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THE LOCATION OF PASS AND STOP BANDS OF AN INFINITE PERIODIC STRUCTURE VERSUS THE EIGENFREQUENCIES OF ITS FINITE SEGMENT CONSISTING OF SEVERAL 'PERIODICITY CELLS'

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Abstract. *On the classical example of a rod (membrane, string) with periodically located inertial inclusions, it is proved that all eigenfrequencies of a finite periodic structure fall into pass-bands of corresponding infinite system in the symmetrical case. In the unsymmetrical case, some eigenfrequencies may not follow this rule.*

The exact equations for eigenfrequencies and the explicit expressions for power flow are obtained. The asymptotic analysis of power flow in an infinite periodic structure is carried out and the structure of pass- and stop-bands is explored. The modes of free vibrations of a finite periodic structure are analyzed with the special attention to edge effects.

1 INTRODUCTION

Wave phenomena in periodic structures are intensively explored during last years [1-4]. The extensive bibliography is represented in [1] for example. In the process of numerical calculations in [4] was noticed that the eigenfrequencies of a finite periodic structure fall into the pass-bands of corresponding infinite system.

In this work on the classical example of a rod (membrane, string) with periodically located inertial inclusions is analytically proved that all eigenfrequencies of a finite periodic structure fall into pass-bands of corresponding infinite system in the symmetrical case. In the unsymmetrical case, some eigenfrequencies may not follow this rule. The stationary problem is considered.

2 INFINITE PERIODIC STRUCTURE

On the interval $x \in (-\infty, +\infty)$ the following equation is considered

$$u''(x) + k^2 \left(1 + \frac{M}{\mu} \sum_{j=-\infty}^{+\infty} \delta(x - jl) \right) u(x) = 0 \qquad (1)$$

It describes propagation of stationary waves in the string (membrane) or longitudinal waves in the rod with periodic point masses (the distance between masses of weight M is equal l). Here $k = \omega / c$, where $c = T / \mu$ - in the case of a string (membrane) or $c = ES / \mu$ - in the case of longitudinal waves in a rod, μ - linear density, T - tension in string, E - Jung modulus, S - cross section of the rod. The dependence on time is $e^{-i\omega t}$ and is omitted. In this model each cell of periodicity consists one mass.

Further we shall use the dimensionless parameters $k := kl$, $m = M / \mu l$, dimensionless variable $x := x / l$ and dimensionless functions $u(x) := u(x) / l$ and $\delta(x) := l\delta(x)$. The equation in terms of dimensionless parameters, variables and functions, having kept former designations is

$$u''(x) + k^2 \left(1 + m \sum_{j=-\infty}^{+\infty} \delta(x - j) \right) u(x) = 0 \qquad (2)$$

The Floquet solution of equation (1) is built by the contact method. The following contact conditions are used in the points x_j where the concentrated masses are located:

The continuity of displacements

$$u(x_j + 0) = u(x_j - 0) \equiv u(x_j)$$

The condition of the jump of the forces

$$u'(x_j + 0) - u'(x_j - 0) = -\frac{k^2}{\mu} M u(x_j)$$

Floquet condition

$$u(x+l)=u(x)e^{i\alpha},\ \forall x\in(-\infty,+\infty),$$

where $e^{i\alpha}$ is the Floquet factor between the cells of periodicity and $\alpha=\alpha(k)$ is determined from equality

$$\cos\alpha=\cos k-\frac{m}{2}k\sin k$$

On the interval $x\in[0,1]$ the solution has the view

$$u(x)=A\left(e^{ikx}-f(k)e^{-ik(x-1)}\right) \tag{3}$$

The general solution in arbitrary point $x\in[-\infty,\infty]$ has the form

$$u(x)=Ae^{ik[x]\alpha}\left(e^{ik\{x\}}-f(k)e^{ik}e^{-ik\{x\}}\right)=Ae^{ik[x]\alpha}e^{ik/2}\left(e^{ik(\{x\}-1/2)}-f(k)e^{-ik(\{x\}-1/2)}\right) \tag{4}$$

where $[x]$ is the floor part of x, $\{x\}$ is the fraction part of x, A is the arbitrary constant and

$$f(k)\equiv f(k,\alpha(k))=\sin\left((\alpha(k)-k)/2\right)/\sin\left((\alpha(k)+k)/2\right) \tag{5}$$

The energy flux (averaged on the period of oscillations) is

$$\Pi(k)=T\frac{\omega}{2}\mathrm{Im}(u'\bar{u})\quad\text{or}\quad\Pi(k)=ES\frac{\omega}{2}\mathrm{Im}(u'\bar{u})$$

in the case of the string (membrane) or the rod correspondingly. The energy flux of the wave (4) normalized on the energy flux of corresponding wave Ae^{ikx} in the homogeneous system has the view

$$\Pi(k)=1-|f(k)|^2 \tag{6}$$

The elementary analysis of the energy flux shows:

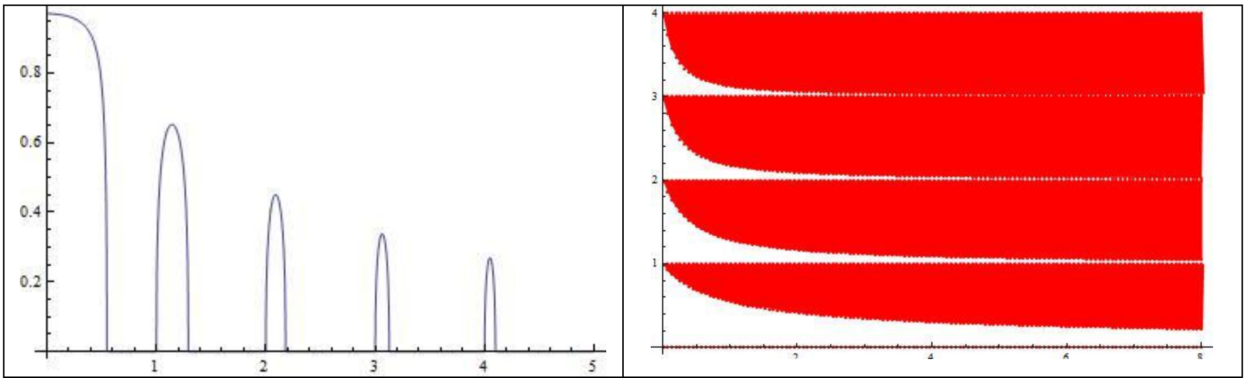
- $\Pi(k)<1$, if $m\neq 0$.

- Asymptotic: $\Pi(k) = \frac{4}{mk}\left(1 + O\left(\frac{1}{k}\right)\right)$, if $k \to \infty$.
- Width of pass bands: $\frac{4}{mk}\left(1 + O\left(\frac{1}{k}\right)\right)$, if $k \to \infty$
- $\exists \lim\limits_{k \to 0} \Pi(k, m) \equiv \Pi_0(m) = 1 - \left|\frac{\sqrt{1+m} - 1}{\sqrt{1+m} + 1}\right|^2$

The typical view of the energy flux dependence on the wave number k (for $m = 1.0$) is shown on the Fig. 1*a* (the numerical results are fulfilled for $k := k/\pi$). The condition of existence of propagating waves in the system will have the form

$$|T(k)| \le 2$$

where $T(k) = mk\sin k - 2\cos k$. The areas on the plain (m, k), where $|T(k)| > 2$ are marked by red color on the Fig. 1*b*. These points are corresponded to nonhomogeneous waves (in the stop bands). The areas outside (where $|T(k)| < 2$) are corresponded homogeneous waves (in the pass bands). The borders between them ($|T(k)| = 2$) are corresponded the special wave regime with zero energy flux.



a) Energy flux, as function of k/π | *b)* k/π as function m (Pass and stop bands)

Figure 1: Infinite periodic structure

3 FINITE PERIODIC STRUCTURE

The finite segment ($x \in [a, b]$), consisting of several (N) cells of periodicity of the infinite periodic system is considered. In the common situation of different masses the equation (2) can be rewritten in the form

$$u''(x) + k^2\left(1 + \sum_{j=0}^{N} m_j \delta(x - x_j)\right)u(x) = 0$$

Let us consider the case of one mass for simplicity (the case of greater number of weights is considered similarly)

$$-u''(x) - \lambda\rho(x)u(x) = 0 \quad \text{or} \quad Lu(x) = \lambda\rho(x)u(x)$$

where $\rho(x) \equiv 1 + m\delta(x)$, $\lambda = k^2$, $L \equiv -\dfrac{d^2}{dx^2}$.

Let's notice that if $\rho(x)$ is continuous positive function then it is a classical case of Sturm-Liouville problem with weight $\rho(x)$. The generalization of this problem on investigated case is based on the following statements.

Operator $L \equiv -\dfrac{d^2}{dx^2}$ is symmetric concerning scalar product in $L_2(a,b)$

$$(Lu,v) = \int_a^b -u''v\,dx = \left(-u'v + uv'\right)\Big|_a^b + (u,Lv) \tag{2a}$$

since extraintegrated members are equal to zero owing to our boundary conditions. For the existence of integral in (2a) it is enough to demand the limitation of energy norm

$$\|u\|^2 = \int_a^b \left(|u|^2 + |u'|^2\right)dx, \quad \text{i.e.} \quad u \in W_2^1(a,b) \equiv H^1(a,b)$$

By the way in our case expression

$$(\rho(x)u,v) = \int_a^b u(x)\overline{v(x)}dx + mu(0)\overline{v(0)}, \quad m \ge 0$$

sets scalar product with weight $\rho(x)$: $(u,v)_\rho \equiv (\rho(x)u,v)$. By using this scalar product the following facts can be obtained on the traditional way.

The spectrum λ_n, $n = 1,2,...$ is discrete and simple. $\lambda_n \ge 0$. Eigenfunctions u_n, v_m are orthogonal via scalar product with weight $\rho(x)$, i.e. $(u_n, v_m)_\rho = 0$, if $n \ne m$.

Now the case of odd and even number of point masses will be considered separately.

3.1 The case of odd number of masses N=2n+1

For odd number of point masses the equation (2) on interval $x \in [a,b]$ is transformed to

$$u''(x) + k^2\left(1 + m\sum_{j=-n}^{n}\delta(x-j)\right)u(x) = 0 \tag{7a}$$

Where $a = -n - 1 + s$, $b = n + s$, $0 < s < 1$.

For the particular case of one mass (n=0, N=1) the following equations on eigenfrequencies can be obtained according to the boundary conditions (BC):

If $u(a) = 0 = u(b)$ (rigid fixing) then $r_1 \equiv r_1(k,s) \equiv mk \sin k(1-s) \sin ks - \sin k = 0$ (7b)

If $u'(a) = 0 = u'(b)$ (free fixing) then $f_1 \equiv f_1(k,s) \equiv mk \cos k(1-s) \cos ks + \sin k = 0$ (7c)

If $u(a) = 0 = u'(b)$ (combined BC) then $g_1 \equiv g_1(k,s) \equiv mk \sin k(1-s) \cos ks - \cos k = 0$ (7d)

In the symmetric case ($s = 1/2$) these equations are simplified:

$$r_1(k,s)\big|_{s=1/2} \equiv \sin\frac{k}{2}\left(\frac{m}{2}k\sin\frac{k}{2} - \cos\frac{k}{2}\right) = 0 \qquad (8)$$

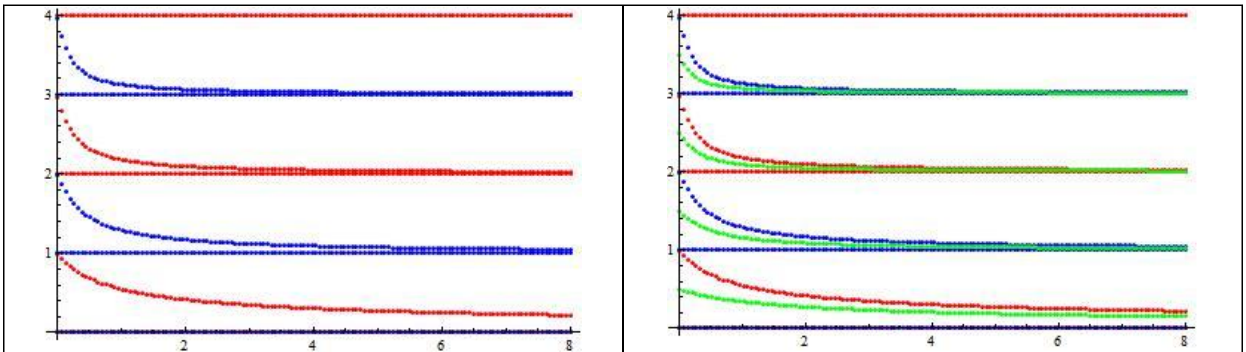
$$f_1(k,s)\big|_{s=1/2} \equiv \cos\frac{k}{2}\left(\frac{m}{2}k\cos\frac{k}{2} + \sin\frac{k}{2}\right) = 0 \qquad (9)$$

$$g_1(k,s)\big|_{s=1/2} \equiv \cos^2\frac{k}{2} - \sin^2\frac{k}{2} - mk\sin\frac{k}{2}\cos\frac{k}{2} = 0 \Leftrightarrow T(k) = 0 \qquad (10)$$

On the Fig. 2*a,b* is shown the corresponding dependencies of eigenfrequencies on parameter m for fixing (8) and free (9) boundary conditions (red and blue lines accordingly). Green curve on the Fig. 2*b* corresponds combined (10) boundary condition. After multiplication of equations (8) and (9) the following equation can be obtained

$$r_1(k,s)\big|_{s=1/2}\, f_1(k,s)\big|_{s=1/2} \equiv \left(T(k)^2 - 4\right)/4 = 0 \Leftrightarrow |T(k)| = 2 \qquad (11)$$

The curves determined by this equation coincide with the borders of pass and stop bands of consequent infinite periodic system (Fig. 1b).



a) Roots of equations (8) and (9) | *b)* Roots of equations (8), (9) and (10)

Figure 2: Eigenfrequencies as functions m

In the case of $n > 1$ the equations on eigenfrequencies have the form

$$r_1(k,s)\prod_{j=1}^{n}\left(T(k)^2 - C_j^2\right) = 0 \quad \text{or} \quad f_1(k,s)\prod_{j=1}^{n}\left(T(k)^2 - C_j^2\right) = 0 \qquad (12)$$

for rigid and free fixing accordingly.

In the symmetric case ($s = 1/2$) the equation for combined fixing has the form

$$T(k)\prod_{j=1}^{n}\left(T(k)^2 - B_j^2\right) = 0 \qquad (13)$$

And multiplication of equations of free and rigid fixing (12) leads to equation

$$\left(T(k)^2 - 4\right)\prod_{j=1}^{n}\left(T(k)^2 - C_j^2\right)^2 = 0 \qquad (14)$$

The multiplication of equations of combined fixing on itself leads to equation

$$T(k)^2\prod_{j=1}^{n}\left(T(k)^2 - B_j^2\right)^2 = 0 \qquad (15)$$

Here $C_j, B_j > 0$ are the constants depending on N. The first values of them are represented in the tables

	C_1	C_2	C_3
$N=3$	1		
$N=4$	$\sqrt{2}$		
$N=5$	$\left(\sqrt{5}-1\right)/2$	$\left(\sqrt{5}+1\right)/2$	
$N=6$	1	$\sqrt{3}$	
$N=7$	0.445042	1.24698	1.80194
$N=8$	$\sqrt{2-\sqrt{2}}$	$\sqrt{2}$	$\sqrt{2+\sqrt{2}}$

Table 1: The values of constants C_j.

	B_1	B_2	B_3
$N=2$	$\sqrt{2}$		
$N=3$	$\sqrt{3}$		

$N=4$	$\sqrt{2-\sqrt{2}}$	$\sqrt{2+\sqrt{2}}$	
$N=5$	$\sqrt{\left(5-\sqrt{5}\right)/2}$	$\sqrt{\left(5+\sqrt{5}\right)/2}$	
$N=6$	$\sqrt{2}$	$\sqrt{2-\sqrt{3}}$	$\sqrt{2+\sqrt{3}}$

Table 2: The values of constants B_j.

3.2 The case of even number of masses N=2n

For even number of point masses the equation on interval $x \in [a,b]$ is transformed to

$$u''(x)+k^2\left(1+m\sum_{j=-n}^{n-1}\delta\left(x-\frac{2j+1}{2}\right)\right)u(x)=0 \tag{16a}$$

where $a=-\frac{2n+1}{2}+s$, $b=\frac{2n-1}{2}+s$, $0<s<1$.

For the particular case of two masses ($n=1$, $N=2$) the following equations on eigenfrequencies can be obtained according to BC:

If $u(a)=0=u(b)$ (rigid fixing) then $r_2 \equiv r_2(k,s)=0$, where

$$r_2 \equiv 2\sin 2k + km(2\cos 2k - \cos(2k(s-1)) - \cos 2ks + 2mk\sin(k(1-s))\sin ks \sin k) \tag{16b}$$

If $u'(a)=0=u'(b)$ (free fixing) then $f_2 \equiv f_2(k,s)=0$, where

$$f_2 \equiv 2\sin 2k + km\left(2\cos 2k + \cos\left(2k(s-1)\right) + \cos 2ks - mk(\cos k + \cos\left(k(1-2s)\right))\right) \tag{16c}$$

If $u(a)=0=u'(b)$ (combined conditions) then $g_2 \equiv g_2(k,s)=0$, where

$$g_2 \equiv \left(k^2m^2-4\right)\cos 2k + km\left(km\left(\cos\left(2k(s-1)\right)-\cos 2ks\right)+4\sin 2k+4\cos k\sin\left(k(1-2s)\right)-km\right)$$

In the symmetric case $s=1/2$ these equations on eigenfrequencies are simplified

$$r_2(k,s)\big|_{s=1/2} \equiv \sin\frac{k}{2}\left(\frac{m}{2}k\sin\frac{k}{2}-\cos\frac{k}{2}\right)T(k)=0 \tag{17}$$

$$f_2(k,s)\big|_{s=1/2} \equiv \cos\frac{k}{2}\left(\frac{m}{2}k\cos\frac{k}{2}+\sin\frac{k}{2}\right)T(k)=0 \tag{18}$$

$$g_2(k,s)\big|_{s=1/2} \equiv 2-\left(2\cos k - km\sin k\right)^2 = 2 - T^2(k) = 0 \tag{19}$$

The multiplication of equations (17) and (18) gives the equation

$$\left(T(k)^2 - 4\right)\ T(k)^2 = 0 \tag{20}$$

The multiplication of equation (19) on itself gives the equation

$$g_2(k,s)\big|_{s=1/2}\, g_2(k,s)\big|_{s=1/2} \equiv \left(T^2(k)-2\right)^2 = 0 \tag{21}$$

In the case of $n>1$ the equations on eigenfrequencies have the form

$$r_2(k,s)\prod_{j=1}^{n-1}\left(T(k)^2 - C_j^2\right) = 0 \quad \text{or} \quad f_2(k,s)\prod_{j=1}^{n-1}\left(T(k)^2 - C_j^2\right) = 0 \tag{22}$$

for rigid and free fixing accordingly. In the symmetric case ($s=1/2$) the equation for combined fixing has the form

$$\prod_{j=1}^{n}\left(T(k)^2 - B_j^2\right) = 0 \tag{23}$$

and the multiplication of equations of free and rigid fixing leads to equation

$$\left(T(k)^2 - 4\right)T(k)^2\prod_{j=1}^{n-1}\left(T(k)^2 - C_j^2\right)^2 = 0 \tag{24}$$

The multiplication of equations of combined fixing on itself leads to equation

$$\prod_{j=1}^{n}\left(T(k)^2 - B_j^2\right)^2 = 0 \tag{25}$$

3.3 The summary for equations on eigenvalues

Further in the table $\Phi_N \equiv \prod_{j=1}^{n-1}\left(T(k)^2 - C_j^2\right)$, $\Omega_N \equiv \prod_{j=1}^{n-1}\left(T(k)^2 - B_j^2\right)$, $\Psi_N \equiv \prod_{j=1}^{n}\left(T(k)^2 - B_j^2\right)$

<table>
<tr><td></td><td></td><td colspan="3">$N = 2n-1$</td><td colspan="3">$N = 2n$</td></tr>
<tr><td rowspan="3">$n=1$</td><td>$s \neq \frac{1}{2}$</td><td>$r_1 = 0$</td><td>$f_1 = 0$</td><td>$g_1 = 0$</td><td>$r_2 = 0$</td><td>$f_2 = 0$</td><td>$g_2 = 0$</td></tr>
<tr><td rowspan="2">$s = \frac{1}{2}$</td><td>$r_1\left(\frac{1}{2}\right) = 0$</td><td>$f_1\left(\frac{1}{2}\right) = 0$</td><td>$T = 0$</td><td>$r_2\left(\frac{1}{2}\right) = 0$</td><td>$f_2\left(\frac{1}{2}\right) = 0$</td><td>$T^2 - 2 = 0$</td></tr>
<tr><td colspan="2">$T^2 - 4 = 0$</td><td>$T^2 = 0$</td><td colspan="2">$\left(T^2 - 4\right) T^2 = 0$</td><td>$\left(T^2 - 2\right)^2 = 0$</td></tr>
<tr><td rowspan="3">$n \geq 2$</td><td>$s \neq \frac{1}{2}$</td><td>$r_1\Phi_N = 0$</td><td>$f_1\Phi_N = 0$</td><td>$G_N = 0$</td><td>$r_2\Phi_N = 0$</td><td>$f_2\Phi_N = 0$</td><td>$G_N = 0$</td></tr>
<tr><td rowspan="2">$s = \frac{1}{2}$</td><td>$r_1\Phi_N = 0$</td><td>$f_1\Phi_N = 0$</td><td>$T\Omega_N = 0$</td><td>$r_2\Phi_N = 0$</td><td>$f_2\Phi_N = 0$</td><td>$\Psi_N = 0$</td></tr>
<tr><td colspan="2">$\left(T^2 - 4\right)\Phi_N^2 = 0$</td><td>$T^2\Omega_N^2 = 0$</td><td colspan="2">$\left(T^2 - 4\right)T^2\Phi_N^2 = 0$</td><td>$\Psi_N^2 = 0$</td></tr>
</table>

Table 3: The view of equations on eigenvalues for different N and s.

4 THE PROOF OF THE STATEMENT

The proof that all eigenfrequencies of a finite periodic structure fall into pass-bands of corresponding infinite system in the symmetrical case ($s = 1/2$) is based on two statements:

- The structure of the equations on eigenfrequencies has the above-stated view for any arbitrary finite number of masses.
- All constants C_j, B_j in equations on eigenfrequencies satisfy the inequality $C_j, B_j \leq 2$.

The first statement can be proved by mathematical induction method. In order to prove the second statement let's consider one of the factors $T(k)^2 - C^2$ from product $\prod_j$. After selecting $\operatorname{tg} k$ the following expression can be obtained

$$T(k)^2 - C^2 = \cos^2 k\left(\left(m^2k^2 - C^2\right)\operatorname{tg}^2 k - 4mk\operatorname{tg} k + 4 - C^2\right) \tag{26}$$

The multiplier with $\operatorname{tg} k$ can be factorized as a square polynomial

$$R(k,m) \equiv \left(m^2k^2 - C^2\right)\operatorname{tg}^2 k - 4mk\operatorname{tg} k + 4 - C^2 = \left(m^2k^2 - C^2\right)\left(\operatorname{tg} k - A_+(k)\right)\left(\operatorname{tg} k - A_-(k)\right) \tag{27}$$

where

$$A_\pm(k) = \frac{2mk \pm C\sqrt{m^2k^2 + 4 - C^2}}{\left(m^2k^2 - C^2\right)} \tag{28}$$

If $4-C^2 \geq 0$ in (28), the simple analyses shows that equation $R(k,m)=0$ has the real roots for every value of m (from geometrical method of solving of the equation at least).
If $4-C^2<0$, then the expression $m^2k^2+4-C^2$ under the radical can be made negative by choosing m. The corresponding real root transforms into the complex one and we obtain the violation with Sturm-Liouville problem. So all constants $C_j, B_j \leq 2$ and thereby all eigenfrequencies of a finite periodic structure fall into pass-bands of corresponding infinite system in the symmetrical case ($s=1/2$).

If $s \neq 1/2$ then some eigenfrequencies may not follow this rule. It will be shown in the next item.

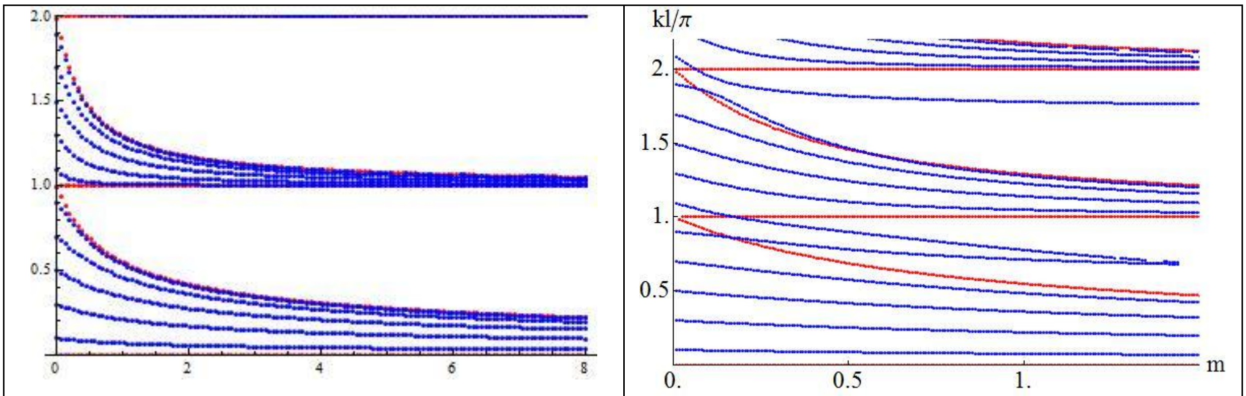
5 THE INFLUENCE OF UNSYMMETRY

In this section the influence of parameter s on the processes is numerically analyzed. The figures are obtained for the case of five point masses and combined BC (the fixed left and free right border). On these figures the curves corresponding dimensionless eigenfrequencies k/π of the finite system as functions of mass m are designated by blue color, and the curves corresponding the boundary of stop bands of the infinite system are designated by red color.

In the case of symmetrical conditions ($s=0.5$, Fig. 3*a*) all the curves corresponding eigenfrequencies do not overstep the bounds of stop bands, according to the proved statement.

In the unsymmetrical case $s=0.27$ as on the Fig. 3b, two curves, in each interval $(n,n+1), n=0,1$ on axes k penetrate into the corresponding stop bands. The modes, corresponding these points of curves when they lie in the stop bands, unlike the corresponding modes for the symmetrical case look like nonhomogeneous waves, i.e. oscillations modulated by decreasing or increasing exponent.

The numerical experiments showed the localization of the mode of finite periodic structure near the boundaries when its eigenfrequency falls into the stop band of corresponding infinite structure. The both direction of decreasing (increasing) are realized for different modes. This "decaying" mode (standing wave) can be interpreted as the combination of non homogeneous waves from the stop band of corresponding infinite periodic structure. This reasoning is analogous to the case of the standing wave as the combination of propagating waves in a finite homogeneous rod.



a) $s=0.5$ | *b)* $s=0.87$

Figure 3: Eigenfrequencies as functions of m for different s. Combined BC

In the case of fixed and free BC the eigenfrequency curves locating in the stop bands are fully determined by the equations (7b), (7c) and (16b), (16c) correspondingly (according to

(12,13) and (22,23)). It can be interpreted as the eigenfrequencies related with the border cells (masses). The calculations show that if the eigenfrequency curve lies in the stop band it lies there wholly in this case.

If combined BC is considered then not only border cells take part in forming of these curves. New opportunities are realized for this situation. For example the curve can cross the border of the stop band several times and interact with another curve in the stop band (as on the Fig. 3b).

6 CONCLUSIONS

These results can be generalised on the case of differential equations of higher order (bending displacements of the ribbed plates, shells e.t.c.). They can be used for design of finite periodical structures with eigenfrequencies located only in the pass bands of corresponding infinite periodical structure. To achieve this goal, the symmetry of cells and special symmetry at the boundaries are requested.

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