UNIFORMLY DISTRIBUTED LOAD

As a second example we consider a simply supported beam with the same geometry and material properties as in Section 4. Its first 10 eigenfrequencies calculated with the coupled beam theory of the present study, and a comparison made with a FE analysis can be found in Table 2.

We apply a uniformly distributed loading $q(x,t) = q_0 \sin(\Omega t)$ with $q_0 = 50\ N/m$ on the FGB.

k	$\omega_{k,CC}$	$\omega_{k,CF}$	$\omega_{k,CC}$ FEM	$\omega_{k,CF}$ FEM
1	51.785111	46.949494	51.973881	46.751925
2	187.65204	187.62006	186.11423	186.08282
3	428.02624	296.17575	422.14209	413.74775
4	744.60570	576.91977	727.96985	559.31659
5	1094.3204	942.38759	1133.3610	741.41587
6	1180.6837	1100.5816	1139.7698	1139.7698
7	1715.5681	1441.8223	1635.5131	1600.3273
8	2141.2072	1983.0178	2117.8733	1704.5653
9	2381.3737	2215.4974	2332.5697	2180.5795
10	3015.1517	2683.8256	2788.6033	2748.5794

Table 2: First 10 eigenfrequencies $\omega_k\ [1/s]$ of the simply supported FGB.

Figure 4 presents the vertical displacements of the FGB for different excitation frequencies Ω. In the first case $a)$ with $\Omega = 50\ s^{-1}$, we have an excitation frequency near the first eigenfrequency, and the amplitudes are hence high. A comparison with the finite element analysis shows good agreements in the frequency of the vibration, but the maximal amplitude is about $16\ \%$ lower. The results given by Li [2] are invalid because they do not take into account the coupling between the longitudinal and the transversal vibrations. The other cases $b)$–$d)$ show the response of the system to the frequencies $\Omega = 100\ s^{-1}$, $\Omega = 150\ s^{-1}$ and $\Omega = 200\ s^{-1}$, respectively. Since the first eigenfrequency has the largest influence on the dynamic response of the system, and the excitation frequencies are far away from it, the vibration amplitudes are very low.

6 CONCLUSIONS

This paper deals with the transient response of functionally graded beams (FGBs) subjected to harmonic and non-periodic impulse loadings. The beams are supposed to be governed by the Euler-Bernoulli beam theory. The free vibration solution of the differential equations of motion has been derived analytically. With the obtained eigenfrequencies and mode shapes, this solution has been used to obtain the forced vibration solution by means of a modal analysis. For a cantilever and a simply supported beam, the eigenfrequencies as well as the forced vibration solution have been derived for different loading and boundary conditions. The present analytical solution has a good agreement with the numerical results obtained with the Finite Element Method.

The present study shows also that there is a significant difference between the present theory and the theory neglecting the coupling between the longitudinal and the transverse vibrations [2]. The most important differences can be found in the dynamic solution of FGBs which are constrained longitudinally at both ends. For future works the present coupled beam theory should be extended to two-dimensional functionally graded structures such as plates and shells.

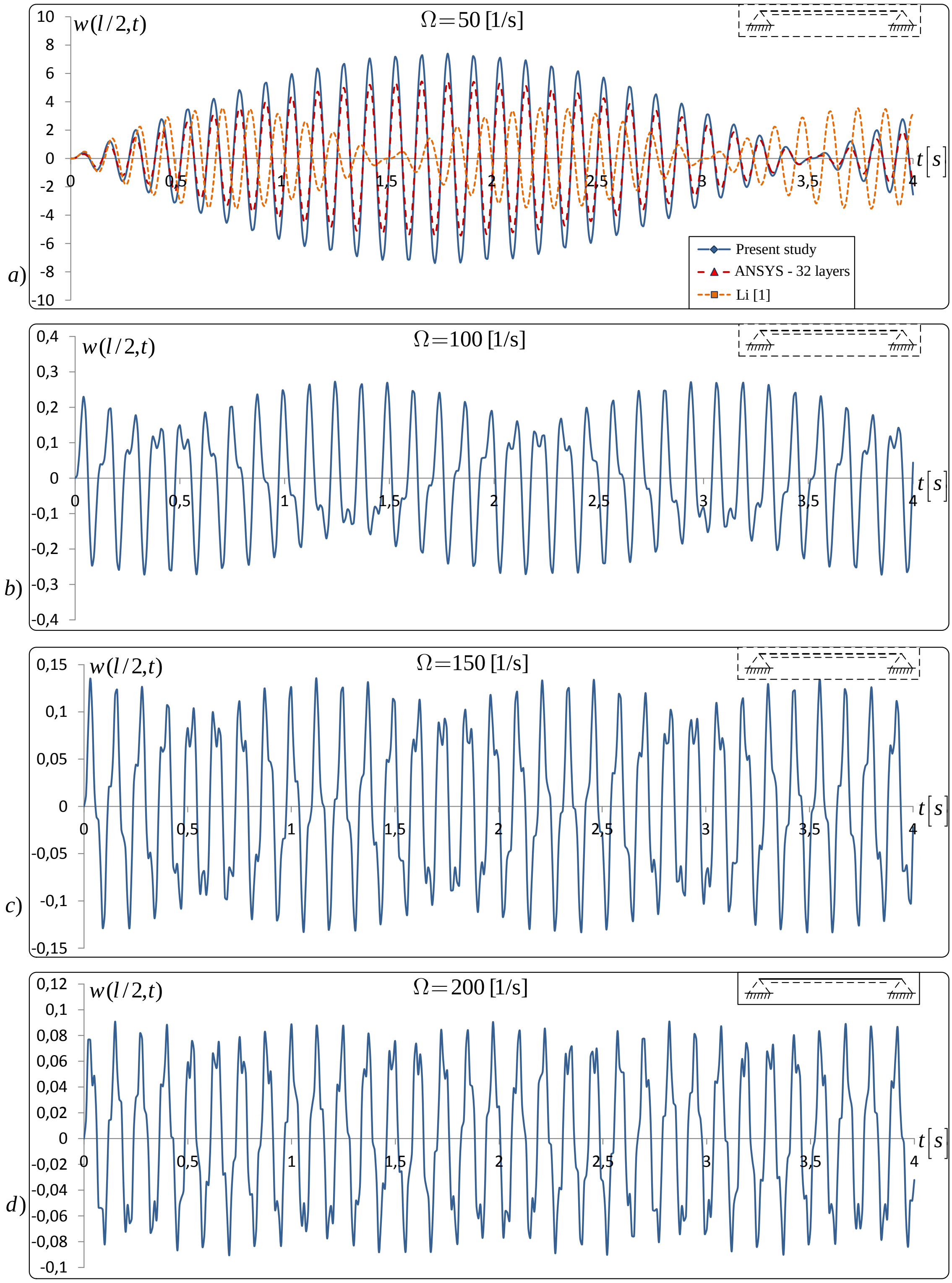

Figure 4: Vertical displacements of the simply supported FGB with CC boundary conditions for different excitation frequencies Ω.

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