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PROBLEM OF NON-STATIONARY WAVES IN VISCOELASTIC ORTHOTROPIC STRIP SOLVED USING ANALYTICAL METHOD

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Abstract. *The transient response of an infinite orthotropic strip subjected to a transverse load is investigated using an analytical method in this work. The material properties of the strip are described by the discrete model of standard linear viscoelastic solid. In this study, the case of special orthotropy is assumed, i.e., the principal material and geometric axes of the strip are coincident. Once the final system of equations describing the plane-stress problem solved is derived, the integral transform method is used to obtain the Laplace transforms of displacement and velocity components. The inversion of the resulted formulae back to time domain is carried out by the help of numerical inverse Laplace transform. In particular, an algorithm based on the FFT and Wynn's epsilon accelerator was used for this purpose. In the last part of this work, the analytical results obtained for selected orthotropic material are compared to those resulted from numerical simulation performed in the finite element code MSC.Marc. The comparison made showed good agreement between analytical and numerical results and proved their correctness. Finally, the efficiency of both approaches is discussed.*

1 INTRODUCTION

In-plane loaded thin orthotropic and general anisotropic structures are of intensive research interest for many years. This interest is related to the use of thin ribs and panels in many industrial applications, e.g. fuselages and wings of aircrafts, stiffeners in car bodies etc. With respect to high demands on the weight reduction, such structural elements are more often made from composite materials. Due to the fact that these components are mostly subjected to dynamic loading (stationary or transient), it is necessary to be able to describe and analyse their behaviour under such operating conditions.

Since the composite materials have low resistance to crack propagation, most of existing works dealing with the elastodynamics of strip-like solids focus on the stationary problems of fracture mechanics. The number of works dealing with the transient response of orthotropic strips of elastic or viscoelastic properties is very small, especially those which present analytical or semi-analytical approaches.

In this work, the transient response of an infinite viscoelastic orthotropic strip subjected to transverse load is investigated. The integral transform method and numerical inverse Laplace transform are used for solving this plane-stress problem. This study follows [1, 2] in which similar isotropic problems of elastic and viscoelastic strips are solved using analytical methods.

2 PROBLEM FORMULATION

Let us assume an infinite thin orthotropic strip of total height $2d$. The lower boundary of the strip will be free of traction and the upper one will be loaded by a pressure which will be non-zero only for $x_1 \in \langle -h, h\rangle$, see Fig. 1. The amplitude of the pressure will be constant through the strip thickness and its dependence on x_1 will be the same as in [2]. This means that the boundary conditions of this problem can be written as:

$$\sigma_{22}(x_1, d, t) = \begin{cases} -\sigma_a \cos\left(\frac{\pi}{2}\frac{x_1}{h}\right) H(t) & \text{for } x_1 \in \langle -h, h\rangle, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$
$$\sigma_{22}(x_1, -d, t) = 0\,, \quad \sigma_{12}(x_1, d, t) = 0\,, \quad \sigma_{12}(x_1, -d, t) = 0\,,$$

where σ_{ij} $(i, j = 1, 2)$ denotes normal $(i = j)$ and shear $(i \neq j)$ stress components. From the above mentioned, it is clear that the plane-stress problem in $x_1 - x_2$ plane will be solved and that σ_{11}, σ_{22} and σ_{12} are the only non-zero stress components. Further, zero initial conditions will be considered for simplicity, i.e., the strip is free of load and in a rest at $t = 0$.

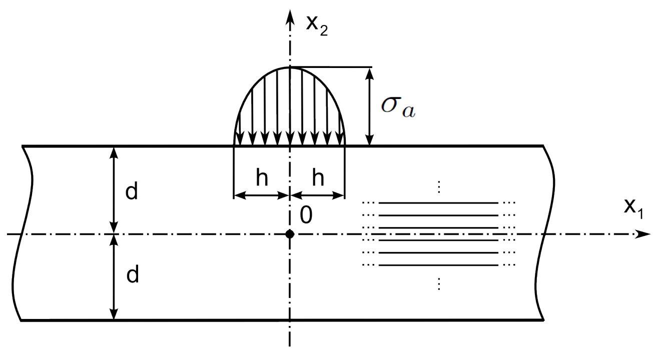


Figure 1: The scheme of the problem.

The orthotropic properties of the strip will be such that the principal geometric and material axes x_1 and x_2 are coincident. Moreover, the memory effect of the material will be taken into account. This will be modelled by using the discrete model of standard linear viscoelastic solid in the Zener configuration [3] in both directions x_1 and x_2. It means that the viscoelastic model consists of a spring and the Maxwell element in parallel, while the Maxwell element contains a spring and a dashpot in series. It is well known that the behaviour of an orthotropic elastic material under the state of plane-stress is described by four independent parameters [4]. They can be denoted by $E_{0,1}$, $E_{0,2}$, $G_{0,12}$ and $\nu_{0,12}$ for the alone-standing spring of the model and they represent two Young's moduli in the direction of principal axes, shear modulus and Poisson's ratio in sequence. Analogously, the parameters of the spring present in the Maxwell element will be introduced in the same way omitting the subscript 0, i.e., E_1, E_2, G_{12} and ν_{12}. Finally, the viscous properties of the dashpot will be characterised by two coefficients of normal viscosity λ_1, λ_2, by the coefficient of shear viscosity η_{12} and by the viscous Poisson's ratio μ_{12}. It is known, see [3, 4], that there hold

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \quad \text{and} \quad \frac{\mu_{12}}{\lambda_1} = \frac{\mu_{21}}{\lambda_2}\,. \tag{2}$$

Additionally, hereinafter we will consider that $\mu_{12} = \nu_{12}$ and $\mu_{21} = \nu_{21}$, i.e., the corresponding viscous and elastic Poisson's ratios are equal for simplicity.

Based on the assumptions made, it is clear that the actual state at an arbitrary point of the strip is described by two independent functions $u_1(x_1, x_2, t)$ and $u_2(x_1, x_2, t)$ which are the displacement components in x_1 and x_2 directions, respectively, and which will represent the solution of the problem solved.

3 ANALYTICAL SOLUTION

3.1 Governing equations

The equations of motion describing the general plane-stress problem of elastodynamics have the form [5]

$$\rho\,\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2}\,, \quad \rho\,\frac{\partial^2 u_2}{\partial t^2} = \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{12}}{\partial x_1}\,, \tag{3}$$

where ρ denotes the density of the strip material. In the following, constitutive and kinematic equations will be introduced into (3).

The kinematic equations giving the relationship between the displacement and strain components can be expressed as [2]

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}\,, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}\,, \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\,. \tag{4}$$

The functions $\varepsilon_{11}(x_1, x_2, t)$, $\varepsilon_{22}(x_1, x_2, t)$ and $\gamma_{12}(x_1, x_2, t)$ represent two normal and one shear strain components in sequence.

The constitutive relations valid for an orthotropic Zener model can be simply derived based on the superposition principle of stresses and strain rates applied to the appropriate parts of the Zener model [3]. Doing so and taking into account the zero initial conditions, the resulting constitutive equations have the form

$$\sigma_{11} = b_{0,11}\left(\varepsilon_{11} + \nu_{0,21}\varepsilon_{22}\right) + b_{11}\left(\varepsilon_{11} + \nu_{21}\varepsilon_{22}\right) - \alpha_1 b_{11}\int_0^t \left(\varepsilon_{11} + \nu_{21}\varepsilon_{22}\right) e^{-\alpha_1(t-\tau)} d\tau\,, \tag{5}$$

$$\sigma_{22} = b_{0,22}\left(\varepsilon_{22} + \nu_{0,12}\varepsilon_{11}\right) + b_{22}\left(\varepsilon_{22} + \nu_{12}\varepsilon_{11}\right) - \alpha_2 b_{22}\int_0^t \left(\varepsilon_{22} + \nu_{12}\varepsilon_{11}\right) e^{-\alpha_2(t-\tau)} d\tau\,,$$

$$\sigma_{12} = \left(b_{0,66} + b_{66}\right)\gamma_{12} - \beta b_{66}\int_0^t \gamma_{12}\, e^{-\beta(t-\tau)} d\tau\,,$$

where b_{ij} are the so-called reduced stiffnesses for a plane-stress state in the plane $x_1 - x_2$ and there hold [4]

$$b_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}\,, \quad b_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} \quad \text{and} \quad b_{66} = G_{12}\,. \tag{6}$$

The parameters $b_{0,ij}$ in (5) corresponding to the alone-standing spring in the Zener model are defined analogously. The coefficients α_i were introduced as $\alpha_i = E_i/\lambda_i$ and they represent the reciprocal values of relaxation times. The constant β is defined in similar way as $\beta = G_{12}/\eta_{12}$. Using the relations (2) and the assumption $\mu_{ij} = \nu_{ij}$ made in the section 2, it is clear that it holds $\alpha_1 = \alpha_2 = \alpha$.

Based on these assumptions and substituting (5) and (4) into (3), the equations of motion take the form

$$\begin{aligned}\frac{\partial^2 u_1}{\partial t^2} &= c_{0,11}^2\left(\frac{\partial^2 u_1}{\partial x_1^2} + \nu_{0,21}\frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) + c_{11}^2\left(\frac{\partial^2 u_1}{\partial x_1^2} + \nu_{21}\frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right)\\ &\quad + c_{0,12}^2\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) + c_{12}^2\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right)\\ &\quad - \int_0^t \left[\alpha c_{11}^2\left(\frac{\partial^2 u_1}{\partial x_1^2} + \nu_{21}\frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) e^{-\alpha(t-\tau)} + \beta c_{12}^2\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_2}\right) e^{-\beta(t-\tau)}\right] d\tau\,,\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{\partial^2 u_2}{\partial t^2} &= c_{0,22}^2\left(\frac{\partial^2 u_2}{\partial x_2^2} + \nu_{0,12}\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right) + c_{22}^2\left(\frac{\partial^2 u_2}{\partial x_2^2} + \nu_{12}\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right)\\ &\quad + c_{0,12}^2\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right) + c_{12}^2\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right)\\ &\quad - \int_0^t \left[\alpha c_{22}^2\left(\frac{\partial^2 u_2}{\partial x_2^2} + \nu_{12}\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right) e^{-\alpha(t-\tau)} + \beta c_{12}^2\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right) e^{-\beta(t-\tau)}\right] d\tau\,,\end{aligned} \tag{8}$$

where the quantities c_{11}, c_{22} and c_{12} are defined as

$$c_{11}^2 = \frac{b_{11}}{\rho}\,, \quad c_{22}^2 = \frac{b_{22}}{\rho} \quad \text{and} \quad c_{12}^2 = \frac{b_{66}}{\rho}\,. \tag{9}$$

Analogous relations hold also for $c_{0,11}$, $c_{0,22}$ and $c_{0,12}$.

Hereinafter we will further assume that $\beta = \alpha$ and that the Poisson's ratios of both springs in the Zener model are the same, i.e., $\nu_{0,ij} = \nu_{ij}$. These assumptions will lead to the simplification of the system (7) - (8).

3.2 Application of Laplace and Fourier transforms

In this subsection, the Laplace transform will be applied to the simplified system (7) - (8). Introducing a new complex variable p and taking into account the zero initial conditions specified in the section 2, the transform of the system (7) - (8) takes the form

$$p^2 U_1 = C_{11}^2 \left(\frac{\partial^2 U_1}{\partial x_1^2} + \nu_{21} \frac{\partial^2 U_2}{\partial x_1 \partial x_2} \right) + C_{12}^2 \left(\frac{\partial^2 U_1}{\partial x_2^2} + \frac{\partial^2 U_2}{\partial x_1 \partial x_2} \right), \tag{10}$$
$$p^2 U_2 = C_{22}^2 \left(\frac{\partial^2 U_2}{\partial x_2^2} + \nu_{12} \frac{\partial^2 U_1}{\partial x_1 \partial x_2} \right) + C_{12}^2 \left(\frac{\partial^2 U_2}{\partial x_1^2} + \frac{\partial^2 U_1}{\partial x_1 \partial x_2} \right),$$

where the functions $U_1(x_1, x_2, p)$ and $U_2(x_1, x_2, p)$ denote the Laplace transforms of the unknown displacement components $u_1(x_1, x_2, t)$ and $u_2(x_1, x_2, t)$, respectively, and the complex functions $C_{11}(p)$, $C_{22}(p)$ and $C_{12}(p)$ are defined as follows:

$$C_{11}(p) = \sqrt{c_{0,11}^2 + \left(1 - \frac{\alpha}{p+\alpha}\right) c_{11}^2}\,, \quad C_{22}(p) = \sqrt{c_{0,22}^2 + \left(1 - \frac{\alpha}{p+\alpha}\right) c_{22}^2}\,, \tag{11}$$
$$C_{12}(p) = \sqrt{c_{0,12}^2 + \left(1 - \frac{\alpha}{p+\alpha}\right) c_{12}^2}\,.$$

In contrast to the problem of isotropic viscoelastic strip solved in [2], the system (10) cannot be rewritten to an uncoupled system of PDEs.

Now we will suppose that the solutions U_1 and U_2 of (10) can be written in the form of Fourier integrals. With respect to the fact that the applied load (1) is an even function of x_1, the functions of displacement components u_1 and u_2, as well as their Laplace transforms, will be odd and even functions of this variable, respectively. Therefore one can write

$$U_1(x_1, x_2, p) = \frac{1}{\pi} \int_0^\infty A(\omega, x_2, p) \sin(\omega x_1)\, d\omega\,, \tag{12}$$
$$U_2(x_1, x_2, p) = \frac{1}{\pi} \int_0^\infty B(\omega, x_2, p) \cos(\omega x_1)\, d\omega\,.$$

Substituting the relations (12) into (10), taking the assumption that all necessary functions are integrable and after rearrangement of terms, the equations (10) give the coupled system

$$\frac{\partial^2 A}{\partial x_2^2} - a_1 A - a_2 \frac{\partial B}{\partial x_2} = 0\,, \quad \frac{\partial^2 B}{\partial x_2^2} - b_1 B + b_2 \frac{\partial A}{\partial x_2} = 0 \tag{13}$$

for the unknown Fourier spectra $A(\omega, x_2, p)$ and $B(\omega, x_2, p)$. The function $a_i(\omega, p)$ and $b_i(\omega, p)$ present in (13) are given by

$$a_1(\omega, p) = \left[\left(\frac{C_{11}}{C_{12}} \right)^2 + \left(\frac{p}{\omega C_{12}} \right)^2 \right] \omega^2\,, \quad a_2(\omega, p) = \left[1 + \nu_{21} \left(\frac{C_{11}}{C_{12}} \right)^2 \right] \omega\,, \tag{14}$$
$$b_1(\omega, p) = \left[\left(\frac{C_{12}}{C_{22}} \right)^2 + \left(\frac{p}{\omega C_{22}} \right)^2 \right] \omega^2\,, \quad b_2(\omega, p) = \left[\nu_{12} + \left(\frac{C_{12}}{C_{22}} \right)^2 \right] \omega\,.$$

After some calculus, it is possible to show that the solution of (13) can be expressed as

$$A(\omega, x_2, p) = P \sinh(\Lambda_1 x_2) + Q \cosh(\Lambda_1 x_2) + R \sinh(\Lambda_2 x_2) + S \cosh(\Lambda_2 x_2)\,, \tag{15}$$

$$B(\omega, x_2, p) = L_1 \left(P \cosh\left(\Lambda_1 x_2\right) + Q \sinh\left(\Lambda_1 x_2\right)\right) + L_2 \left(R \cosh\left(\Lambda_2 x_2\right) + S \sinh\left(\Lambda_2 x_2\right)\right),$$

where

$$L_1(\omega, p) = \frac{\Lambda_1^2 - a_1}{\Lambda_1 a_2}, \quad L_2(\omega, p) = \frac{\Lambda_2^2 - a_1}{\Lambda_2 a_2}, \tag{16}$$

$$\Lambda_1(\omega, p) = \sqrt{\frac{1}{2}\left(k + \sqrt{s}\right)}, \quad \Lambda_2(\omega, p) = \sqrt{\frac{1}{2}\left(k - \sqrt{s}\right)},$$

$$k(\omega, p) = a_1 + b_1 - a_2 b_2, \quad s(\omega, p) = a_2^2 b_2^2 - 2a_2 b_2 \left(a_1 + b_1\right) + \left(a_1 - b_1\right)^2.$$

The complex functions $P(\omega, p)$, $Q(\omega, p)$, $R(\omega, p)$ and $S(\omega, p)$ present in (15) are unknown at this moment and they will be determined by using the boundary conditions (1).

3.3 Final formulae for displacement and velocity components in Laplace domain

In this subsection we will focus on finding the functions P, Q, R and S and then on the derivation of final formulae for displacement and velocity components in Laplace domain.

First, the Laplace transform needs to be applied to the boundary conditions (1) and the Laplace transforms of stress components σ_{22} and σ_{12} have to be determined in terms of U_1 and U_2. Let us denote $\Sigma_{22}(x_1, x_2, p)$ and $\Sigma_{12}(x_1, x_2, p)$ the transforms of stresses. Then using the constitutive equations (5) and the zero initial conditions, one can write

$$\Sigma_{22} = h_{22}\left(\frac{\partial U_2}{\partial x_2} + \nu_{12}\frac{\partial U_1}{\partial x_1}\right), \quad \Sigma_{12} = h_{12}\left(\frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1}\right), \tag{17}$$

where the functions $h_{11}(p)$ and $h_{12}(p)$ were introduced by the following relations:

$$h_{22}(p) = b_{0,22} + \frac{p}{p + \alpha} b_{22}, \quad h_{12}(p) = b_{0,66} + \frac{p}{p + \alpha} b_{66}. \tag{18}$$

When we introduce (17) into the left hand sides of transformed conditions (1) and substituting (12) and (15), a non-homogeneous system of four equations for the unknown functions $P(\omega, p)$, $Q(\omega, p)$, $R(\omega, p)$ and $S(\omega, p)$ is obtained. If we define the functions

$$k_1(\omega, p) = h_{22}\left(\nu_{12}\omega + L_1\Lambda_1\right), \quad k_2(\omega, p) = h_{22}\left(\nu_{12}\omega + L_2\Lambda_2\right), \tag{19}$$

$$k_3(\omega, p) = h_{12}\left(\Lambda_1 - \omega L_1\right), \quad k_4(\omega, p) = h_{12}\left(\Lambda_2 - \omega L_2\right),$$

$$k_5(\omega, p) = k_2 k_3 \cosh\left(d\Lambda_1\right)\sinh\left(d\Lambda_2\right) - k_1 k_4 \cosh\left(d\Lambda_2\right)\sinh\left(d\Lambda_1\right),$$

$$k_6(\omega, p) = k_2 k_3 \cosh\left(d\Lambda_2\right)\sinh\left(d\Lambda_1\right) - k_1 k_4 \cosh\left(d\Lambda_1\right)\sinh\left(d\Lambda_2\right),$$

the solution of the mentioned system can be written in the following compact form:

$$P(\omega, p) = -\frac{1}{2}\,\frac{a(\omega)k_4\cosh\left(d\Lambda_2\right)}{pk_5}, \quad Q(\omega, p) = -\frac{1}{2}\,\frac{a(\omega)k_4\sinh\left(d\Lambda_2\right)}{pk_6}, \tag{20}$$

$$R(\omega, p) = \frac{1}{2}\,\frac{a(\omega)k_3\cosh\left(d\Lambda_1\right)}{pk_5}, \quad S(\omega, p) = \frac{1}{2}\,\frac{a(\omega)k_3\sinh\left(d\Lambda_1\right)}{pk_6},$$

where

$$a(\omega) = \frac{4\pi\sigma_a h\cos(\omega h)}{\pi^2 - 4\omega^2 h^2}. \tag{21}$$

The final formulae for the Laplace transforms of displacement components $u_1(x_1, x_2, t)$ and $u_2(x_1, x_2, t)$ can be now simply obtained by the substitution of (20) and (15) into the relations (12). The Laplace transforms $V_1(x_1, x_2, p)$ and $V_2(x_1, x_2, p)$ of the velocity components $v_1(x_1, x_2, t)$ and $v_2(x_1, x_2, t)$ then can be derived analogously because it holds

$$V_1(x_1, x_2, p) = p\, U_1(x_1, x_2, p) \quad \text{and} \quad V_2(x_1, x_2, p) = p\, U_2(x_1, x_2, p). \tag{22}$$

4 ANALYTICAL AND NUMERICAL RESULTS

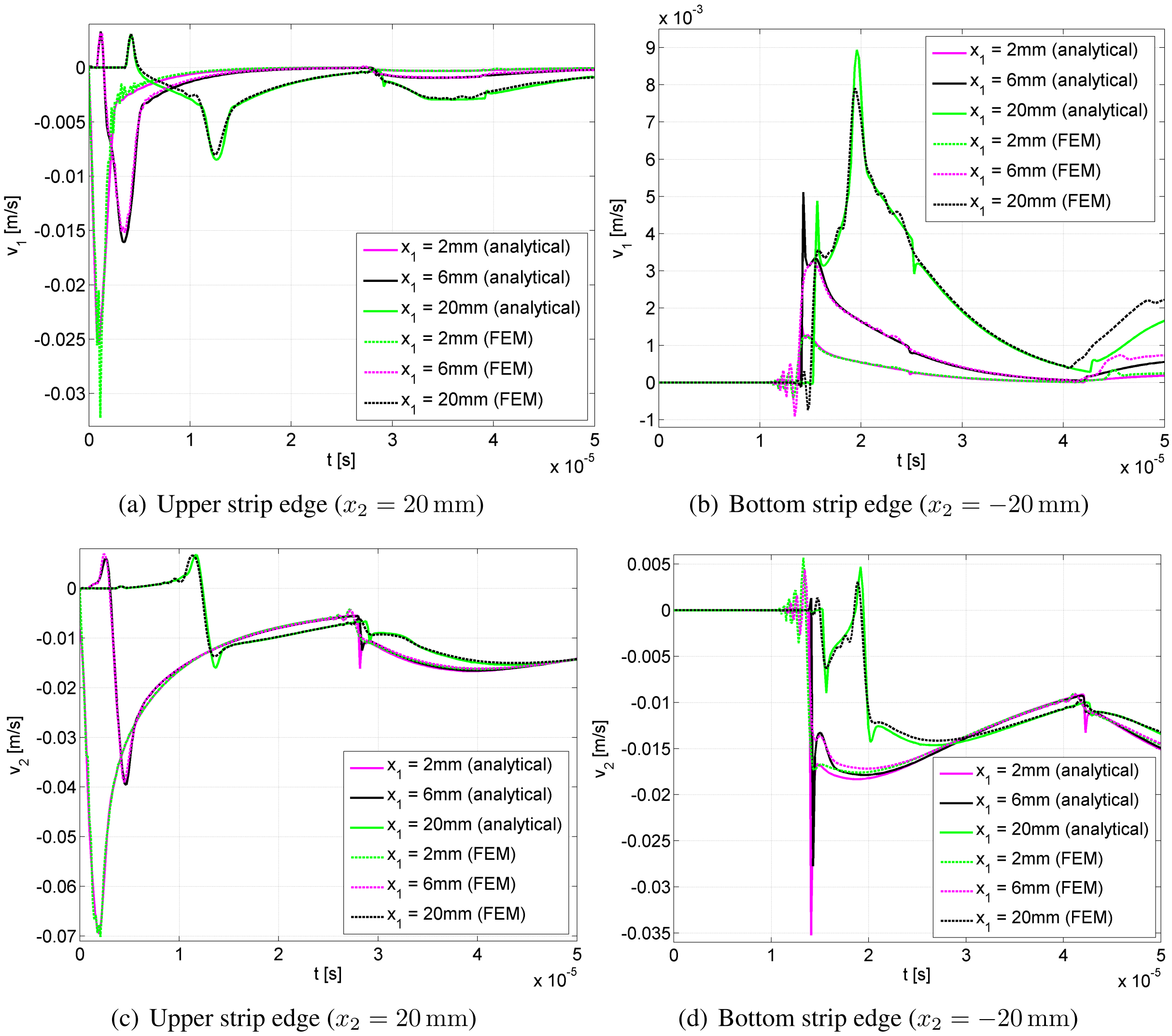
To obtain analytical results in time domain, the inverse Laplace transform (ILT) of U_1 and U_2 (or V_1 and V_2) has to be performed. Basically, there exist two possible approaches. The analytical one is based on the evaluation of the Bromwich integral defining the ILT by using the theorem of residue [6]. This approach is feasible in this case but it involves the process of finding the solution of dispersion equation, i.e., the dispersion curves need to be investigated, which is very computer time consuming process. The application of a suitable and sufficiently precise numerical algorithm to ILT is the next possibility. With respect to the complexity of the formulae derived and to authors' previous good experiences with this procedure, the numerical code based on FFT and Wynn's epsilon accelerator was used for ILT of required functions. This code has been previously tested and verified by authors on different transient wave problems (e.g. radial impact on a thin isotropic viscoelastic disc [7], transverse impact on an isotropic viscoelastic strip [2], impact on functionally graded beams [8] etc.) such that it can be said that its use does not mean the loss of analytical results accuracy.

The results obtained by this procedure were then compared to those of numerical simulation performed in the commercial FE code MSC.Marc. This comparison served both for the verification of the derivation process of the final formulae in the Laplace domain and for finding the limits of the FE model used.

Considering the symmetry of the problem, see Fig. 1, the geometry of the finite element model consisted only of one-half of the strip of total length $200\,\mathrm{mm}$. The spatial discretisation was done by regular bilinear isoparametric elements for 2D plane-stress problem. The basic size of elements was $0.4 \times 0.4\,\mathrm{mm}$ and the mesh was once refined in the area close to the applied load such that the total number of elements was about 25000. Using the procedure described in [9], the frequency limit of the model can be estimated as $0.9\,\mathrm{MHz}$. The implicit Newmark algorithm with constant step $4 \times 10^{-8}\,\mathrm{s}$ was used for the integration in time domain.

The evaluation of analytical solution and the numerical simulation were performed for the following parameters: $d = 20\,\mathrm{mm}$, $\sigma_a = 1\,\mathrm{MPa}$, $h = 2\,\mathrm{mm}$. The material properties of the strip were assumed to be the same as in [10], i.e., $\rho = 2250\,\mathrm{kg\,m^{-3}}$, $E_{0,1} = 35 \times 10^9\,\mathrm{Pa}$, $E_{0,2} = 11.584 \times 10^9\,\mathrm{Pa}$, $G_{0,12} = 4 \times 10^9\,\mathrm{Pa}$, $\nu_{0,12} = 0.278$, $E_1 = 18.48 \times 10^9\,\mathrm{Pa}$, $G_{12} = 1.83 \times 10^9\,\mathrm{Pa}$ and $\lambda_1 = 5 \times 10^4\,\mathrm{Pa\,s^{-1}}$. The other parameters were then determined by using the assumptions mentioned in the previous sections. All the constants have been chosen in such a way to roughly correspond to the parameters of orthotropic material Gevetex (E-Glass 21xK43) which can be found in [11].

The comparison of analytical and numerical results was made at several points lying on the upper ($x_2 = 20\,\mathrm{mm}$) and bottom ($x_2 = -20\,\mathrm{mm}$) edge of the strip. Figs. 2(a) - 2(d) show this comparison for horizontal v_1 and vertical v_2 velocity component for $x_1 = \{2, 6, 20\}\,\mathrm{mm}$. It is clear from these figures that the analytical (*solid lines*) and the numerical (*dotted lines*) solutions are in a very good agreement. Additionally, it is clear that this agreement is better for $x_2 = 20\,\mathrm{mm}$ than in the case of $x_2 = -20\,\mathrm{mm}$. This is mainly caused by too large finite elements which cannot suppress the dispersion due to spatial discretisation when the waves are



(a) Upper strip edge ($x_2 = 20\,\mathrm{mm}$)

(b) Bottom strip edge ($x_2 = -20\,\mathrm{mm}$)

(c) Upper strip edge ($x_2 = 20\,\mathrm{mm}$)

(d) Bottom strip edge ($x_2 = -20\,\mathrm{mm}$)

Figure 2: Comparison of horizontal v_1 and vertical v_2 velocities determined by analytical and FEM approaches for $x = \{2, 6, 20\}\,\mathrm{mm}$ at the upper and the bottom edge of the strip.

propagated through the whole strip height. The numerical results could be further improved, especially in the vicinity of the steep wave fronts (see Figs. 2(b) and 2(d)), when the element size is reduced. Naturally, this improvement of numerical model will leads to significant increase of computational time. For comparison, the numerical simulation for $t \in \langle 0, 50\rangle\,\mu\mathrm{s}$ takes about $48\,\mathrm{mins}$ whereas the evaluation of analytical solution for the same spatial grid only about $7\,\mathrm{mins}$.

5 CONCLUSIONS

Non-stationary wave problem of an infinite viscoelastic strip of special orthotropic properties was solved for a specific type of external load. The method used was based on the combination of the classical method of integral transforms and the numerical inverse Laplace transform. The analytical results were compared to the numerical ones obtained by using an existing FE code. The agreement of both types of results proved the correctness of the derivation process and showed the limits of the FE model used. Due to the efficiency and accuracy of the algorithm used for the evaluation of analytical solution, the presented solution can be used e.g. for solving the problems of material parameters identification. For instance, the evaluation of the time history of velocity at one particular point composed of 256 time steps takes less than $13\,\mathrm{s}$ using one core of $2.4\,\mathrm{MHz}$ CPU. Moreover, this solution will be used for solving the problem of general orthotropic viscoelastic strip in future, which will find even more practical applications.

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