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MODE SHAPE ASSEMBLY FOR AMBIENT MODAL ANALYSIS USING A TWO-STAGE BAYESIAN SPECTRAL DENSITY APPROACH

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Abstract. *A two-stage Bayesian spectral density approach was formulated for ambient modal analysis recently. The interaction between spectrum variables (e.g., frequency, damping ratio as well as the magnitude of modal excitation and prediction error) and spatial variables (e.g., mode shape components) can be decoupled so that they can be identified separately. The proposed method can be implemented in the environment of a wireless sensor network through a distributed computing strategy so that local mode shapes as well as their uncertainties confined to different clusters can be identified. However, the difficulty on how to assemble these local mode shapes estimated from multiple clusters is still a problem required to be resolved properly. In this study, a Bayesian mode shape assembly methodology is proposed so that the weight for different clusters is accounted for properly according to their data quality. The optimal values for the global mode shape components corresponding to all measured dofs can be obtained by a fast iterative scheme, while the associated uncertainties can be derived analytically. There is no need to share the same set of reference dofs for all clusters for scaling purpose when using ambient vibration data. The proposed mode shape assembly method is investigated with a shear building model. Results show that the overall mode shape can be effectively identified by the proposed method.*

1 INTRODUCTION

Ambient modal analysis, which aims to identify structural modal properties by using output-only measured response without the knowledge of the input, has aroused increasing attention in industrial applications due to its efficiency and economic implementation. It has also been one of the most important issues in wireless sensor network (WSN) based structural health monitoring (SHM) systems. Recent interest has arisen to calculate the uncertainties of identified modal parameters by using Bayesian approaches [1]. In the context of ambient modal analysis, a number of Bayesian approaches have been proposed [2, 3, 4]. These methods provide rigorous means for obtaining optimal modal properties as well as their uncertainties. However, computational difficulty has severely hindered the wider application of these approaches in real-life engineering applications.

A breakthrough was made by Au [5, 6, 7] recently to address the computational challenges of conventional Bayesian FFT approach. Motivated by Au's work, a two-stage fast Bayesian spectral density approach has been proposed for ambient modal analysis when there are separated modes and closely spaced modes [8]. The proposed method is able to separate spectrum variables (e.g., frequency, damping ratio as well as the magnitude of modal excitation and prediction error) and spatial variables (e.g., mode shape components) so that these two kinds of variables can be identified independently in two stages. The proposed method can be implemented through a distributed computing strategy in the environment of wireless sensor networks. As a result, a group of local mode shapes identified from different sensor clusters sharing some reference sensors can be obtained. The local mode shapes identified from different clusters may have different senses and scaling factors since they are normalized individually [9]. The assembly of these local modes shapes to form a global mode shape is an important issue in modal analysis. A novel global least square method with an automated procedure for determining the global mode shape by minimizing a measure-of-fit function was proposed recently by Au [9]. However, the global least square approach assigns the same weights for all setups. There is still room for improvement since the quality of originally well-identified clusters may be corrupted by the quality of some more problematic setups [10]. Therefore, it is reasonable to try to incorporate the uncertainty information of different clusters into the mode shape assembly procedure. In particular, local mode shapes not well identified in particular clusters should be assigned less weight due to their relatively unreliable data quality. As a sequel to the global least square method [9], a Bayesian mode shape assembly approach is developed in this study so that the weight for different clusters can be accounted for properly according to the data quality.

2 REVISITING THE TWO-STAGE BAYESIAN APPROACH FOR AMBIENT MODAL ANALYSIS

Bayesian spectral density approach (BSDA) [4] proposed previously is novel since it can consider different kinds of uncertainties and provides a rigorous means for obtaining modal properties as well as their uncertainties which is useful for further risk assessment. However, there are some challenges related to its practical implementation: (1) BSDA requires solving a high-dimensional numerical optimization problem whose computational effort grows with the number of measured dofs and the number of modes to be identified. (2)When calculating the posterior covariance function, BSDA involves computing the inverse of a Hessian matrix, for which the computational effort and required memory space grow with the number of measured dofs and the number of modes. (3) BSDA involves repeated evaluations of the determinant and inverse of a rank deficient (singular) matrix. Therefore, the minimization problem may be ill-conditioned. (4) The dofs of interest will be always measured and processed sepa-

rately in different setups or clusters, resulting in different tour size among different setups. Therefore, BSDA encounters the difficulty on how to combine the parameters and their uncertainties identified from different setups.

To address the problems aforementioned, a two-stage Bayesian spectral density approach has been developed [8]. Following a “divide and conquer” strategy, the frequency band can be divided into multiple sub-bands and one can only focus on a specified sub-band each time. There are two possible cases over each selected frequency band: case of separated modes and case of closely spaced modes. For both cases, the interaction between spectrum variables (e.g., frequency, damping ratio as well as the magnitude of modal excitation and prediction error) and spatial variables (e.g., mode shape components) can be decoupled so that they can be conquered in two consecutive stages. In the first stage, the spectrum variables as well as their uncertainties can be identified through ‘fast Bayesian spectrum trace approach’ (FBSTA) by employing the statistical properties of the trace of the spectral density matrix. The information contained in all measured dofs from different setups can be collected together, thus avoiding the difficulty of posterior data fusion. Once the spectrum variables are extracted, the spatial variables as well as their uncertainties can be identified in a second stage by ‘fast Bayesian spectrum density approach’ (FBSDA) using the statistical information of the spectrum density matrix. In this stage, the matrix determinant lemma and matrix inverse lemma are employed to avoid the ill-conditioning of conventional BSDA. The proposed method can deal with the practical difficulties of conventional Bayesian spectrum density approach even for a large number of measured dofs. The case of separated modes and the case of closely spaced modes are discussed separately in [8].

3 HIERARCHICAL ARCHITECTURE OF WSN

As shown in Figure 1, the proposed two-stage Bayesian spectrum density approach for ambient modal analysis can be implemented through a distributed computing strategy, which is formed as a three-level hierarchical architecture. Wireless sensors can be divided into hierarchical communities, with each community composed of a cluster head node and several leaf nodes. All cluster head nodes can report to the manager node, while the manager node can report to the base station node directly connected to a PC. The ambient modal analysis can be implemented in two stages. In the first stage, the auto-spectrum density of all measured dofs is collected centrally in the manager node so that the spectrum variables and their associated uncertainties can be identified by FBSTA. The identified results will be transmitted to the cluster head nodes from the manager node in the second stage. Then the optimal local mode shapes and their covariance matrix for each community can be identified by FBSDA. Global mode shapes and their uncertainties can then be assembled with the aid of the overlapping nodes public for different clusters, as will be shown in next section.

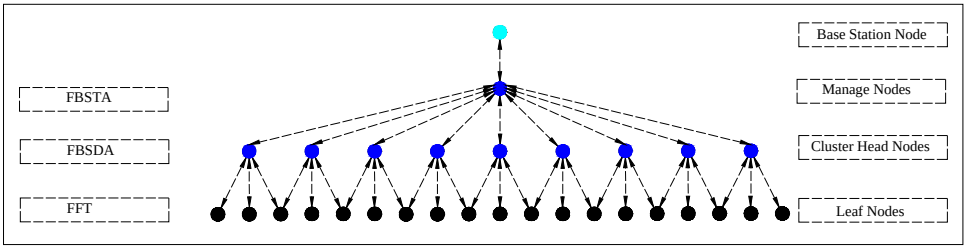

Figure 1: Three-level hierarchical architecture of WSN.

4 ASSEMBLING MODE SHAPE FROM MULTIPLE CLUSTERS

4.1 Negative Logarithm Likelihood Function

For the r-th given mode, let $\hat{\boldsymbol{\psi}}_{r,i} \in \mathbb{R}^{n_i}$ and $\mathbf{C}_{\boldsymbol{\psi}_{r,i}} \in \mathbb{R}^{n_i \times n_i}$ be the optimal values and covariance matrix of the mode shape confined to the measured dofs of the i-th cluster $(i = 1, 2, \cdots, n_t)$; n_i is the number of sensors measured in the i-th cluster and n_t is the total number of clusters included in the ambient vibration test. Assume that n_l is the total number of distinct measured dofs from all clusters, which should satisfy $n_l < \sum_{i=1}^{n_t} n_i$ since some dofs are shared by more than one cluster.

Let $\boldsymbol{\varphi}_r \in \mathbb{R}^{n_l}$ be the r-th given global mode shape covering all measured dofs, and $\boldsymbol{\varphi}_{r,i} \in \mathbb{R}^{n_i}$ be the components of $\boldsymbol{\varphi}_r$ corresponding to the measured dofs in the i-th cluster. The local mode shape $\boldsymbol{\varphi}_{r,i}$ can be mathematically related to the assembled global mode shape $\boldsymbol{\varphi}_r$ as [9]

$$\boldsymbol{\varphi}_{r,i} = \mathbf{L}_i \boldsymbol{\varphi}_r \tag{1}$$

where $\mathbf{L}_i \in \mathbb{R}^{n_i \times n_l}$ is a selection matrix, where $\mathbf{L}_i(j,k) = 1$ if the j-th sensor in the i-th cluster gives the k-th dof of the global mode shape and zero otherwise. The mode shape assembly problem under the Bayesian framework is to determine the global mode shape so that it can best fit the identified counterparts by assigning different weight for various clusters according to their data quality. It is worth noting that $\hat{\boldsymbol{\psi}}_{r,i}$ is normalized to unity when it is identified by FBSDA. Therefore, the measure-of-fit should be implemented based on the discrepancy between $\boldsymbol{\varphi}_{r,i} / \|\boldsymbol{\varphi}_{r,i}\|$ and $\hat{\boldsymbol{\psi}}_{r,i}$ both of unit norms. Since the identified local mode shape can be well-approximated by a Gaussian PDF with mean $\hat{\boldsymbol{\psi}}_{r,i}$ and covariance matrix $\mathbf{C}_{\boldsymbol{\psi}_{r,i}}$, the likelihood function of the i-th local mode shape is given by,

$$p(\hat{\boldsymbol{\psi}}_{r,i}, \mathbf{C}_{\boldsymbol{\psi}_{r,i}} | \boldsymbol{\varphi}_{r,i}) = \exp[-\frac{1}{2}(\boldsymbol{\varphi}_{r,i} / \|\boldsymbol{\varphi}_{r,i}\| - \hat{\boldsymbol{\psi}}_{r,i})^T (\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1})(\boldsymbol{\varphi}_{r,i} / \|\boldsymbol{\varphi}_{r,i}\| - \hat{\boldsymbol{\psi}}_{r,i})] \tag{2}$$

It is assumed that local mode shapes identified from different setups are statistically independent. Therefore, the likelihood function of the local mode shapes measured from all setups is given by,

$$p(\{\hat{\boldsymbol{\psi}}_{r,i}, \mathbf{C}_{\hat{\boldsymbol{\psi}}_{r,i}} : i = 1, \cdots, n_t\} | \boldsymbol{\varphi}_r) = \prod_{i=1}^{n_t} p(\hat{\boldsymbol{\psi}}_{r,i}, \mathbf{C}_{\boldsymbol{\psi}_{r,i}} | \boldsymbol{\varphi}_{r,i}) \tag{3}$$

Under the Bayesian framework, the updated probabilities of the global mode shape given the measured local mode shapes $\wp = \{\hat{\boldsymbol{\psi}}_{r,i}, \mathbf{C}_{\hat{\boldsymbol{\psi}}_{r,i}} : i = 1, \cdots, n_t\}$ should be

$$p(\boldsymbol{\varphi}_r | \wp) = c_0 p(\boldsymbol{\varphi}_r) p(\wp | \boldsymbol{\varphi}_r) \tag{4}$$

In the case where a non-informative prior is used, the posterior PDF of $\boldsymbol{\varphi}_r$ is proportional to the likelihood function $p(\boldsymbol{\varphi}_r | \wp)$, which can be written in terms of the 'negative logarithm likelihood function' (NLLF), where the index 'as' stands for 'assembly',

$$L_{as} = \frac{1}{2} \sum_{i=1}^{n_t} (\boldsymbol{\varphi}_{r,i} / \|\boldsymbol{\varphi}_{r,i}\| - \hat{\boldsymbol{\psi}}_{r,i})^T \mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1} (\boldsymbol{\varphi}_{r,i} / \|\boldsymbol{\varphi}_{r,i}\| - \hat{\boldsymbol{\psi}}_{r,i}) \tag{5}$$

Writing (5) explicitly in terms of the global mode shape $\boldsymbol{\varphi}_r$ gives

$$L_{as} = \frac{1}{2}\sum_{i=1}^{n_t}(\mathbf{L}_i\boldsymbol{\varphi}_r / \|\mathbf{L}_i\boldsymbol{\varphi}_r\| - \hat{\boldsymbol{\psi}}_{r,i})^T(\mathbf{C}_{\boldsymbol{\psi}_{r,i}})^{-1}(\mathbf{L}_i\boldsymbol{\varphi}_r / \|\mathbf{L}_i\boldsymbol{\varphi}_r\| - \hat{\boldsymbol{\psi}}_{r,i}) \tag{6}$$

To enforce the norm constraint of $\boldsymbol{\varphi}_r$, i.e. $\|\boldsymbol{\varphi}_r\|^2 = 1$, (6) can be rearranged by using the Lagrange multipliers [9],

$$L_{as} = \frac{1}{2}\sum_{i=1}^{n_t}(\mathbf{L}_i\boldsymbol{\varphi}_r / \|\mathbf{L}_i\boldsymbol{\varphi}_r\| - \hat{\boldsymbol{\psi}}_{r,i})^T(\mathbf{C}^{-1}_{\boldsymbol{\psi}_{r,i}})(\mathbf{L}_i\boldsymbol{\varphi}_r / \|\mathbf{L}_i\boldsymbol{\varphi}_r\| - \hat{\boldsymbol{\psi}}_{r,i}) + \gamma_r(1-\boldsymbol{\varphi}_r^T\boldsymbol{\varphi}_r) \tag{7}$$

where γ_r is Lagrange multiplier that enforces $\|\boldsymbol{\varphi}_r\|^2 = 1$. As seen, (7) is not a quadratic function of $\boldsymbol{\varphi}_r$, thus the optimal value of $\boldsymbol{\varphi}_r$ cannot be determined analytically. To avoid the above difficulty, the auxiliary variables $\chi_{r,1},\cdots,\chi_{r,n_t}$ similar to [9] are introduced

$$\chi_{r,i}^2 = 1/\|\mathbf{L}_i\boldsymbol{\varphi}_r\|^2 \tag{8}$$

As a result, the objective function can be re-formulated by using the Lagrange multipliers approach:

$$L_{as} = \sum_{i=1}^{n_t}\frac{1}{2}(\chi_{r,i}\mathbf{L}_i\boldsymbol{\varphi}_r - \hat{\boldsymbol{\psi}}_{r,i})^T(\mathbf{C}^{-1}_{\boldsymbol{\psi}_{r,i}})(\chi_{r,i}\mathbf{L}_i\boldsymbol{\varphi}_r - \hat{\boldsymbol{\psi}}_{r,i}) + \gamma_r(1-\boldsymbol{\varphi}_r^T\boldsymbol{\varphi}_r) + \sum_{i=1}^{n_t}\beta_{r,i}(\chi_{r,i}^2\|\mathbf{L}_i\boldsymbol{\varphi}_r\|^2 - 1) \tag{9}$$

where $\beta_{r,i}$ are Lagrange multipliers that enforce equation (8).

4.2 Most Probable Values

The full set of parameters to be identified includes $\boldsymbol{\lambda}_{as} = \{\boldsymbol{\varphi}_r, \gamma_r, \beta_{r,i}, \chi_{r,i} : i = 1,2,\cdots,n_t\}$. Direct solution for the optimal global mode shapes from the objective function (9) is not trivial due to its high-dimensional as well as nonlinear features. In this study, an iterative solution strategy will be employed. The optimal values of $\chi_{r,i}$ and $\beta_{r,i}$ in terms of $\boldsymbol{\varphi}_r$ and γ_r are first derived analytically firstly, following which the optimal value of $\boldsymbol{\varphi}_r$ and γ_r given the remaining parameters are derived. A sequence of iterations comprised of the following linear optimization problems can be implemented:

(1) Optimization for $\beta_{r,i}$ and $\chi_{r,i}$

The gradient of $L_{ass}(\boldsymbol{\varphi}_r)$ with respect to $\chi_{r,i}$ is given by

$$\frac{\partial L_{as}}{\partial \chi_{r,i}} = \chi_{r,i}(\mathbf{L}_i\boldsymbol{\varphi}_r)^T(\mathbf{C}^{-1}_{\boldsymbol{\psi}_{r,i}})(\mathbf{L}_i\boldsymbol{\varphi}_r) - (\hat{\boldsymbol{\psi}}_{r,i})^T(\mathbf{C}^{-1}_{\boldsymbol{\psi}_{r,i}})(\mathbf{L}_i\boldsymbol{\varphi}_r) + 2\beta_{r,i}\chi_{r,i}\|\mathbf{L}_i\boldsymbol{\varphi}_r\|^2 \tag{10}$$

Setting $\frac{\partial L_{as}}{\partial \chi_{r,i}} = 0$ and solving for $\chi_{r,i}$ gives

$$\chi_{r,i} = \frac{(\hat{\boldsymbol{\psi}}_{r,i})^T(\mathbf{C}^{-1}_{\boldsymbol{\psi}_{r,i}})(\mathbf{L}_i\boldsymbol{\varphi}_r)}{(\mathbf{L}_i\boldsymbol{\varphi}_r)^T(\mathbf{C}^{-1}_{\boldsymbol{\psi}_{r,i}})(\mathbf{L}_i\boldsymbol{\varphi}_r) + 2\beta_{r,i}\|\mathbf{L}_i\boldsymbol{\varphi}_r\|^2} \tag{11}$$

Substituting (11) into (8) leads to two roots for $\beta_{r,i}$

$$\beta_{r,i} = -\frac{(\mathbf{L}_i\hat{\boldsymbol{\varphi}}_r)^T(\mathbf{C}^{-1}_{\boldsymbol{\psi}_{r,i}})(\mathbf{L}_i\boldsymbol{\varphi}_r)}{2\|\mathbf{L}_i\boldsymbol{\varphi}_r\|^2} \pm \frac{(\hat{\boldsymbol{\psi}}_{r,i})^T(\mathbf{C}^{-1}_{\boldsymbol{\psi}_{r,i}})(\mathbf{L}_i\boldsymbol{\varphi}_r)}{2\|\mathbf{L}_i\boldsymbol{\varphi}_r\|} \tag{12}$$

It is worth noting that the Hessian of L_{as} with respect to $\chi_{r,i}$ is given by

$$\frac{\partial^2 L_{as}}{\partial \chi_{r,i}^2} = (\mathbf{L}_i \boldsymbol{\varphi}_r)^T (\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1})(\mathbf{L}_i \boldsymbol{\varphi}_r) + 2\beta_{r,i} \left\| \mathbf{L}_i \boldsymbol{\varphi}_r \right\|^2 \tag{13}$$

The minimum of L_{as} occurs only when $\frac{\partial^2 L_{as}}{\partial \chi_{r,i}^2} > 0$, which implies that

$$\beta_{r,i} = -\frac{(\mathbf{L}_i \boldsymbol{\varphi}_r)^T (\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1})(\mathbf{L}_i \boldsymbol{\varphi}_r)}{2\left\| \mathbf{L}_i \boldsymbol{\varphi}_r \right\|^2} + \left| \frac{(\hat{\boldsymbol{\psi}}_{r,i})^T (\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1})(\mathbf{L}_i \boldsymbol{\varphi}_r)}{2\left\| \mathbf{L}_i \boldsymbol{\varphi}_r \right\|} \right| \tag{14}$$

Substituting (14) into (11) leads to,

$$\chi_{r,i} = \frac{\hat{\boldsymbol{\psi}}_{r,i}^T \mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1} (\mathbf{L}_i \boldsymbol{\varphi}_r)}{\left| \hat{\boldsymbol{\psi}}_{r,i}^T \mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1} \mathbf{L}_i \boldsymbol{\varphi}_r \right| \left\| \mathbf{L}_i \boldsymbol{\varphi}_r \right\|} = \operatorname{sgn}\left(\hat{\boldsymbol{\psi}}_{r,i}^T \mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1} \mathbf{L}_i \boldsymbol{\varphi}_r \right) \left\| \mathbf{L}_i \boldsymbol{\varphi}_r \right\|^{-1} \tag{15}$$

Here $\operatorname{sgn}(\cdot)$ denotes the signum function.

(2) Optimization for $\boldsymbol{\varphi}_r$ and γ_r

The gradient of L_{as} with respect to $\boldsymbol{\varphi}_r$ is given by

$$\frac{\partial L_{as}}{\partial \boldsymbol{\varphi}_r} = \sum_{i=1}^{n_t} (\chi_{r,i} \mathbf{L}_i)^T (\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1})(\chi_{r,i} \mathbf{L}_i)\boldsymbol{\varphi}_r - \sum_{i=1}^{n_s} (\chi_{r,i} \mathbf{L}_i)^T (\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1}) \hat{\boldsymbol{\psi}}_{r,i} - 2\gamma_r \boldsymbol{\varphi}_r + \sum_{i=1}^{n_t} 2\beta_{r,i} \chi_{r,i}^2 (\mathbf{L}_i)^T (\mathbf{L}_i)\boldsymbol{\varphi}_r \tag{16}$$

Setting $\frac{\partial L_{as}}{\partial \boldsymbol{\varphi}_r} = 0$ and solving for $\boldsymbol{\varphi}_r$ gives

$$\mathbf{A}_r \boldsymbol{\varphi}_r + \mathbf{b}_r = \gamma_r \boldsymbol{\varphi}_r \tag{17}$$

where $\mathrm{A}_r = \frac{1}{2}\sum_{i=1}^{n_t} (\chi_{r,i} \mathbf{L}_i)^T (\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1})(\chi_{r,i} \mathbf{L}_i) + \sum_{i=1}^{n_t} \beta_{r,i} \chi_{r,i}^2 (\mathbf{L}_i^T)(\mathbf{L}_i)$ and $\mathbf{b}_r = -\frac{1}{2}\sum_{i=1}^{n_t} (\chi_{r,i} \mathbf{L}_i)^T (\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1}) \hat{\boldsymbol{\psi}}_{r,i}$. Equation (17) is subject to the constraint $\left\| \boldsymbol{\varphi}_r \right\|^2 = 1$, which forms a constrained eigenvalue problem different than the conventional eigenvalue problem. It can be solved by constructing an augmented vector that satisfies the standard eigenvalue equation [9]. As a result, all optimal parameters can be obtained in groups given the remaining ones until convergence is achieved instead of optimizing the full set of parameters simultaneously.

4.3 Posterior Uncertainties

The posterior uncertainty of the global mode shape can also be obtained by inversing the Hessian matrix of L_{as} with respect to $\boldsymbol{\lambda}_{as}$. The uncertainty of model parameters well approximated by a Gaussian distribution centered at the most probable parameter values and with covariance matrix equal to the inverse of the Hessian of the function $\boldsymbol{\Gamma}_{as}$ calculated at the optimal parameters $\hat{\boldsymbol{\lambda}}_{as}$. This Hessian matrix is given by

$$\boldsymbol{\Gamma}_{as} = \begin{pmatrix} \mathbf{L}^{(\varphi_r \varphi_r)} & \mathbf{L}^{(\varphi_r \chi_r)} & \mathbf{L}^{(\varphi_r \beta_r)} & \mathbf{L}^{(\varphi_r \gamma_r)} \\ \mathbf{L}^{(\chi_r \varphi_r)} & \mathbf{L}^{(\chi_r \chi_r)} & \mathbf{L}^{(\chi_r \beta_r)} & \mathbf{L}^{(\chi_r \gamma_r)} \\ \mathbf{L}^{(\beta_r \varphi_r)} & \mathbf{L}^{(\beta_r \chi_r)} & \mathbf{L}^{(\beta_r \beta_r)} & \mathbf{L}^{(\beta_r \gamma_r)} \\ \mathbf{L}^{(\gamma_r \varphi_r)} & \mathbf{L}^{(\gamma_r \chi_r)} & \mathbf{L}^{(\gamma_r \beta_r)} & \mathbf{L}^{(\gamma_r \gamma_r)} \end{pmatrix} \tag{18}$$

In (18), $\boldsymbol{\chi}=[\boldsymbol{\chi}_1^T,\cdots,\boldsymbol{\chi}_r^T,\cdots,\boldsymbol{\chi}_{n_m}^T]^T$ with the r-th block $\boldsymbol{\chi}_r^T=[\chi_{r,1},\cdots,\chi_{r,i},\cdots,\chi_{r,n_t}]^T$; $\boldsymbol{\beta}=[\boldsymbol{\beta}_1^T,\cdots,\boldsymbol{\beta}_r^T,\cdots,\boldsymbol{\beta}_{n_m}^T]^T$ with the r-th block $\boldsymbol{\beta}_r^T=[\beta_{r,1},\cdots,\beta_{r,i},\cdots,\beta_{r,n_t}]^T$. $\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\varphi}_r)}$ denotes the second order derivatives of L_{as} with respect to $\boldsymbol{\varphi}_r$. Similar explanation can be given to other blocked members in (18). $\boldsymbol{\Gamma}_{as}$ is a symmetrical matrix, and only the blocked members in the upper triangle $\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\varphi}_r)}$, $\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\chi}_r)}$, $\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\beta}_r)}$, $\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\gamma}_r)}$, $\mathbf{L}^{(\boldsymbol{\chi}_r\boldsymbol{\chi}_r)}$, $\mathbf{L}^{(\boldsymbol{\chi}_r\boldsymbol{\beta}_r)}$, $\mathbf{L}^{(\boldsymbol{\chi}_r\boldsymbol{\gamma}_r)}$, $\mathbf{L}^{(\boldsymbol{\beta}_r\boldsymbol{\beta}_r)}$, $\mathbf{L}^{(\boldsymbol{\beta}_r\boldsymbol{\gamma}_r)}$ and $\mathbf{L}^{(\boldsymbol{\gamma}_r\boldsymbol{\gamma}_r)}$ need to be computed analytically. Among these blocks, $\mathbf{L}^{(\boldsymbol{\chi}_r\boldsymbol{\gamma}_r)}\in\mathbb{R}^{n_t}$, $\mathbf{L}^{(\boldsymbol{\beta}_r\boldsymbol{\beta}_r)}\in\mathbb{R}^{n_t\times n_t}$, $\mathbf{L}^{(\boldsymbol{\beta}_r\boldsymbol{\gamma}_r)}\in\mathbb{R}^{n_t}$ and $\mathbf{L}^{(\boldsymbol{\gamma}_r\boldsymbol{\gamma}_r)}\in\mathbb{R}^{1}$ are all zero matrixes. The non-zero blocks can be derived analytically as follows:

(1) Derivatives of $\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\varphi}_r)}$

$\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\varphi}_r)}\in\mathbb{R}^{n_l\times n_l}$ can be obtained by taking the derivative of (9) with respect to $\boldsymbol{\varphi}_r$, which can be formulated as

$$\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\varphi}_r)}=\sum_{i=1}^{n_t}(\chi_{r,i}\mathbf{L}_i)^T(\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1})(\chi_{r,i}\mathbf{L}_i)+\sum_{i=1}^{n_t}2\beta_{r,i}\chi_{r,i}^2\mathbf{L}_i^T\mathbf{L}_i-2\gamma_r\mathbf{I}_{n_l} \tag{19}$$

(2) Derivatives of $\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\chi}_r)}$

$\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\chi}_r)}\in\mathbb{R}^{n_l\times n_t}$ is a matrix formulated as

$$\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\chi}_r)}=[\mathbf{L}^{(\boldsymbol{\varphi}_r\chi_{r,1})},\cdots,\mathbf{L}^{(\boldsymbol{\varphi}_r\chi_{r,i})},\cdots,\mathbf{L}^{(\boldsymbol{\varphi}_r\chi_{r,n_t})}]_{n_l\times n_t} \tag{20}$$

where $\mathbf{L}^{(\boldsymbol{\varphi}_r\chi_{r,i})}$ can be expressed as

$$\mathbf{L}^{(\boldsymbol{\varphi}_r\chi_{r,i})}=2\chi_{r,i}(\mathbf{L}_i^T)(\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1})(\mathbf{L}_i)\boldsymbol{\varphi}_r-(\mathbf{L}_i^T)(\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1})\hat{\boldsymbol{\psi}}_{r,i}+4\beta_{r,i}\chi_{r,i}(\mathbf{L}_i^T)(\mathbf{L}_i)\boldsymbol{\varphi}_r \tag{21}$$

(3) Derivatives of $\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\beta}_r)}$

$\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\beta}_r)}\in\mathbb{R}^{n_l\times n_t}$ is a matrix formulated as

$$\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\beta}_r)}=[\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\beta}_{r,1})},\cdots,\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\beta}_{r,i})},\cdots,\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\beta}_{r,n_t})}]_{n_l\times n_t} \tag{22}$$

where the i-th block $\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\beta}_{r,i})}\in\mathbb{R}^{n_l}$ can be expressed as:

$$\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\beta}_{r,i})}=2\chi_{r,i}^2\mathbf{L}_i^T\mathbf{L}_i\boldsymbol{\varphi}_r \tag{23}$$

(4) Derivatives of $\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\gamma}_r)}$

$\mathbf{L}^{(\boldsymbol{\varphi}_r\boldsymbol{\gamma}_r)}\in\mathbb{R}^{n_l}$ is a vector which can be expressed as

$$\mathbf{L}^{(\boldsymbol{\varphi}_r\gamma_{r,i})}=-2\boldsymbol{\varphi}_r \tag{24}$$

(5) Derivatives of $\mathbf{L}^{(\boldsymbol{\chi}_r\boldsymbol{\chi}_r)}$

$\mathbf{L}^{(\boldsymbol{\chi}_r\boldsymbol{\chi}_r)}\in\mathbb{R}^{n_t\times n_t}$ is a diagonal matrix shown as follows

$$\mathbf{L}^{(\boldsymbol{\chi}_r\boldsymbol{\chi}_r)}=diagonal(\mathbf{L}^{(\chi_{r,i}\chi_{r,i})}) \tag{25}$$

whose i-th diagonal entry can be derived analytically as follows,

$$\mathbf{L}^{(\chi_{r,i}\chi_{r,i})}=(\mathbf{L}_i\boldsymbol{\varphi}_r)^T(\mathbf{C}_{\boldsymbol{\psi}_{r,i}}^{-1})(\mathbf{L}_i\boldsymbol{\varphi}_r)+2\beta_{r,i}\left\|\mathbf{L}_i\boldsymbol{\varphi}_r\right\|^2 \tag{26}$$

(6) Derivatives of $\mathbf{L}^{(\boldsymbol{\chi}_r\boldsymbol{\beta}_r)}$

$\mathbf{L}^{(\chi_r,\boldsymbol{\beta}_r)} \in \mathbb{R}^{n_t \times n_t}$ is a diagonal matrix shown as follows

$$\mathbf{L}^{(\chi_r,\boldsymbol{\beta}_r)} = diagonal(\mathbf{L}^{(\chi_{r,i},\boldsymbol{\beta}_{r,i})}) \tag{27}$$

whose i-th diagonal entry can be derived analytically as follows,

$$\mathbf{L}^{(\chi_{r,i},\boldsymbol{\beta}_{r,i})} = 2\chi_{r,i}\left\|\mathbf{L}_i\boldsymbol{\varphi}_r\right\|^2 \tag{28}$$

5 NUMERICIAL STUDY

To illustrate the accuracy of the proposed approach, simulated data of a linear and time-invariant system are processed. The system is a 15-story shear building with a uniformly distributed mass and stiffness at each floor. The stiffness to mass ratio is chosen to be $2500s^{-2}$. Rayleigh damping is assumed here, and the damping ratios for the first two modes are set to be 1%. The structure is assumed to be excited with a ground acceleration $\ddot{x}_g$ which can be adequately modeled as Gaussian white noise with auto spectral intensity $0.25m^2s^{-3}$. The prediction error level is taken to be 10%. It is assumed that the 15 dofs are measured using sensors arranged in 3 clusters as shown in Table 1.

Setup	Measured dofs
1	1, 2, 3, 4, 5,6,7
2	6, 7, 8, 9, 10,11,12
3	11, 12, 13, 14, 15

Table 1: Setup information.

It is assumed that 20 sets of acceleration data of 500-seconds duration are available in each dof. Figure 2 shows the most probable global mode shapes of the first four modes of the structure by using the method introduced in section 4. The solid line denotes the exact mode shapes, whereas the squares represent the identified global mode shape assembled from local ones. These two mode shapes almost coincide. The MAC between the identified global mode shape and the exact one is calculated to be 0.999, 0.999, 0.995 and 0.985 for the first to fourth modes, respectively. The numericial example indicates that the Bayesian mode shape assembly approach yields satisfactory results. Moreover, there is no need to share the same reference dofs for all setups.

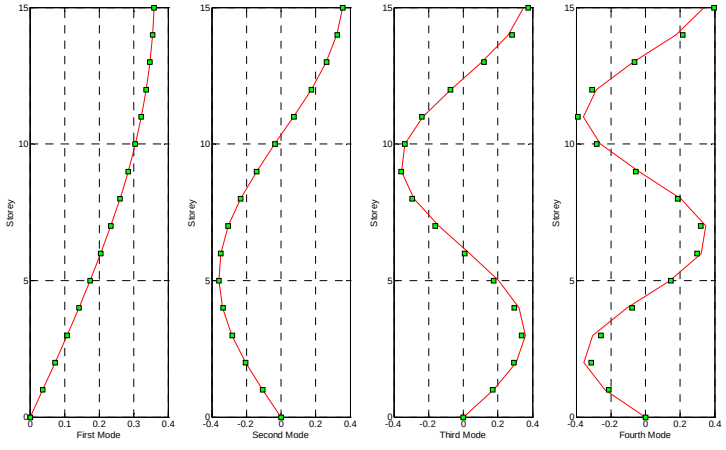

Figure 2: Assembled global mode shape from multiple setups.

6 CONCLUDING REMARKS

A two-stage Bayesian spectrum density approach proposed for ambient modal analysis can be implemented in the environment of wireless sensor networks through a distributed computing strategy. The local mode shapes as well as their uncertainties confined to different clusters can be identified. To assemble the local mode shapes, a Bayesian assembly methodology is proposed in this study so that the weights for different clusters can be accounted for properly according to their data qualities. The optimal values for the global mode shape confined to all measured dofs can be obtained by a fast iterative scheme. The associated uncertainties can also be obtained analytically. A shear building model subject to ground motion is employed to demonstrate and verify the proposed method. Results show that the global mode shapes can be effectively and accurately identified by the proposed method.

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