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NONLINEAR DYNAMICS OF A 2D ARRAY OF COUPLED PENDULUMS UNDER PARAMETRIC EXCITATION

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Abstract. *In this paper, we investigate the frequency response of a 2D array of coupled pendulums under parametric excitation. We developed a computational model while considering the main source of nonlinearities. The nonlinear equations of motion are derived and solved using the Harmonic Balance Method (HBM) coupled with the Asymptotic Numerical Method (ANM). Numerical simulation are performed in the case of 3X3 array to investigate the effect of changing the coupling coefficients in the 2D direction and adding an imperfection on the collective dynamics of the array.*

1 INTRODUCTION

The sine-Gordon model and its discrete analog are ubiquitous models in mathematical physics with a wide range of applications extending from chains of coupled pendulums [1] and Josephson junction arrays [2] to gravitational and high-energy physics models [3, 4]. For instance, Ikeda et al. investigated the behavior of intrinsic localized modes (ILMs) for an array of coupled pendulums subjected to horizontal [5] and verical [6] sinusoidal excitation. Thakur et al. [7] analyzed the collective dynamical behavior of an array of coupled pendulums with a small fraction of random long-range connection under parametric excitation. Moreover Alexeeva et al. [8] demonstrated the stabilizing effect of adding length impurities on a chain of pendulums parametrically excited. The collective dynamics has been studied for several one-dimensional discrete systems. However, to our knowledge, few analyses were devoted to bidimensional systems like Josephson junction oscillator arrays [9]. In this context, a computational model for the nonlinear dynamics of a 2D array of coupled pendulums under parametric excitation is proposed. The principal goal is to track the frequency responses of the considered system in terms of bifurcation topologies and energy transfer with respect to the excitation amplitude and to investigate the effects of adding an impurity to the system.

The coupled nonlinear equations of motion have been solved using the harmonic balance method coupled with the asymptotic numerical continuation technique [10]. Several numerical simulations have been performed for a particular set of design parameters and two configurations with respect to the coupling coefficients in both directions have been considered. It is shown that the number of branches, their stability and the bifurcation topology of the frequency response of each pendulum can be tuned with respect to an added impurity.

2 PROBLEM FORMULATION

The discreet system, depicted in Figures 1 and 2, is composed of n equidistant horizontal axle A_i. Along these axles, at equally spaced intervals, there are p similar pendulums. Each pendulum consists of a rigid rod, with a mass m attached at the end. At rest, all the pendulums point down the vertical. $\theta_{i,j}$ the angle between the (i,j) pendulum and the downward vertical, k_x and k_y are the linear torque constant respectively in x and y axis. By neglecting the mass of the rigid rods, all the pendulums have the same moment of inertia $I = ml^2$, where l is the length of the pendulum. The considered periodic structure is excited by vibrating the support that holds the system. The driven frequency of the parametric force is equal to 2ω and with amplitude A_e. The boundary conditions are $\theta_{0,j} = \theta_{i,0} = \theta_{n+1,j} = \theta_{i,n+1} = 0$. The potential and kinetic energy of the system can be written as:

$$V = \sum_{i,j} \tfrac{1}{2}k_x(\theta_{i,j} - \theta_{i-1,j})^2 + \tfrac{1}{2}k_x(\theta_{i,j} - \theta_{i+1,j})^2 + \tfrac{1}{2}k_y(\Delta_{i,j-1} - d)^2 + \tfrac{1}{2}k_y(\Delta_{i,j+1} - d)^2$$
$$mg\left(l - l\cos(\theta_{i,j}) - A_e\cos(2\omega t)\right) \tag{1}$$

$$T = \sum_{i,j} \frac{1}{2} m v_{i,j}^2 \tag{2}$$

Where $\Delta_{i,j+1}$ is defined as the distance between the mass attached to the pendulum (i,j) and (i,j+1) and can be written as:

$$\Delta_{i,j+1} = d + lsin(\theta_{i,j+1}) - lsin(\theta_{i,j}) \approx d + l\theta_{i,j+1} - l\theta_{i,j} \tag{3}$$

and $v_{i,j}$ is the velocity of the moving mass of (i,j) pendulum,

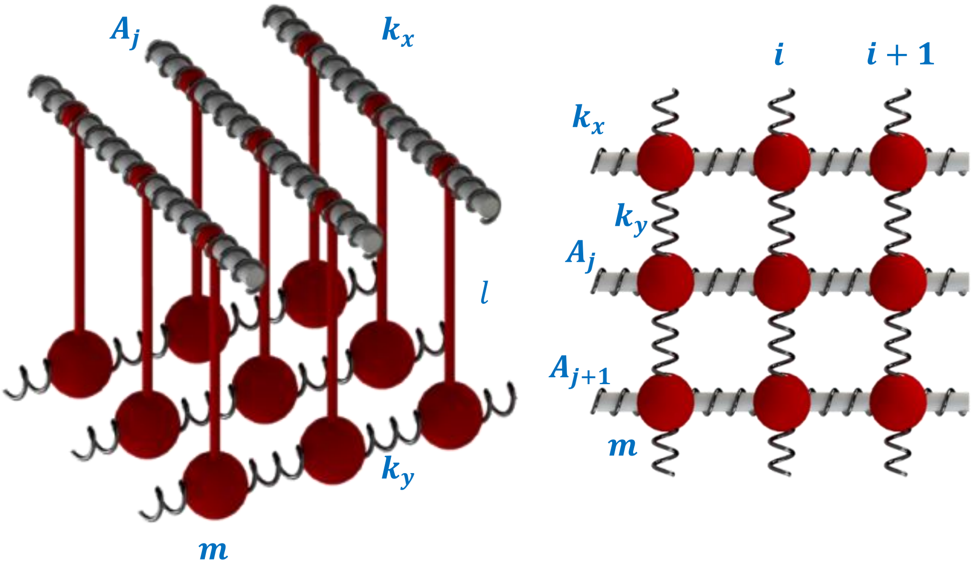

Figure 1: Schematic of the 2D array of coupled pendulums

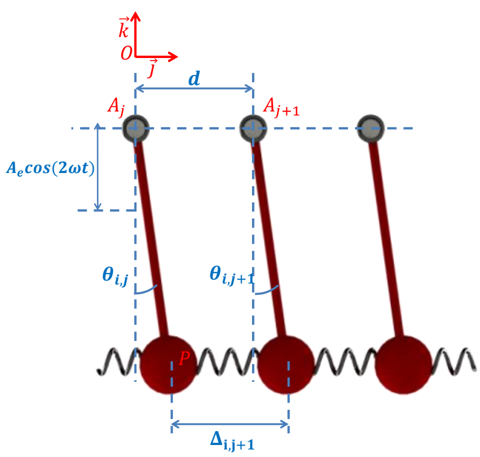

Figure 2: Front view of the system

$$\overrightarrow{v_{i,j}} = \overrightarrow{\dot{r}_{OA}} + \vec{\omega} \times \overrightarrow{r_{AP}} \tag{4}$$

where

$$\overrightarrow{\dot{r}_{OA}} = 2A_e\omega\sin(2\omega t)\vec{k} \qquad \vec{\omega} = \dot{\theta}_{i,j}\vec{i} \qquad \overrightarrow{r_{AP}} = l\left(\sin\left(\theta_{i,j}\right)\overrightarrow{j} - \cos\left(\theta_{i,j}\right)\overrightarrow{k}\right)$$

The Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_{i,j}}\right) - \frac{\partial L}{\partial \theta_{i,j}} = Q_{i,j} \tag{5}$$

where $L = T - V$ and $Q_{i,j} = \alpha l \dot{\theta}_{i,j}$ is the generalized forces applied to the (i, j) pendulum and α is the damping coefficient.

The resulting equation of motion of the (i, j) pendulum is obtained as:

$$\begin{aligned} &ml^2\ddot{\theta}_{i,j} + \alpha l\dot{\theta}_{i,j} + k_x\left(2\theta_{i,j} - \theta_{i+1,j} - \theta_{i-1,j}\right) + l^2k_y\left(2\theta_{i,j} - \theta_{i,j+1} - \theta_{i,j-1}\right) \\ &+ml\left[g + 4A_e\omega^2\cos\left(2\omega t\right)\right]\sin(\theta_{i,j}) = 0 \end{aligned} \tag{6}$$

By expanding $\sin(\theta_{i,j})$ in Taylor series up to the third order, the equation (6) can be written as:

$$\begin{aligned} &ml^2\ddot{\theta}_{i,j} + \alpha l\dot{\theta}_{i,j} + k_x\left(2\theta_{i,j} - \theta_{i+1,j} - \theta_{i-1,j}\right) + l^2k_y\left(2\theta_{i,j} - \theta_{i,j+1} - \theta_{i,j-1}\right) \\ &+ml\left[g + 4A_e\omega^2\cos\left(2\omega t\right)\right]\left(\theta_{i,j} - \tfrac{1}{6}\theta_{i,j}^3\right) = 0 \end{aligned} \tag{7}$$

Equation (7) describes a system of Duffing oscillators coupled by linear springs k_x and k_y in the x and y directions and subjected to parametric excitation ($4mlA_e\omega^2\cos(2\omega t)$). To determine the natural frequencies and their eigenvectors, Equation (7) can be written as:

$$M\ddot{\theta} + B\dot{\theta} + K\theta + G\left(\theta, t\right) = 0 \tag{8}$$

with

$$M = \begin{bmatrix} ml^2 & & 0 \\ & \ddots & \\ 0 & & ml^2 \end{bmatrix}; B = \begin{bmatrix} \alpha l & & 0 \\ & \ddots & \\ 0 & & \alpha l \end{bmatrix};$$

$$K = \begin{bmatrix} mgl + 2k_x + 2l_0^2k_y & -k_x & 0 & -l_0^2k_y & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -l_0^2k_y & 0 & -k_x & mgl + 2k_x + 2l_0^2k_y \end{bmatrix}$$

Where M denotes the mass matrix, B the damping matrix, K the stiffness matrix with dimension $N_{pen} \times N_{pen}$ and G contains the nonlinear terms with dimension N_{pen}, where N_{pen} is the number of pendulums in the array. The natural frequencies ω_ν and the eigenvectors θ_ν of the corresponding linear system can be computed by solving the following eigenvalue problem

$$\left(K - \omega_\nu^2 M\right)\theta_\nu = 0 \tag{9}$$

3 SOLVING PROCEDURE

The goal of this section is to determine numerically the periodic solutions of the 2D periodic system. Therefore, we use the harmonic balance method and the asymptotic numerical method (HBM and ANM). But first, we rewrite the nonlinearity of the system (7) in a quadratic form. This technique is not applicable for any equation of motion, because some nonlinearity cannot be transformed into quadratic terms [10]. Consequently, we transformed the equation (6) into equation (7) by developing the sin term to the third order Taylor series expansion. Let us consider an autonomous system of differential equations:

$$\dot{Y} = f(Y, \lambda, t) \tag{10}$$

where Y is a vector of unknowns, f is a smooth nonlinear vector valued function and λ is a real parameter. The dot represents the derivative with respect to time t. First, we transform the system (10) into a new system where the nonlinearities are quadratic, which can be written as follows:

$$m\left(\dot{Y}\right) = c(\lambda, t) + l(Y) + q(Y, Y) \tag{11}$$

where $c(\lambda, t)$ is a constant vector with respect to Y, $l(Y)$ is the linear vector and $q(Z, Z)$ is the quadratic vector. After writing the system in a quadratic form, we decompose the solution into Fourier series:

$$Y(t) = Y_0 + \sum_{k=1}^{H} Y_{c,k}\cos(k\omega t) + Y_{s,k}\sin(k\omega t) \tag{12}$$

By replacing this solution in Equation (10) and expanding $f(Y, \lambda)$ into Fourier series, we obtain an algebraic system with $2H + 1$ vector of unknowns Y_i the unknown pulsation ω, and the parameter λ. The components of the Fourier series are collected into column vector U that contains all the unknowns Y_i with a size $(2H + 1)N_eq$; $U = [Y_0^T, Y_{c,1}^T, Y_{s,1}^T, \ldots, Y_{c,H}^T, Y_{c,H}^T]$ where N_{eq} is the number of equation in (10). The system (12) is transformed into a quadratic one with $(2H + 1)N_{eq}$ and can be written as:

$$\omega M(U) = C + L(U) + Q(U, U) \tag{13}$$

where $M(.)$, C, $L(.)$ and $Q(.,.)$ are operators depending on $m(.)$, c, $l(.)$ and $q(.,.)$. Once the algebraic system is obtained, we use the asymptotic numerical method (ANM) to solve it. In order to apply the ANM, Equation (12) is reformed into:

$$R(U, \omega) = C + L(U) + Q(U, U) - \omega M(U) = 0 \tag{14}$$

4 RESULTS AND DISCUSSION

In this section, we consider a small array of 3×3 coupled oscillators where the extreme pendulums are coupled to the frame.

$$\theta_{0,j} = \theta_{i,0} = \theta_{3,j} = \theta_{i,3} = 0, i, j \in \{1, 2, 3\} \tag{15}$$

Figure 3 displays the frequency responses of 3×3 arrays of pendulums parametrically excited and having the same coupling springs in the x and y direction ($k_x = l^2 k_y$. The physical parameters are listed in Table 1. The dotted and the solid lines denote the unstable and the

configuration	mass	length	excitation	damping	k_x	k_y	Impurity
1	$0.1kg$	$0.2m$	$0.01m$	$0.01Ns/m$	0.5	$0.5/{l_0}^2$	0%
2	$0.1kg$	$0.3m$	$0.0075m$	$0.01Ns/m$	0.15	$0.5/{l_0}^2$	0%
3	$0.1kg$	$0.3m$	$0.0075m$	$0.01Ns/m$	0.15	$0.5/{l_0}^2$	5%

Table 1: Design parameters of the array configurations

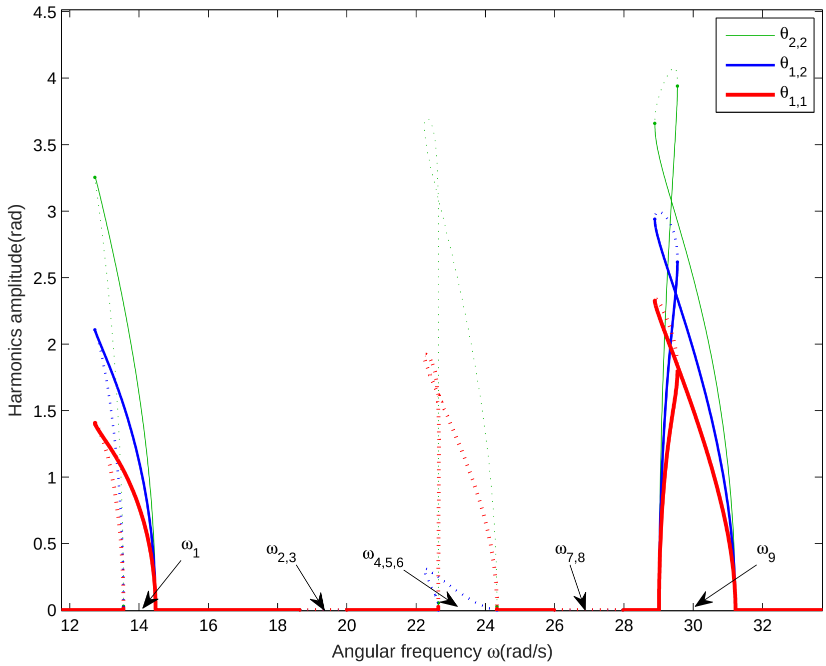

Figure 3: Frequency responses of the 2D array of pendulums for the first configuration listed in Table 1

stable steady state solutions, respectively. The equidistant pendulums with respect to the center of the array have the same response which is illustrated by the three resonance regions. As we can notice the curves bend to the left and exhibit softening characteristics due to the negative sign of the cubic nonlinearity ($-\frac{mgl}{6}\theta_{i,j}^3$). For this configuration, we notice that there are triple and double resonant frequency and the apparent modes are equal to three because the other eigen-modes are perpendicular to the excitation vector.

The second configuration of Table 1 considers two different coupling springs in the x and y directions. In this case, the system becomes symmetric with respect to the central pendulum. Therefore, the number of stable regions increases as shown in Figure 4,. By solving the system (8), no triple or double eigen-frequencies are found and the number of excited modes is equal to four.

To create the disorder in the array, we introduce an impurity in the system. We change the length of one pendulum in the array. To do so, we choose to increase the length of the pendulum located in the position (1,2) by 5%. By comparing this configuration to the pervious one, we remark that the system loses its symmetry and each pendulum has a specific frequency response. In this case, the number of resonant regions increases, which leads to the augmentation of the stable solutions (Figure 5).

5 CONCLUSION

In this paper, the nonlinear dynamics of a small 2D array of pendulums was modeled by determining the equation of motion of each pendulum using the Lagrange equations. The obtained system has been solved using the Harmonic Balance Method (HBM) coupled with the Asymptotic Numerical Method (ANM). The collective dynamic was investigated by plotting the frequency responses for several configurations. We demonstrated that for an axisymmetric

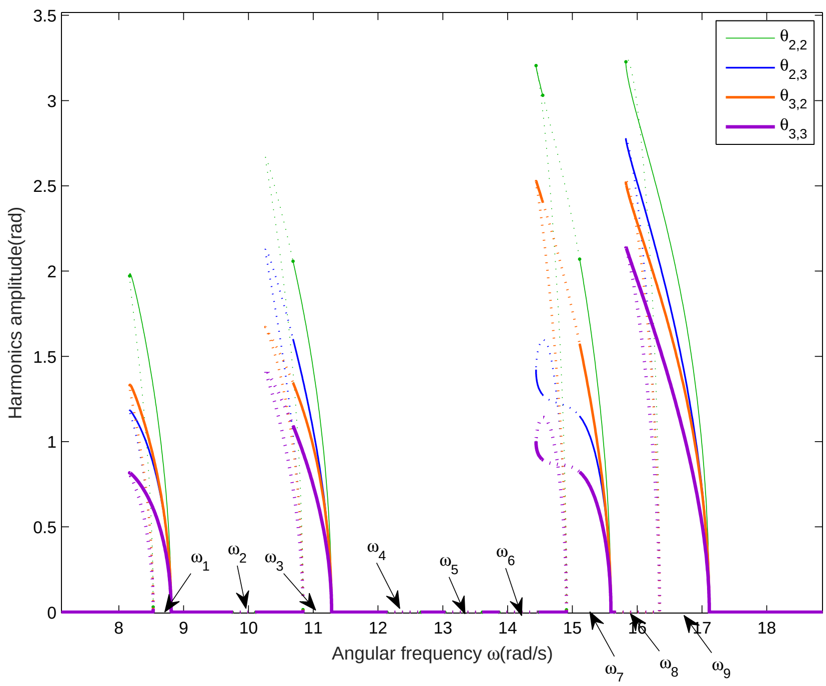

Figure 4: Frequency responses of the 2D array of pendulums for the second configuration listed in Table 1

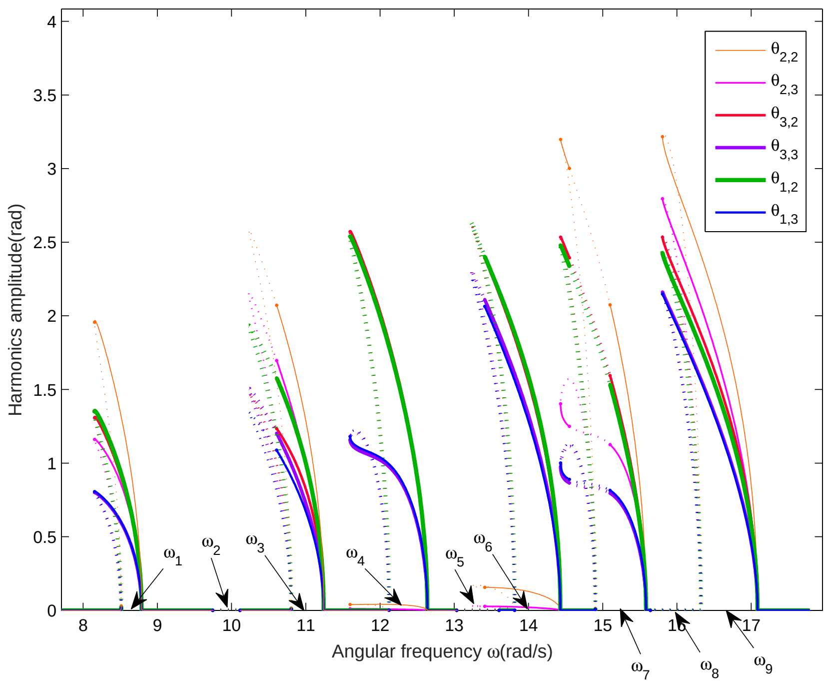

Figure 5: Frequency responses of the 2D array of pendulums for the third configuration listed in Table 1

array, where the coupling coefficient are equal ($k_x = l^2 k_y$), the pendulums that are equidistant with respect to the center of the array has the same frequency response. However, by changing the coupling coefficients, the system becomes symmetric and this leads to an augmentation of the number of the steady state solutions which can be tuned with respect to an added impurity. In a future work, the intrinsic localized modes of the proposed system will be analyzed in terms of existence and stability in presence of impurities.

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