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GRAPHICAL MODAL ANALYSIS AND EARTHQUAKE STATICS OF LINEAR ONE-WAY ASYMMETRIC SINGLE-STORY STRUCTURE

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Abstract. *Graphical Static methods were developed in 1800s by European engineers and were a fundamental practice for generations of structural designers. Today they remain generally a valuable tool for representing and visualizing problems of structural mechanics that are governed by linear equations and rigid body kinematics, i.e. beam section analysis, mechanics of rigid bodies, etc. In this work we explore an application of graphical methods to the static earthquake response and vibrations of a single story elastic structure with a rigid-body diaphragm and a one-way asymmetry. We show that, based on involutory relations with respect to the floor ellipse of elasticity it is possible to graphically determine the lateral deflection under bi-directional earthquake loads, and the vibrational mode shapes. The method proves effective to gain insight into parameters that govern the earthquake response, and may be further explored to study more complex and general structures.*

1 INTRODUCTION

The use of graphics to solve static problems is found from some sketches made by Leonardo Da Vinci to some more systematic applications by Varignon and Poncelet [1]. A robust graphical static theory was first introduced in the 1800s by the work of Culmann [2], and further developed by other European engineers. Graphical methods became then an important tool for design of structures. Among them, the theory of the ellipse of elasticity was further developed by Ritter [2] and was used to solve elastic problems of beam and frame analysis, [4, 5]. The theory applies to any structural system that is elastic and whose kinematics can be described by rigid body deformations. This assumption is normally made for the cross section of a beam or for a building with rigid floor diaphragms. In some works of the literature of the last century [6] the ellipse of elasticity of a foundation block on an elastic soil was used to derive the stiffness matrix for vibrational analysis. The theory of the ellipse of elasticity can also be applied to the multi-directional and torsionally coupled response of three-dimensional frame structures, such as buildings under horizontal earthquake loads.

As it was observed after earthquake events the torsionally coupled behavior of irregular and plan-asymmetric structures can result in unbalanced demand on structural components and non-optimal or poor earthquake performance. Several authors have studied the behavior of torsionally coupled one-way asymmetric linear systems [7], and proposed methods for estimating the multidirectional nonlinear response or gave recommendations on how to approach the nonlinear response of one-way and two-ways eccentric torsionally coupled structures [8,9,10].

In this paper we highlight some graphical properties of the static earthquake response of an elastic single-story structure with a one-way plan asymmetry, that are useful to gain insight into the torsionally coupled behavior and into some key parameters that govern the seismic response. We introduce a graphical dynamic application of the theory of the ellipse of elasticity for dynamic modal analysis. We show that the static earthquake response can be handled through antipolarity relations based on the shear-type floor ellipse of elasticity. An additional antipolarity with respect to the circle of mass gyration is used for the modal analysis. We then introduce a simple graphical method to compute the modes of vibration, and use the properties of the ellipse of elasticity to obtain the modal frequencies.

2 GRAPHICAL EARTHQUAKE STATICS

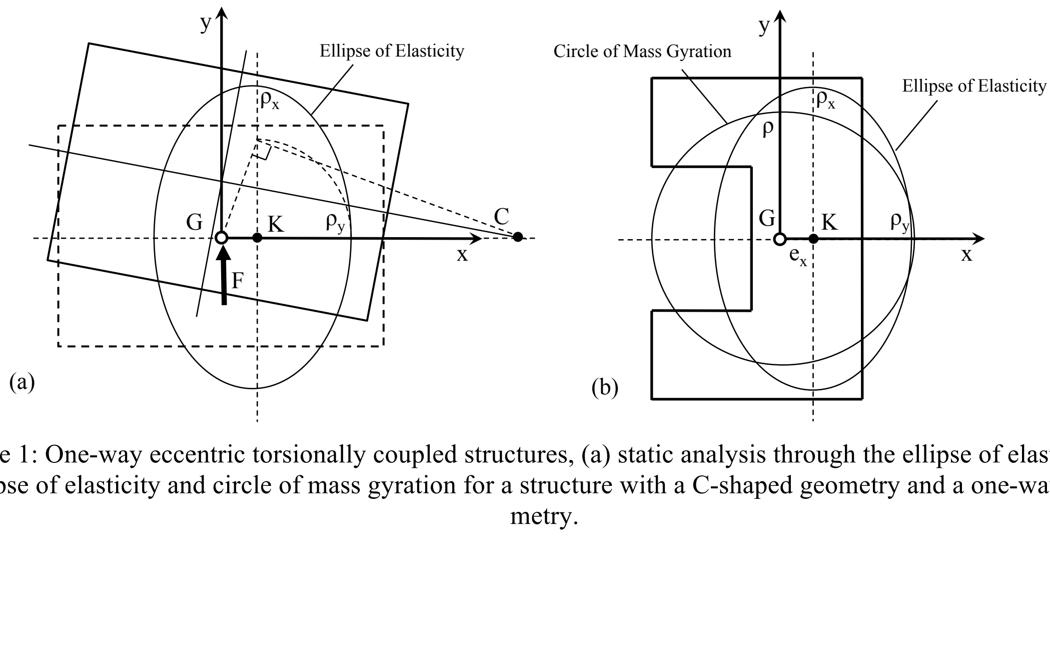

Figure 1: One-way eccentric torsionally coupled structures, (a) static analysis through the ellipse of elasticity. (b) Ellipse of elasticity and circle of mass gyration for a structure with a C-shaped geometry and a one-way asymmetry.

We consider herein a general single-story elastic structure with a one-way asymmetry. Both the center of mass G and the shear-type center of stiffness K are located on a principal axis. Therefore one of the three modes of vibrations has the direction of the eccentricity, i.e. along the principal axis that connects the floor center of mass with the center of shear-type stiffness. Without loss of generality, we will consider this principal axis as the X-axis, and restrict our analysis to a 2DOFs rigid diaphragm structure loaded in the Y-direction.

We select as DOFs the displacement in the Y-direction and rotation of the center of mass, and then we express them as a function of the rotation angle θ and of the position of the center of rotation x_c. This leads to decoupling the shape (represented by the position) from the amplitude, represented by the rotation angle. Also the force vector is expressed in term of the intensity of the external force F and of the distance x_F of the force from the center of stiffness. The static equations become

$$\mathbf{F}_e = \mathbf{K}\mathbf{u} = \begin{bmatrix} k_y & e_x k_y \\ e_x k_y & e_x^2 k_y + k_\theta \end{bmatrix} \begin{Bmatrix} -x_c \\ 1 \end{Bmatrix} \theta = \begin{Bmatrix} 1 \\ x_F \end{Bmatrix} F \tag{1}$$

Where k_y is the stiffness in the Y direction, k_θ is the rotational stiffness about the center of stiffness K, e_x is the eccentricity. A graphical static method of analysis can be used based on the ellipse of elasticity that is defined by its center K, and by the two principal semi-diameters

$$\begin{aligned} \rho_x &= \sqrt{k_\theta / k_x} \\ \rho_y &= \sqrt{k_\theta / k_y} \end{aligned} \tag{2}$$

The circle of mass gyration, which will be used later in the graphical dynamic analysis, can be defined through its radius ρ using the floor mass m and the polar moment of inertia I_p

$$\rho = \sqrt{I_p / m} \tag{3}$$

Figure 1a illustrates the graphical static analysis. The floor deformation for effect of a force F applied in the center of mass G and acting in the Y direction is identified by the position of the center of rotation C, and by the angle of rotation θ. According to the theory of the ellipse of elasticity the center of rotation is the antipole of the line of action of the force F with respect to the ellipse of elasticity. In this case the center C is the conjugated antipole of the vertical line passing through the center of mass G, and can be found graphically applying the involutory relation following the steps: 1) we rotate the X semi-diameter ρ_y of the ellipse by 90 degrees, 2) we draw a line from G to the end point of the ρ_y segment along the vertical line that passes through G, 3) we then draw from this point an orthogonal line whose intersection with the X-axis is the center of rotation. The points G and C are in mutual correspondence therefore the procedure can be inverted and used to determine the forces associated to a given deformation.

Figure 2 depicts the general case of a rotational displacement field and the relevant graphical determination of the elastic forces in the structure based on the properties of the ellipse of elasticity.

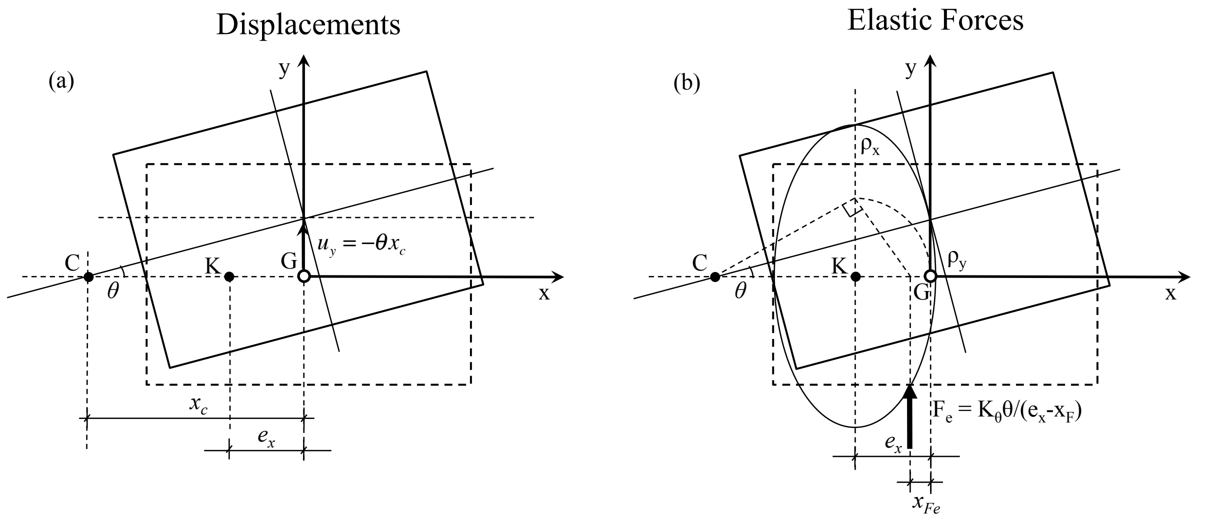

Figure 2: DOFs of one-way symmetric structure loaded with seismic forces orthogonal to a principal axis. (a) Rotational rigid-body displacements, (b) corresponding elastic forces F_e based on the antipolarity with respect to the ellipse of elasticity.

By using the equations (1) we can find the analytical confirmation of the antipolarity that allows to expresses the center of rotation as the antipole of the line of action of the force with respect to the ellipse of elasticity:

$$-(x_c - e_x) = \frac{\rho_y^2}{(x_{Fe} - e_x)} \tag{4}$$

Equation (4) represents an involutory relation from which x_c can be obtained, once x_F is known and vice-versa. The same static equations give the value of the floor rotation as a function of the intensity of the force, in terms of the rotational stiffness k_θ about the center of stiffness K.

$$\theta = \frac{F}{k_\theta}(x_{Fe} - e_x) \tag{5}$$

It can be noted that the equation (5) is one of the properties of the ellipse of elasticity, and k_θ represents the inverse of the so-called "elastic weight" encountered in the theory of the ellipse of elasticity.

3 GRAPHICAL DYNAMICS

Under dynamic undamped free vibration conditions the systems of the elastic forces $\mathbf{F}_e$ and of the inertial forces $\mathbf{F}_m$ must be in equilibrium. The equations of the free vibration of the structure are described by the previously defined stiffness matrix and by the mass matrix:

$$\mathbf{F}_m + \mathbf{F}_e = \mathbf{M\ddot{u}} + \mathbf{Ku} = \mathbf{0} \tag{6}$$

The total response vector can be expressed in terms of modal contributions as a summation of the products of each mode shape function times the relevant time-dependent function:

$$\mathbf{u}(t) = \sum \Phi_n(x)\theta_n(t) = \sum \begin{Bmatrix} -x_{c,n} \\ 1 \end{Bmatrix} \theta_n(t) \tag{7}$$

The system of equations can then be written expressing both the modal rotational accelerations and displacements in terms of the positions of the modal centers of rotation and of the rotational displacement and acceleration angles:

$$m\begin{bmatrix} 1 & 0 \\ 0 & \rho^2 \end{bmatrix}\sum\begin{Bmatrix} -x_{c,n} \\ 1 \end{Bmatrix}\ddot{\theta}_n(t)+\begin{bmatrix} k_y & ek_y \\ ek_y & e^2k_y+k_\theta \end{bmatrix}\sum\begin{Bmatrix} -x_{c,n} \\ 1 \end{Bmatrix}\theta_n(t)=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8}$$

The use of rotational kinematic degrees of freedom allows to decouple the shape from the rotational amplitude, therefore: 1) the mode shapes can be univocally identified through the positions $x_{c,n}$ of their centers of rotation, and their position remains fixed throughout the free vibration response; 2) the time-dependent functions are the time functions $\theta_n(t)$ of the modal angles of rotation.

Figure 3a shows a generic modal displacement and acceleration field. The latter is represented by a rigid body rotational acceleration about the center of rotation C_n, and is used to define the inertial forces that can be obtained through the terms of the mass matrix, as shown in Figure 3b. It is important to note that the moment of the inertial forces about the center of mass is equivalent to that generated by a force equal to $m\rho d^2\theta_n/dt^2$ applied at a distance from the center of mass G equal to the radius of gyration ρ.

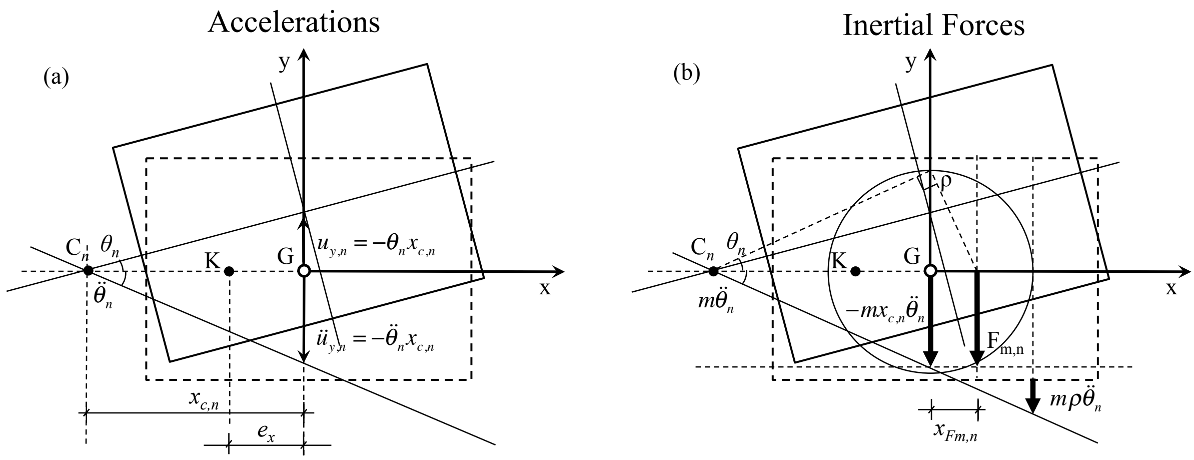

Figure 3: Definition of the floor accelerations based on rotational kinematics for a single modal component of the response (a), corresponding inertial forces F_m (b) expressed in terms of antipolarity with respect to the circle of mass gyration.

This is equivalent to saying that the line of action of the modal inertial force is the antipolar line of the center of rotation C_n with respect to the circle of mass gyration, as we can find from the inertial contribution of the equations (8) that yields the involutory relation:

$$-x_{c,n} = \frac{\rho^2}{x_{F_m,n}} \tag{9}$$

Therefore we have found that both the elastic forces and the inertial forces can be determined graphically using the involutory properties of the ellipse of elasticity and of the circle of mass gyration respectively. This allows the introduction of a graphical modal analysis method. Considering that the vectors of the modal inertial forces must be in equilibrium with the elastic forces, and since the modal centers of rotation are fixed, we can say that each mode's elastic and inertial forces must act on the same line. If we write both equations (4) and (7) for the modal centers of rotation we can find the following important property: the modal centers must be mutually conjugated in both the antipolarity with respect to the ellipse of elasticity and in the antipolarity with respect to the circle of mass gyration. This implies that also the two directions of the modal and inertial forces are mutually conjugated in both the antipolarites. In other terms, in the graphical representation, the property of orthogonality of the mode shapes with respect to both the mass matrix and the stiffness matrix, can be translated into the property of mutual graphical/involutory conjugation of the centers and of the lines of action of the forces with respect to the circle of mass gyration and to the ellipse of elasticity respectively.

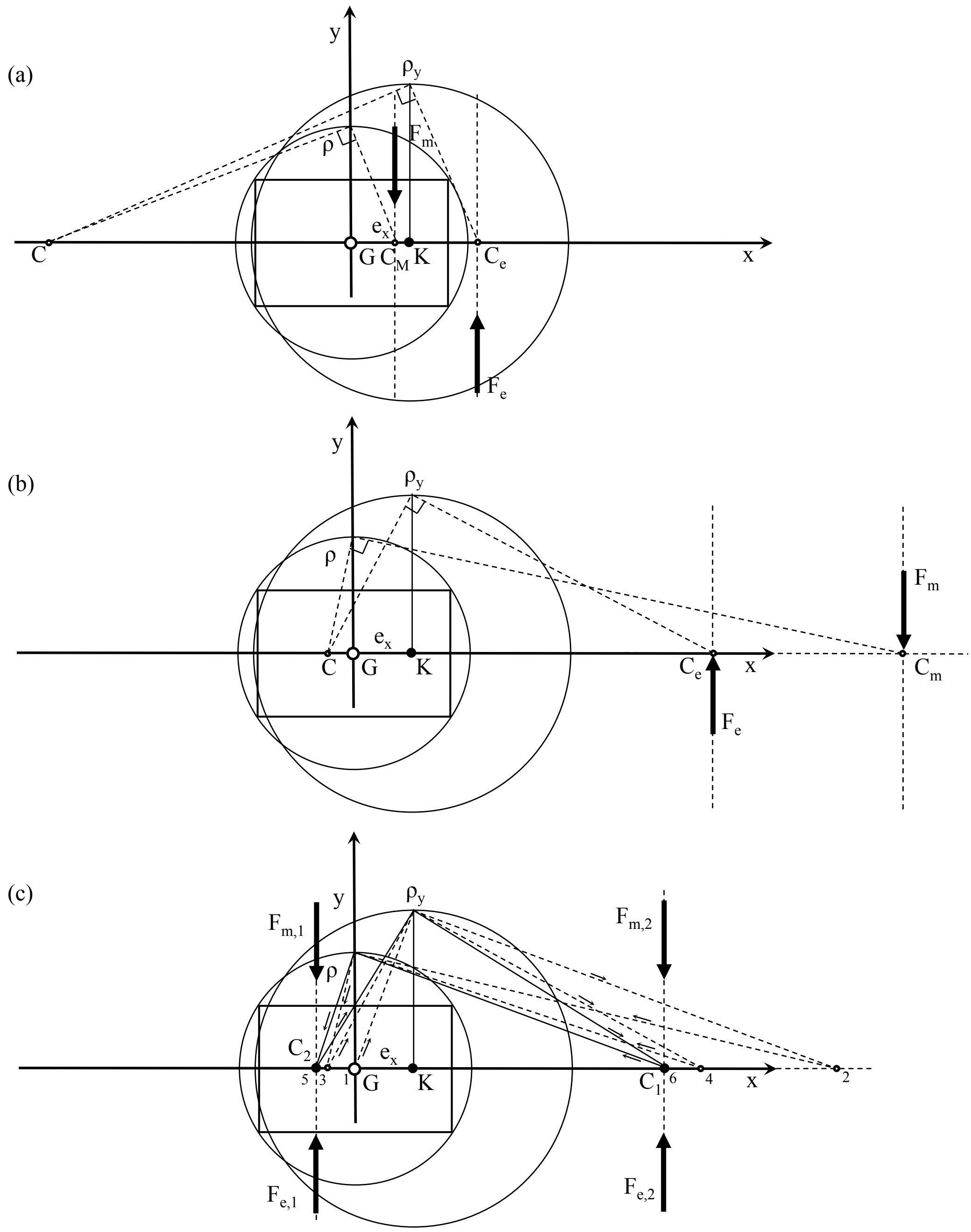

Figure 5: For a general deformed shape of center C there is an offset (a) between the inertial force and the elastic force; (b) the forces reverse as the center of rotation C approaches the center of mass; the offset is zero in correspondence of the modal centers of rotation. (c) Recursive graphical determination of C_1 and C_2 starting from the center of mass.

This leads to a convenient method to search and locate the modes of vibration through their centers of rotation, allowing a graphical procedure for modal analysis. Normally the mode shapes can be determined by matrix analysis finding the eigenvectors of the system of dynamic free vibration equations. In other cases they can be determined by direct methods choosing a starting displacement vector and recursively generating force vectors, such as in the Ritz method.

Here we present a graphical recursive approach for finding the mode shapes. In Figure 5 the ellipse of elasticity is for ease of description omitted and replaced by the circle of radius ρ_y.

This is done without losing generality since the X-axis is a principal axis and both the vibrations of the reference points K and G and the forces are in the Y direction. Therefore the centers of rotation are contained in the X-axis. Figure 5a shows that if a trial point that is external to both the ellipse of elasticity and the circle of mass gyration is chosen as center of deflection/acceleration, in general there is an offset between the corresponding inertial forces and elastic forces, as shown by the graphical forces determination. Therefore the corresponding forces cannot be in equilibrium, and the elastic forces fall beyond the corresponding inertial forces with respect to the center of rotation.

Figure 5b shows that if a trial deformation is chosen with center of rotation internal to both the circle and the ellipse and close enough to the center of mass G, there is again an offset between the inertial and elastic forces. However in this case the outer-most force is the inertial force. This means that as the trial center of rotation moves on the X axis approaching the center of mass, there is a position corresponding to which the offset is zero and the two forces act on the same line. Such middle zero-offset or reversal point is one of the two modal centers of rotation. Figure 5c shows the graphical procedure for finding the modal centers of rotation C_1 and C_2. The procedure consists of recursively generating elastic and inertial force vectors by applying the antipolarity with respect to the mass and to the stiffness, until convergence is achieved. Any trial point can be assumed as starting center of rotation. In this case we start from a rotational deflected shape centered in the center of mass G, labeled as 1. The conjugate point of 1 with respect to the stiffness is point 2, whose conjugate point with respect to the mass does is not point 1 but point 3. Then the conjugate of 3 with respect to the stiffness is found in point 4, and so on. At the step corresponding to point 6 we find that the procedure falls very close to a convergence loop represented by the double mutual antipolarity (conjugated points and directions) of points C_1 and C_2 that are the modal centers of rotation sought. The line of action of the Mode 2 inertial and elastic forces is the vertical line passing through C_1, and vice-versa the line of action of the Mode 1 inertial and elastic forces is the vertical line passing through C_2.

Finally the mode shapes are represented in Figure 6. It can be noted that from the analytical standpoint the procedure described above is equivalent to applying alternatively equations (4) and (9), therefore the convergence points C_1 and C_2 can be also found analytically, i.e. finding the values of x_c that satisfy both equations. This yields the two solutions

$$x_{c1,2} = \frac{(\rho^2 - d^2) \pm \sqrt{(\rho^2 - d^2)^2 + 4e_x^2\rho^2}}{2e_x} \tag{10}$$

Where d is the segment that connects the center of mass G with the end of the semi-axis ρ_x oriented along a vertical line

$$d = \sqrt{e_x^2 + \rho_y^2} \tag{11}$$

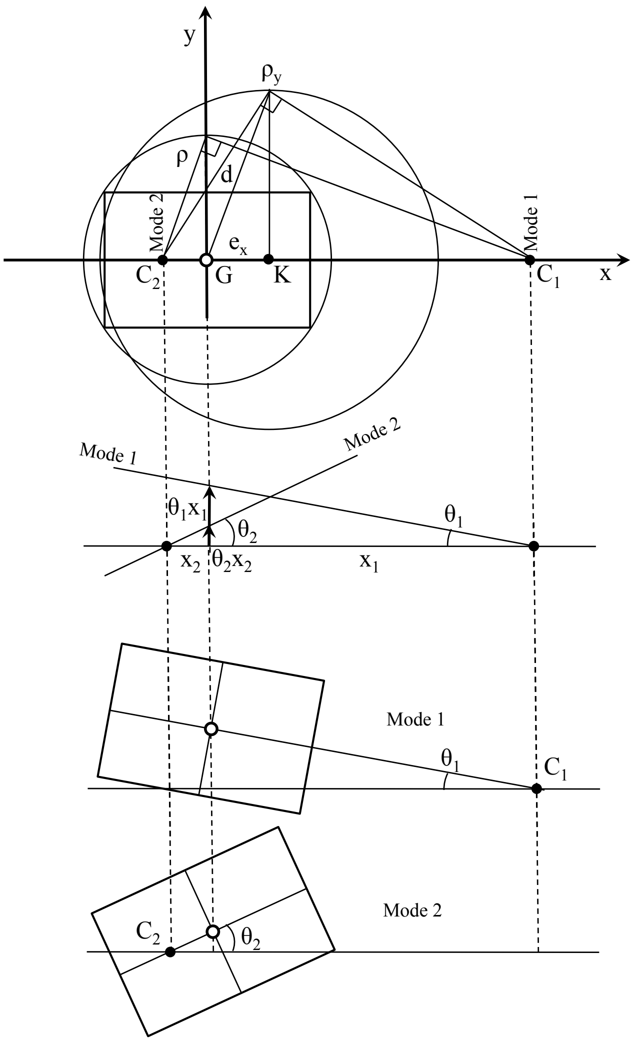

Figure 6: Torsionally coupled mode shapes. The centers C_1 and C_2 are double-conjugated in both the antipolarity with respect to the ellipse of elasticity and the antipolarity with respect to the circle of mass gyration.

Once the positions x_{c1} and x_{c2} are known, due to the equilibrium we can impose for each mode the equal intensity of the modal inertial forces and of the modal elastic forces, and compute the rotation amplitudes according to the properties of the ellipse of elasticity. This yields the values of the two frequencies of vibration (12) as a function of x_{c1} and x_{c2}, of the semi-diameters ρ and ρ_y, and of the other structural properties.

$$\begin{aligned} \omega_1^2 &= \frac{k_y}{m} \frac{\rho_y^2}{(\rho^2 + e_x x_{c1})} \\ \omega_2^2 &= \frac{k_y}{m} \frac{\rho_y^2}{(\rho^2 + e_x x_{c2})} \end{aligned} \tag{12}$$

We can also note that the mode shapes and frequencies can be normally found solving the corresponding eigenproblem. However the use of rotational kinematic DOFs for modal displacements and the inherent decoupling of the shape (centers) from the amplitude (rotation) leads to a simplification of the equations so that they can easily be solved by substitution, and gives the chance to introduce the involutory relations on which the graphical procedure described here is based.

4 CONCLUSIONS

We presented the application of graphical static methods to the dynamics of one-way symmetric torsionally coupled elastic single-story structures, provided that the positions of the center of mass and of the center of stiffness are known, and given the ellipse of elasticity and the floor mass radius of gyration. In alternative to static analysis thorough the stiffness matrix, and in analogy with the method used in sectional analysis for determination of the neutral axis, we introduce rigid body rotational kinematics to solve the statics of the structure under horizontal loads using the involutory properties of the antipolarity with respect to the ellipse of elasticity. After introducing the dynamic equations of free vibrations, we find that the antipolarity with respect to the circle of radius equal to the floor gyrator of the mass can be used to graphically determine also the inertial forces. Therefore both the dynamic inertial forces and the static elastic forces can be determined based on the ellipse and on the circle. These properties are used to introduce a graphical method for direct determination of the mode shapes through a recursive generation of the lines of action of the elastic forces and of the inertial forces. The modal frequencies are then determined using the other properties of the ellipse of elasticity imposing the equilibrium of the modal inertial forces and elastic forces. Instead of regular eigenvector analysis, the graphical procedure described here relies on a number of graphical static analyses to obtain the modal frequencies, the mode shapes and the intensity and lines of action of the modal forces. The method is effective to gain insight into properties and parameters that govern the dynamic earthquake response, and its application to more complex structures can be object of further investigation.

REFERENCES

[1] E. Benvenuto, La Scienza delle Costruzioni e il suo sviluppo storico, Edizioni di Storia e Letteratura, Roma, 2006. (In Italian).

[2] K. Culmann, Die Graphische Statik, 1865. (In german).

[3] W. Ritter, Anwendungen der Graphischen Statik, **3**, (Appendix), Zurich, 1865. (In German).

[4] O. Belluzzi, Scienza delle Costruzioni, Bologna: Zanichelli, 1942. (In Italian).

[5] V. Franciosi, Fondamenti di Scienza delle Costruzioni, vol. **III** Teoria delle Strutture, Napoli, Liguori ed., 1988. (In Italian).

[6] A. Galli, Su di una applicazione dinamica dell'ellisse di elasticità. *Ricerche di Ingegneria*, **2**, 106-109, 1934. (In Italian).

[7] T.L. Trombetti, J.P. Conte, New insight into and simplified approach to seismic analysis of torsionally coupled one-story, elastic systems, *Journal of Sound and Vibration*, **286**, 265-312, 2005.

[8] A.K. Chopra, R.K. Goel. A modal pushover analysis procedure to estimate seismic demands for un-symmetric-plan buildings*, Earthquake Engineering and Structural Dynamics*, **33**, 903-927, 2004.

[9] J.L. Lin, K.C. Tsai, Seismic analysis of two-way asymmetric building systems under bidirectional seismic ground motions, *Earthquake Engineering and Structural Dynamics*, **37**, 305-328, 2008.

[10] J.C. Reyes, E. Kalkan. Relevance of Normal-Fault and Maximum Components of Ground-Motions on Nonlinear Behavior of Symmetric and Asymmetric Buildings in Near Fault Sites. *15WCEE, 15th World Conference in Earthquake Engineering.* Lisbon, Portugal, 2012.