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TWO-DIMENSIONAL LINEAR MODEL OF ELASTIC SHELL MADE OF ANISONROPIC HETEROGENEOUS MATERIAL

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Abstract. *A thin linearly elastic shell made of heterogeneous anisotropic material is studied. The general anisotropy described by 21 elastic moduli is examined. The material may be functionally graduated or multi-layered. The asymptotic expansions in powers of the relative shell thickness of the three-dimensional elasticity equations are used to deliver the two-dimensional shell model. In the zeroth-order approximation the model of 8th differential order is obtained. The asymptotic precision of this model is the same as the precision of the classic Kirchhoff–Love model for isotropic shell. Influence of the elastic moduli variation in the thickness direction is examined. As an example the Donnell system for shallow shells is presented, and this system is used to discuss the free transversal shell vibrations spectrum.*

1 INTRODUCTION

Numerous studies are devoted to derive the two-dimensional (2D) models of plates and shells from the three-dimensional (3D) equations of the theory of elasticity. The main used methods are the following: the method of hypotheses about distributions of unknown functions in the thickness direction z [1-4], the methods of expansions in the series of the Legendre polynomials in z [5,6], and the asymptotic expansions in series in the small parameter h_* which is equal to the dimensionless thickness [7-11].

An anisotropy of the material introduces additional difficulties. Some 2D models including a transverse shear for a transversely isotropic material are proposed in [12]. The Timoshenko–Reissner hypotheses are used and the resulting system of the 10th order is obtained.

In the present paper we study the case of the general anisotropy with 21 elastic moduli. It is shown [13,14] that for plates with the general anisotropy (in contrast to isotropic or orthotropic materials) the 2D models based on the Kirchhoff–Love (KL) or on the Timoshenko–Reissner (TR) hypotheses are inconsistent in the principal terms with respect to h_*. It is illustrated by comparison with the asymptotic 3D solutions for the test problems. To obtain the correct 2D plate model the generalized TR hypotheses are suggested [13,14]. The same hypotheses are used to deliver the 2D shell model in the case of the general anisotropy [10,11]. The system of 10th differential order is obtained. The main defect of this system consists in its high order and as a result in the presence of boundary layers among its solutions. The main principle of the correct 2D shell model is that the length of tangential waves is much longer than the shell thickness [15]. The boundary layers do not satisfy this principle.

In the present paper it is assumed that all elastic moduli are of the identical orders, we expect that in this case the KL model is better. But in connection with the generalized TR hypotheses we have the TR model. In the studied case the TR model gives asymptotically the same results as the more simple KL model. That is why it is desirable to exclude the boundary layers from the TR model and to decrease the differential order of system to 8. This simplification of model is partly made in [10], but some difficulties appear.

On the other hand if the transversal elastic moduli are relatively small the TR model gives the essentially better results than the KL model [16,17].

In this paper the asymptotic expansions in powers of the relative shell thickness of the 3D elasticity equations are used to deliver the 2D shell model. In the zeroth-order approximation the model of 8th differential order is obtained. The asymptotic precision of this model is of the order h_*, and is the same as the precision of the classic KL for isotropic shell. The accepted way allows us to avoid any hypotheses and to obtain the desired model. Additionally this model is acceptable for multi-layered (sandwich) and functionally graduated shells. For plates with the general anisotropy the similar model is proposed in [18], and for transversely isotropic material it is given in [19,20].

2 THE MAIN ASSUMPTIONS, 3D EQUATIONS AND ELASTICITY RELATIONS

Consider a thin shell of the constant thickness h with smooth mid-surface. We introduce curvilinear co-ordinates x_1, x_2 in the mid-surface coinciding with lines of curvature. The third co-ordinate $x_3 = z$ ($|z| \leq h/2$) coincides with the mid-surface normal (Fig. 1). The Lame coefficients of a shell body are $H_i = A_i(1 + k_i z),\ i = 1, 2$, $H_3 = 1$, where A_i, k_i are the Lame coefficients of the mid-surface and its main curvatures. Further with the error of order h_* we put approximately $H_i \simeq A_i$, but $\partial H_i/\partial z = A_i k_i$.

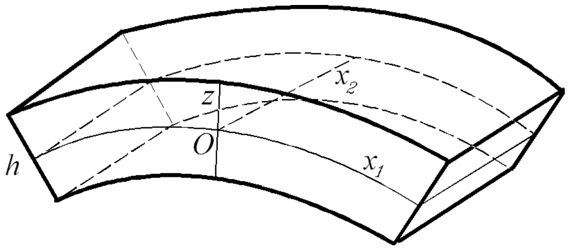

Figure 1: The shell.

In these co-ordinates the equilibrium equations are [21]

$$
\begin{aligned}
&\frac{\partial(A_2\sigma_{11})}{A_2\partial y_1}+\frac{\partial(A_1\sigma_{12})}{A_1\partial y_2}+\frac{\partial A_1}{A_1\partial y_2}\sigma_{12}-\frac{\partial A_2}{A_2\partial y_1}\sigma_{22}+\frac{\partial \sigma_{13}}{\partial z}+k_1\sigma_{13}+f_1=0, \quad (1\leftrightarrow 2),\\
&\frac{\partial(A_2\sigma_{13})}{A_2\partial y_1}+\frac{\partial(A_1\sigma_{23})}{A_1\partial y_2}+\frac{\partial \sigma_{33}}{\partial z}-k_1\sigma_{13}-k_2\sigma_{23}+f_3=0,\\
&\partial y_1=A_1\partial x_1, \quad \partial y_2=A_2\partial x_2,
\end{aligned}
\tag{1}
$$

where σ_{ij} are stresses, $f_i(x_1,x_2,z,t)$ are the components of the external body forces densities.

Consider the material with general anisotropy containing 21 elastic moduli. To describe the elasticity relations it is more convenient to use the matrix designations instead of the 4th rank tensor. We divide stresses σ_{ij} and strains ε_{ij} into groups of tangential $\boldsymbol{\sigma}_t,\ \boldsymbol{\varepsilon}_t$ and non-tangential $\boldsymbol{\sigma}_n,\ \boldsymbol{\varepsilon}_n$ stresses and strains,

$$
\begin{aligned}
&\boldsymbol{\sigma}_t=(\sigma_{11},\,\sigma_{12},\,\sigma_{22})^T, \qquad &&\boldsymbol{\sigma}_n=(\sigma_{13},\,\sigma_{23},\,\sigma_{33})^T,\\
&\boldsymbol{\varepsilon}_t=(\varepsilon_{11},\,\varepsilon_{12},\,\varepsilon_{22})^T, \qquad &&\boldsymbol{\varepsilon}_n=(\varepsilon_{13},\,\varepsilon_{23},\,\varepsilon_{33})^T;
\end{aligned}
\tag{2}
$$

the superscript ^{T} designates a transposition. Then the relations between stresses and strains are written in the matrix form

$$
\boldsymbol{\sigma}_t=\mathbf{A}\cdot\boldsymbol{\varepsilon}_t+\mathbf{B}\cdot\boldsymbol{\varepsilon}_n, \quad \boldsymbol{\sigma}_n=\mathbf{B}^T\cdot\boldsymbol{\varepsilon}_t+\mathbf{C}\cdot\boldsymbol{\varepsilon}_n,
\tag{3}
$$

where $\mathbf{A}=\{A_{ij}\},\ \mathbf{B}=\{B_{ij}\},\ \mathbf{C}=\{C_{ij}\}$ are the matrices of the elastic moduli, the matrices $\mathbf{A}$ and $\mathbf{C}$ being symmetric. The relations (3) contain 21 elastic moduli. It is assumed that the matrix of 6th order

$$
\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{pmatrix}
\tag{4}
$$

is positively definite.

The strains ε_{ij} (in assumption that $H_i=A_i$) are

$$
\begin{aligned}
&\varepsilon_{11}=\varepsilon_{11}(u_1,u_2,w)=\frac{\partial u_1}{\partial y_1}+\frac{\partial A_2}{A_1\partial y_2}u_2+k_1w, \quad (1\leftrightarrow 2);\\
&\varepsilon_{12}=\varepsilon_{12}(u_1,u_2,w)=A_1\frac{\partial}{\partial y_2}\left(\frac{u_1}{A_1}\right)+A_2\frac{\partial}{\partial y_1}\left(\frac{u_2}{A_2}\right),\\
&\varepsilon_{13}=\frac{\partial w}{\partial y_1}+\frac{\partial u_1}{\partial z}-k_1u_1, \ (1\leftrightarrow 2), \qquad \varepsilon_{33}=\frac{\partial w}{\partial z},
\end{aligned}
\tag{5}
$$

where $u_i,\ i=1,2$ and w are the tangential and normal displacements respectively. With an error of order h_* the terms k_iu_i may be neglected.

Assume that the upper and lower shell surfaces $z = \pm h/2$ are free from external forces,

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0, \quad z = \pm h/2. \tag{6}$$

If the surface external forces are presented, we assume that the conditions (6) are fulfilled and these forces are included into the body external forces.

3 ASYMPTOTIC SIMPLIFICATION OF THE EQUILIBRIUM EQUATIONS

Relations (1–6) contain the full 3D description of the problem. To obtain the approximate 2D model we use the asymptotic expansions in powers of the relative shell thickness h_*. Introduce the dimensionless variables (with the sign ˆ).

$$\begin{aligned} &\{x_i, u_i, w\} = l\{\hat{x}_i, \hat{u}_i, \hat{w}\}, \quad i = 1,2, \quad x_3 = hz, \quad h_* = \frac{h}{l}, \\ &\{A_{ij}, B_{ij}, C_{ij}, \sigma_{ij}\} = E\{\hat{A}_{ij}, \hat{B}_{ij}, \hat{C}_{ij}, \hat{\sigma}_{ij}\}, \quad \{f_i\} = (E/l)\{\hat{f}_i\}, \quad i,j = 1,2,3, \end{aligned} \tag{7}$$

where l is the typical length of shell, E is the typical value of the elastic moduli, h_* is the small parameter. Further the sign ˆ is omitted.

As a result we get the system with small parameter h_* on the right sides

$$\frac{\partial w}{\partial z} = h_* \varepsilon_{33}; \tag{8.1}$$

$$\frac{\partial u_i}{\partial z} = h_* \left(\varepsilon_{i3} - \frac{\partial w}{\partial y_i} + k_i u_i \right), \quad i = 1,2, \tag{8.2}$$

$$\frac{\partial \sigma_{i3}}{\partial z} = -h_* \hat{L}_i, \quad \hat{L}_i = L_i(\boldsymbol{\sigma}_t) + k_i \sigma_{i3} + f_i, \quad i = 1,2, \tag{8.3}$$

$$\frac{\partial \sigma_{33}}{\partial z} = -h_* \hat{L}_3, \quad \hat{L}_3 = L_3(\boldsymbol{\sigma}_n) - k_1 \sigma_{11} - k_2 \sigma_{22} + f_3, \tag{8.4}$$

where

$$\begin{aligned} &L_1 = L_1(\boldsymbol{\sigma}_t) = \frac{\partial (A_2 \sigma_{11})}{A_2 \partial y_1} + \frac{\partial (A_1 \sigma_{12})}{A_1 \partial y_2} + \frac{\partial A_1}{A_1 \partial y_2} \sigma_{12} - \frac{\partial A_2}{A_2 \partial y_1} \sigma_{22}, \quad (1 \leftrightarrow 2), \\ &L_3 = L_3(\sigma_{13}, \sigma_{23}) = \frac{\partial (A_2 \sigma_{13})}{A_2 \partial y_1} + \frac{\partial (A_1 \sigma_{23})}{A_1 \partial y_2}. \end{aligned} \tag{9}$$

The main unknowns in the system (8) are $u_1, u_2, w, \sigma_{13},\ \sigma_{23},\ \sigma_{33}$. For deriving the right sides we use the relations (see(3))

$$\boldsymbol{\varepsilon}_n = \boldsymbol{C}^{-1} \cdot \boldsymbol{\sigma}_n - \boldsymbol{C}^{-1} \cdot \mathrm{B}^T \cdot \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\sigma}_t = \boldsymbol{A}^* \cdot \boldsymbol{\varepsilon}_t + \boldsymbol{B} \cdot \boldsymbol{C}^{-1} \cdot \boldsymbol{\sigma}_n, \quad \boldsymbol{A}^* = \boldsymbol{A} - \boldsymbol{B} \cdot \boldsymbol{C}^{-1} \cdot \boldsymbol{B}^T. \tag{10}$$

We seek the solution of the system (8) that satisfies conditions (6)

$$\sigma_{i3} = 0, \quad i = 1,2,3, \quad z = \pm 1/2 \tag{11}$$

in assumption that all elastic moduli are of the identical orders.

Let $\sigma_t,\ \sigma_s,\ \sigma_n$ be the typical values of the tangential stresses, $\sigma_{11}, \sigma_{12}, \sigma_{22}$, of the transversal shear stresses, σ_{13}, σ_{23}, and of the normal stresses, σ_{33}, respectively. From equations (8.3) and (8.4) the asymptotic estimates follow

$$\sigma_s \sim h_* \sigma_t, \quad \sigma_n \sim h_* \sigma_s \sim h_*^2 \sigma. \tag{12}$$

Accepting $\sigma_s = \sigma_n = 0$ we get the classic Kirchhoff–Love model. If we hold the stresses σ_s, then we obtain the Timoshenko–Reissner model.

4 ASYMPTOTIC SOLUTION OF THE SYSTEM (8)

As in [19,20], to find the approximate solution we use the method of iteration consisting in the successive solution. The equations (8). After integration at z equations (8.1) (8.2) the arbitrary functions $w^0(x_1, x_2)$ and $u_i^0(x_1, x_2)$ are introduced. Then these functions may be found from the compatibility conditions of equations (8.3), (8.4) and boundary conditions (11), which consist in the equality to zero of the average values of the right sides of equations (8.3), (8.4)

$$\left\langle \hat{L}_i \right\rangle = 0, \quad i = 1, 2, 3, \qquad \langle Z \rangle \equiv \int_{-1/2}^{1/2} Z dz. \tag{13}$$

In zeroth-order approximations equations (8.1) and (8.2) give

$$w = w^0(x_1, x_2), \qquad u_i = u_i^0(x_1, x_2) - h_* z \varphi_i, \quad \varphi_i = \frac{\partial w^0}{\partial y_i}, \quad i = 1, 2. \tag{14}$$

From relation (10) omitting the small summand with $\boldsymbol{\sigma}_n$ we get

$$\begin{aligned} &\boldsymbol{\sigma}_t^0 = (\sigma_{11}^0, \sigma_{12}^0, \sigma_{22}^0)^T = \boldsymbol{A}^* \cdot (\boldsymbol{\varepsilon}_t^0 - h_* z \boldsymbol{\varepsilon}_t^\varphi), \\ &\boldsymbol{\varepsilon}_t^0 = (\varepsilon_{11}(u_1^0, u_2^0, w^0),\ \varepsilon_{12}(u_1^0, u_2^0, w^0),\ \varepsilon_{22}(u_1^0, u_2^0, w^0))^T, \\ &\boldsymbol{\varepsilon}_t^\varphi = (\varepsilon_{11}(\varphi_1, \varphi_2, 0),\ \varepsilon_{12}(\varphi_1, \varphi_2, 0),\ \varepsilon_{22}(\varphi_1, \varphi_2, 0))^T. \end{aligned} \tag{15}$$

The compatibility conditions (13) with $i = 1, 2$ give two equations for the unknown functions $u_i^0(x_1, x_2),\ w^0(x_1, x_2)$

$$\begin{aligned} &L_i(\boldsymbol{a}^{(0)} \cdot \boldsymbol{\varepsilon}_t^0) - h_* L_i(\boldsymbol{a}^{(1)} \cdot \boldsymbol{\varepsilon}_t^\varphi) + k_i \sigma_{i3} + F_i = 0, \quad i = 1, 2, \\ &\boldsymbol{a}^{(k)} = \left\langle z^k \boldsymbol{A}^* \right\rangle, \quad k = 0, 1, 2, \qquad F_i = \langle f_i \rangle. \end{aligned} \tag{16}$$

The term $k_i \sigma_{i3}$ is small and it may be omitted.

Here we study the general case in which the elastic moduli in matrix $\boldsymbol{A}^*$ depend on z. Therefore the multi-layered shells and functionally graduated shells are not excluded from consideration. If matrix $\boldsymbol{A}^*$ is constant in z or it is symmetric, $\boldsymbol{A}^*(-z) = \boldsymbol{A}^*(z)$, then $\boldsymbol{a}^{(1)} = \boldsymbol{0}$ (further we refer to this case as to symmetric one, the opposite case is asymmetric).

Now we find stresses σ_{13}^0 and σ_{23}^0 in zeroth-order approximation

$$\begin{aligned} &\sigma_{i3}^0(x_1, x_2, z) = -h_* \left(L_i(\tilde{\boldsymbol{a}}^{(0)} \cdot \boldsymbol{\varepsilon}_t^0) - h_* L_i(\tilde{\boldsymbol{a}}^{(1)} \cdot \boldsymbol{\varepsilon}_t^\varphi) + \tilde{F}_i \right), \quad i = 1, 2, \\ &\tilde{\boldsymbol{a}}^{(k)}(z) = \textstyle\int_{-1/2}^{z} z^k \boldsymbol{A}^* dz, \quad \tilde{F}_i(z) = \int_{-1/2}^{z} f_i dz. \end{aligned} \tag{17}$$

Lastly the compatibility condition for equation (8.4) gives the third equation for functions $u_i^0(x_1, x_2),\ w^0(x_1, x_2)$

$$\begin{aligned} &h_* L_3 \left(L_1(\boldsymbol{a}^{(0)} \cdot \boldsymbol{\varepsilon}_t^0)), L_2(\boldsymbol{a}^{(0)} \cdot \boldsymbol{\varepsilon}_t^0)) \right) - h_*^2 L_3 \left(L_1(\boldsymbol{a}^{(1)} \cdot \boldsymbol{\varepsilon}_t^\varphi)), L_2(\boldsymbol{a}^{(1)} \cdot \boldsymbol{\varepsilon}_t^\varphi)) \right) + \\ &+ h_* L_3(\langle z F_1 \rangle, \langle z F_2 \rangle) - k_1 \sigma_{11}^0 - k_2 \sigma_{22}^0 + F_3 = 0. \end{aligned} \tag{18}$$

Three equations (16) and (18) give the system from which the unknown functions $u_i^0(x_1, x_2)$, $w^0(x_1, x_2)$ are to be found. The differential order of this system is 8 as in the Kirchhoff–Love model.

5 COMPARISON WITH THE KIRCHHOFF–LOVE MODEL

We introduce the tangential stress-resultants, $\hat{\boldsymbol{T}}$, the stress-couples, $\hat{\boldsymbol{M}}$, and the shear stress-resultants, $\hat{\boldsymbol{Q}}$, by relations

$$\begin{aligned}
&\hat{\boldsymbol{T}} = (\hat{T}_{11}, \hat{T}_{12}, \hat{T}_{22})^T = \langle \boldsymbol{\sigma}_t \rangle = \boldsymbol{T} + \boldsymbol{N}^\varphi, \quad \boldsymbol{T} = \boldsymbol{a}^{(0)} \cdot \boldsymbol{\varepsilon}_t^0, \quad \boldsymbol{N}^\varphi = -h_* \boldsymbol{a}^{(1)} \cdot \boldsymbol{\varepsilon}_t^\varphi, \\
&\hat{\boldsymbol{M}} = (\hat{M}_{11}, \hat{M}_{12}, \hat{M}_{22})^T = \langle z \boldsymbol{\sigma}_t \rangle = \boldsymbol{M} + \boldsymbol{N}^t, \quad \boldsymbol{M} = -h_*^2 \boldsymbol{a}^{(2)} \cdot \boldsymbol{\varepsilon}_t^\varphi, \quad \boldsymbol{N}^t = h_* \boldsymbol{a}^{(1)} \cdot \boldsymbol{\varepsilon}_t^0, \\
&\boldsymbol{Q} = (Q_1, Q_2)^T = (\langle \sigma_{13} \rangle, \langle \sigma_{23} \rangle)^T.
\end{aligned} \tag{19}$$

From (8.3), (11), (17) it follows

$$Q_i = \langle \sigma_{i3} \rangle = -\left\langle z \frac{\partial \sigma_{13}}{\partial z} \right\rangle = h_* \left\langle z(L_i(\boldsymbol{\sigma}_t^0) + f_i) \right\rangle = L_i(\hat{\boldsymbol{M}}) + h_* \langle z f_i \rangle \tag{20}$$

Therefore equations (16) and (18) coincide with the corresponding Kirchhoff–Love equations, respectively

$$\begin{aligned}
&L_i(\hat{\boldsymbol{T}}) + F_i = 0, \quad i = 1, 2, \\
&\frac{\partial (A_2 Q_1)}{A_2 \partial y_1} + \frac{\partial (A_1 Q_2)}{A_1 \partial y_2} - k_1 \hat{T}_{11} - k_2 \hat{T}_{22} + F_3 = 0.
\end{aligned} \tag{21}$$

In the non-symmetric case equations (21) are complex enough. In the symmetric case $\boldsymbol{a}^{(1)} = \boldsymbol{0}$ and $\boldsymbol{N}^t = \boldsymbol{N}^\varphi = \boldsymbol{0}$, $\hat{\boldsymbol{T}} = \boldsymbol{T}$, $\hat{\boldsymbol{M}} = \boldsymbol{M}$, and the equations (21) accept the ordinary form of the Kirchhoff–Love equations.

The surface potential energy density is

$$U = \frac{1}{2} \sum_{i,j=1}^{3} \langle \sigma_{ij} \varepsilon_{ij} \rangle \tag{22}$$

and after the accepted simplifications we get approximately

$$U \approx U^0 = \frac{1}{2} \left\langle (\boldsymbol{\varepsilon}_t^0 - h_* z \boldsymbol{\varepsilon}_t^\varphi)^T \cdot \boldsymbol{A}^* \cdot (\boldsymbol{\varepsilon}_t^0 - h_* z \boldsymbol{\varepsilon}_t^\varphi) \right\rangle. \tag{23}$$

The variation of the full potential energy $\Pi = \int \int_S U^0 \, dS$ (here S is the shell mid-surface) with respect to functions $u_1^0(x_1, x_2)$, $u_2^0(x_1, x_2)$, $w^0(x_1, x_2)$ gives the equations (16), (18) or the equations (21) and the boundary conditions.

6 THE DONNELL TYPE SIMPLIFICATION

The more complex system (than the system (21)) of 10th differential order of the Timoshenko–Reissner type is obtained in [10,11], and the main stress-strain states (SSS) are described. After excluding the boundary layer the order of this system is decreased to 8, the obtained system coincides with the system (21) in the symmetric case.

Here we study the system (21) in the asymmetric case in assumptions which lead to the Donnell system. We suppose that the Lame coefficients, A_1, A_2, the curvatures, k_1, k_2, of the mid-surface, and the matrices of elasticity, $\boldsymbol{a}^{(0)}, \boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}$, $(\boldsymbol{a}^{(1)} \neq \boldsymbol{0})$ are constant, the external forces $f_1 = f_2 = 0$. These assumptions are valid for the shallow shell, for the cylindrical shell, and for the SSS with the large variability (see [7] and the relation (33) below). Under these assumptions the governing relations are

$$\begin{aligned}&\boldsymbol{\varepsilon}_t^0 = (p_1u_1 + k_1w,\, p_1u_2 + p_2u_1,\, p_2u_2 + k_2w)^T,\\&\boldsymbol{\varepsilon}_t^\varphi = (p_1^2w,\, 2p_1p_2w,\, p_2^2w)^T,\end{aligned}\qquad p_1 = \frac{\partial}{\partial y_1},\quad p_2 = \frac{\partial}{\partial y_2}. \tag{24}$$

$$\begin{aligned}&p_1\hat{T}_{11} + p_2\hat{T}_{12} = 0,\quad p_1\hat{T}_{12} + p_2\hat{T}_{22} = 0,\\&\hat{\boldsymbol{T}} = (\hat{T}_{11}, \hat{T}_{12}, \hat{T}_{22})^T = \boldsymbol{a}^{(0)}\cdot\boldsymbol{\varepsilon}_t^0 - h_*\boldsymbol{a}^{(1)}\cdot\boldsymbol{\varepsilon}_t^\varphi.\end{aligned} \tag{25}$$

$$\begin{aligned}&p_1Q_1 + p_2Q_2 - k_1\hat{T}_{11} - k_2\hat{T}_{22} + F_3 = 0,\\&Q_1 = p_1\hat{M}_{11} + p_2\hat{M}_{12},\quad Q_2 = p_1\hat{M}_{12} + p_2\hat{M}_{22} = 0,\\&\hat{\boldsymbol{M}} = (\hat{M}_{11}, \hat{M}_{12}, \hat{M}_{22})^T = h_*\boldsymbol{a}^{(1)}\cdot\boldsymbol{\varepsilon}_t^0 - h_*^2\boldsymbol{a}^{(2)}\cdot\boldsymbol{\varepsilon}_t^\varphi.\end{aligned} \tag{26}$$

The system (24)–(26) may be reduced to the system with two unknown functions, $w(x_1, x_2)$,$\Phi(x_1, x_2)$,

$$\begin{aligned}&L_4(p_1, p_2)\Phi - (k_1p_2^2 + k_2p_1^2)w + h_*\hat{L}_4(p_1, p_2)w = 0,\\&h_*\hat{N}_4(p_1, p_2)\Phi - (k_1p_2^2 + k_2p_1^2)\Phi - h_*^2N_4(p_1, p_2)w + F_3 = 0,\end{aligned} \tag{27}$$

where differential operators $L_4, \hat{L}_4, N_4, \hat{N}_4$ of forth order are

$$\begin{aligned}&L_4 = \boldsymbol{P}_1^T\cdot\boldsymbol{b}\cdot\boldsymbol{P}_1,\quad \hat{L}_4 = \boldsymbol{P}_1^T\cdot\boldsymbol{c}\cdot\boldsymbol{P}_2,\qquad \boldsymbol{b} = \left(\boldsymbol{a}^{(0)}\right)^{-1},\quad \boldsymbol{c} = \boldsymbol{b}\cdot\boldsymbol{a}^{(1)}\\&N_4 = \boldsymbol{P}_2^T\cdot\hat{\boldsymbol{a}}^{(2)}\boldsymbol{P}_2,\quad \hat{N}_4 = \boldsymbol{P}_2^T\cdot\boldsymbol{c}^T\cdot\boldsymbol{P}_1 = \hat{L}_4,\\&\hat{\boldsymbol{a}}^{(2)} = \boldsymbol{a}^{(2)} - \boldsymbol{a}^{(1)}\cdot\boldsymbol{b}\cdot\boldsymbol{a}^{(1)},\qquad \boldsymbol{P}_1 = (p_2^2,\, -p_1p_2,\, p_1^2)^T,\quad \boldsymbol{P}_2 = (p_1^2,\, 2p_1p_2,\, p_2^2)^T.\end{aligned} \tag{28}$$

Function Φ is the stress-function connected with the stress-resultants by relations

$$\hat{T}_{11} = p_2^2\Phi,\quad \hat{T}_{12} = -p_1p_2\Phi,\quad \hat{T}_{22} = p_1^2\Phi. \tag{29}$$

The first equation (27) (the so called compatibility equation) is obtained after excluding u_1 and u_2 from the relations (25). The second equation (27) is the equilibrium equation (26) after simplifications.

In the symmetric case

$$\boldsymbol{a}^{(1)} = 0,\quad \hat{L}_4 = \hat{N}_4 = 0,\quad \hat{\boldsymbol{a}}^{(2)} = \boldsymbol{a}^{(2)} \tag{30}$$

and the system (27) coincides with the earlier obtained Donnell system [22-24].

7 THE SHORT ASYMPTOTIC ANALYSIS OF THE DONNELL SYSTEM (27)

In the symmetric case the analysis of the Donnell system for anisotropic shells is carried out in [10]. Here we pay attention to the difference between the symmetric and the asymmetric cases.

Firstly, we estimate the possible values of the stiffness matrices. Assume that all elements of the matrix $\boldsymbol{A}^*(z)$ depend of z as

$$\boldsymbol{A}^*(z) = f(z)\boldsymbol{A}_0^*,\quad \langle f(z)\rangle = 1. \tag{31}$$

If the function $f(z)$ is linear then the most difference between the linear and the non-linear cases has place for $f(z) = 1 - 2z$. For this $f(z)$ we get

$$\begin{aligned} &\boldsymbol{a}^{(0)} = \boldsymbol{A}_0^*, \quad \boldsymbol{a}^{(1)} = -\frac{1}{6}\boldsymbol{a}^{(0)}, \quad \boldsymbol{b}\cdot\boldsymbol{a}^{(1)} = -\frac{1}{6}\boldsymbol{E}, \\ &\boldsymbol{a}^{(2)} = \frac{1}{12}\boldsymbol{a}^{(0)}, \quad \hat{\boldsymbol{a}}^{(2)} = \frac{1}{18}\boldsymbol{a}^{(0)} = \frac{2}{3}\boldsymbol{a}^{(2)}, \end{aligned} \tag{32}$$

where $\boldsymbol{E}$ is the unit matrix of the third order. Therefore, for a functionally graduated material with $f(z) = 1 - 2z$ the bending stiffness matrix $\hat{\boldsymbol{a}}^{(2)}$ is 2/3 times of the matrix $\boldsymbol{a}^{(2)}$ for an isotropic material.

To fulfill the asymptotic analysis of the system (27) or of the system (24–(26), and to describe the asymptotic behavior of functions at $h_* \to 0$ let us introduce partial variability indices t_1, t_2 of the state (SSS) by the relations

$$\frac{\partial Z}{\partial x_1} \sim h_*^{-t_1} Z, \quad \frac{\partial Z}{\partial x_2} \sim h_*^{-t_2} Z, \tag{33}$$

where Z is any unknown function. We refer to $t = \max(t_1, t_2)$ as to a common variability indix. From (24) it follows that

$$p_1 \sim h_*^{-t_1}, \quad p_2 \sim h_*^{-t_2}. \tag{34}$$

The asymptotic approach allows us simply to estimate the orders of the unknown variables and to construct the approximate equations and their solutions (see also ([7,10,11,25]).

For the momentless (membrane) SSS the variability of unknowns is small ($t = 0$) and with the error of the order h_* the system (27) accepts the form

$$L_4(p_1, p_2)\Phi - (k_1 p_2^2 + k_2 p_1^2) w = 0, \quad (k_1 p_2^2 + k_2 p_1^2)\Phi = F_3. \tag{35}$$

As it is clear the effects of the asymmetry are not contained in (35). Also the equations (35) may be approximately used to describe the SSS with $t < 1/2$, but the error is larger and it is of the order h_*^{1-2t}.

The differential order of the system (35) is not enough to satisfy all 4 boundary conditions at the shell edges, and it is necessary to construct the additional SSS, namely the edge effect solutions. Let us study the edge effect near the edge $x_1 = 0$, for which the variability indices are $t_1 = 1/2,\ t_2 = 0$. In this case with the error of the order $h_*^{(1/2)}$ the edge effect satisfies to the equation

$$h_*^2 (b_{33} \hat{a}_{11}^{(2)} + c_{13}^2) \frac{\partial^4 w}{\partial y^4} - 2 h_* k_2 c_{13} \frac{\partial^2 w}{\partial y^2} + k_2^2 w = 0, \tag{36}$$

where we hold only the largest elements of operators (28) and

$$\boldsymbol{b} = \{b_{ij}\}, \quad \boldsymbol{c} = \{c_{ij}\}, \quad \hat{\boldsymbol{a}}^{(2)} = \{\hat{a}_{ij}^{(2)}\}. \tag{37}$$

According to (36) the asymmetry influence on the edge effect is essential.

If $t_1 = t_2 = 1/2$ then all terms in the system (27) are of the identical order and the asymmetry influence is also essential.

For plates $k_1 = k_2 = 0$, and the equations (27) are

$$L_4(p_1, p_2)\Phi + h_* \hat{L}_4(p_1, p_2) w = 0, \qquad h_* \hat{N}_4(p_1, p_2)\Phi - h_*^2 N_4(p_1, p_2) w + F_3 = 0. \tag{38}$$

In the symmetric case (with $\hat{L}_4 = \hat{N}_4 = 0$) the equations (38) may be solved separately, but in the general (asymmetric) case these equations are connected.

8 FREE VIBRATIONS OF A SHALLOW SHELL

As an example of the Donnell equations applications we study the free transversal vibrations of a shallow shell. According to the assumptions, in which the Donnell equations are acceptable, we hold only the normal inertia forces. In equation (27) we put

$$F_3 = \Lambda w, \quad \Lambda = \langle \rho \rangle \, l^2 \omega^2 E, \tag{39}$$

where Λ is the unknown dimensionless frequency parameter, ω is the natural frequency, $\langle \rho \rangle$ is the average mass density.

We seek the solution of system (27) in the form

$$w(x_1, x_2) = w_0 \exp\{i(q_1 y_1 + q_2 y_2)\}, \quad \Phi(x_1, x_2) = \Phi_0 \exp\{i(q_1 y_1 + q_2 y_2)\}, \quad i = \sqrt{-1}. \tag{40}$$

Then equations (27) give

$$\Lambda = h_*^2 N_4(q_1, q_2) + \frac{\left(k_1 q_2^2 + k_2 q_1^2 + h_* \hat{L}_4(q_1, q_2)\right)^2}{L_4(q_1, q_2)}. \tag{41}$$

This expression (through the term $\hat{L}_4$) indicates influence the asymmetry of elastic properties on the natural frequencies. To use the expression (41) for calculations it is necessary to know the wave numbers q_1 and q_2. They may be found satisfying the boundary conditions at the shell edges. In the general case it is a very complex problem. For the rectangular shallow shell ($0 \le x_1 \le l_1, \ 0 \le x_2 \le l_2$) this problem is solved in [26] by adding to the form (40) solutions of the so-called dynamic edge effects decreasing away from the shell edges.

The expression (41) allows us to get a lot of qualitative conclusions about the dependence of natural frequencies on the shell curvatures k_1 and k_2, and on the variability indices t_1 and t_2 of the form (40) ($q_i \sim h_*^{-t_i}$).

For the circular cylindrical shell we take $q_2 = m$ (m is integer).

The equations (27) and the expression (41) do not describe the tangential vibrations, and frequencies of these vibrations are of the order $\Lambda \sim 1$. That is why we discuss the lowest part of the shell spectrum containing frequencies with $\Lambda << 1$.

The functions $L_4(q_1, q_2)$ and $N_4(q_1, q_2)$ are positively defined. Taking into account that all elements of matrices $\hat{\boldsymbol{a}}^{(2)}, \ \boldsymbol{b}, \ \boldsymbol{c}$ are of the order of 1 we deliver from the expression (41) the following estimate

$$\Lambda \sim h_*^{2-4t} + \left((k_1 h_*^{-2t_2} + k_2 h_*^{-2t_1}) h_*^{2t} + j_c h_*\right)^2, \qquad t = \max(t_1, t_2), \quad t_1 \ge 0, \ t_2 \ge 0, \tag{42}$$

where $j_c = 0$ in the symmetric case and $j = 1$ in the asymmetric case.

From (42) it follows that

— inequality $\Lambda \ll 1$ is fulfilled only if $t < 1/2$,
— for the cylindrical shell (with $k_1 = 0$) $\min \Lambda \sim h_* k_2$ at $t_1 = 0, \ t_2 = 1/4$,
— for the shell with negative Gaussian curvature (with $k_1 k_2 < 0$) $\min \Lambda \sim h_*^{2/3}$ at $t_1 = t_2 = 1/3$,
— for the convex shell (with $k_1 k_2 < 0$) inequality $\Lambda \ll 1$ is possible only if $k_1^2 + k_2^2 \ll 1$.

For shells made of isotropic homogeneous material the similar conclusions are obtained in [27]. In the accepted assumptions the anisotropy and the asymmetry of material effects on the value of Λ, but its asymptotic order at $h_* \to 0$ remains the same.

9 BUCKLING OF THE CIRCULAR CYLINDRICAL SHELL

The same method may be applied in study of the buckling of shallow shell. Consider for example the circular cylindrical shell under normal pressure. For this case, we put in equation (27)

$$F_3 = \Lambda p_2^2 w, \tag{43}$$

where Λ is the unknown loading parameter.

For the cylindrical shell the curvilinear coordinates x_1, x_2 coincide with the arc lengths in the longitudinal and circumferential directions, $0 \leq x_2 \leq 2\pi$ and $k_1 = 0, k_2 = 1$. For the cylindrical shell the SSS is described with the variability indices $t_1 = 0,\ t_2 = 1/4$ (semi-momentless SSS).

We seek the solution of system (27) in the form

$$w(x_1,x_2) = w(y_1)\exp\{imy_2\}, \quad \Phi(x_1,x_2) = \Phi(y_1)\exp\{imy_2\}, \quad i = \sqrt{-1}. \tag{44}$$

where m is integer.

Since $t_2 = 1/4$, we will seek the solutions $w(y_1), \Phi(y_1)$ and Λ as an asymptotic series of a small parameter $\mu = h_*^{1/4}$.

Let us assume, for example, that the boundary conditions are of clamped edge

$$w = u_1 = u_2 = \gamma = 0, \tag{45}$$

where γ is average angle of the normal fibers rotation.

In zeroth-order approximation, equations (27) give

$$p_1^4 w_0 - r^4 w_0 = 0, \quad r^4 = m^4 b_{11}\left(m^2\Lambda_0 - m^4 h_*^2 \hat{a}_{33}^{(2)}\right), \tag{46}$$

with boundary conditions

$$w_0 = p_1 w_0 = 0, \quad y_1 = \pm 1/2. \tag{47}$$

For $r^* \approx 4.73$ equation (46) with boundary conditions (47) has a non-zero solution

$$w_0(y_1) = \frac{\cos(ry_1)}{\cos(0.5r)} - \frac{\cosh(ry_1)}{\cosh(0.5r)}. \tag{48}$$

Opposed to the free vibrations case, we are interested in only finding minimal loading parameter. The expression for Λ_0 and minimal m can be derived from (46):

$$\Lambda_0 = \min_m \frac{1}{m^6}\left(\frac{r^4}{b_{11}} + m^4 h_*^2 \hat{a}_{33}^{(2)}\right).$$

As it was said earlier, the solution of equation (46) might not satisfy all 4 boundary conditions because the differential order of equation (46) is less than order of the system (27). Nonetheless, solution (48) satisfies the particular boundary conditions (45).

The equation in first approximation is

$$p_1^4 w_1 - r^4 w_1 = \Lambda_1 m^2 b_{11} w_0 + 2irm^7\left(\hat{a}_{23}^{(2)} b_{11} - \hat{a}_{33}^{(2)} b_{12} + \frac{\Lambda_0}{m^5} b_{12}\right) p_1 w_0 \tag{49}$$

with following boundary conditions

$$w_1 = 0, \quad p_1 w_1 = i\frac{b_{12}}{m^3 b_{11}} p_1^2 w_0, \quad y_1 = \pm 1/2.$$

Here, for equation (49) to have a solution, Λ_1 should satisfy the compatibility condition:

$$\int_{-1/2}^{1/2} \left(\Lambda_1 m^2 b_{11} w_0 + 2irm^7 \left(\hat{a}_{23}^{(2)} b_{11} - \hat{a}_{33}^{(2)} b_{12} + \frac{\Lambda_0}{m^5} b_{12} \right) p_1 w_0 \right) w_0 dy_1 = 0. \tag{50}$$

It is easy to see that $\Lambda_1 = 0$.

The solution $w_1(y_1)$ can be written in form

$$w_1 = i \left(\frac{\alpha_1 y_1}{4r^4} p_1^2 w_0 + \alpha_2 p_1 w_0 + \alpha_3 p_1^3 w_0 \right), \tag{51}$$

where

$$\alpha_1 = 2rm^7 \left(\hat{a}_{23}^{(2)} b_{11} - \hat{a}_{33}^{(2)} b_{12} + \frac{\Lambda_0}{m^5} b_{12} \right),$$

$$\alpha_2 = \frac{\alpha_1 \left(16r^4 - 4 + r\tan\left(r/2\right) - r\tanh\left(r/2\right)\right)}{16r^4}, \alpha_3 = \frac{\alpha_1}{4r^5\left(\tan\left(r/2\right) - \tanh\left(r/2\right)\right)}.$$

Let us examine the material with elastic moduli of symmetric case discussed in [10] with the following matrix A_0^*:

$$A_0^* = \begin{pmatrix} 1.0577 & 0.0185 & 0.3056 \\ 0.0185 & 0.3903 & -0.0103 \\ 0.3056 & -0.0103 & 1.0482 \end{pmatrix}. \tag{52}$$

For the cylindrical shell of the relative thickness $h_* = 0.01$ we will have the loading parameter $\Lambda \approx 4.388 \times 10^{-4}$ with number of half-waves in circumferential direction $m = 8$.

On figure 2 the buckling form of first-order approximation is presented. Unlike the isotropic case, the buckling form has an incline on circumferential direction.

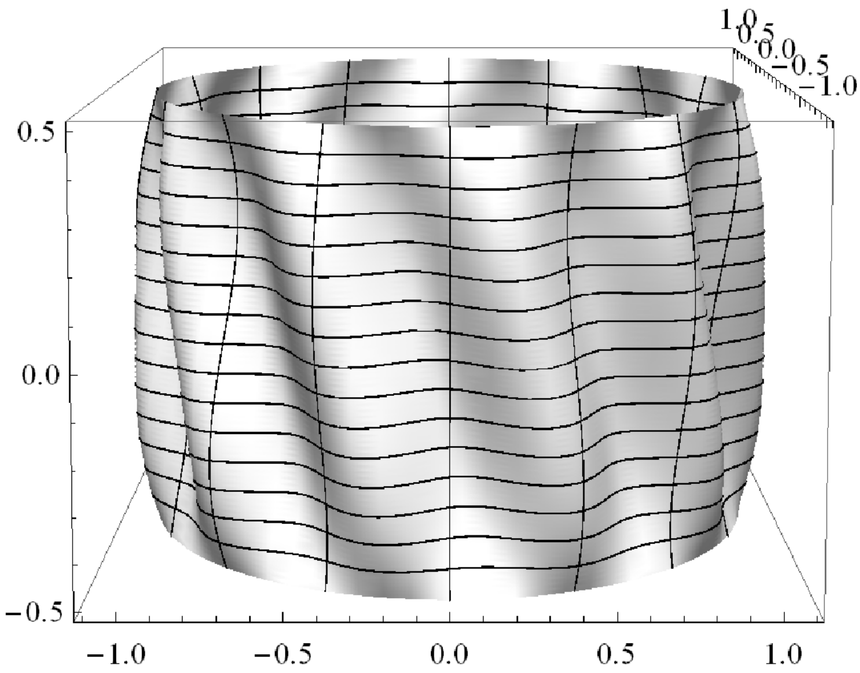

Figure 2: The buckling form of the cylindrical shell.

10 CONCLUSIONS

- For thin anisotropic heterogeneous in the thickness direction shell the 2D model is built. To deliver this model the asymptotic expansions of the 3D equations of elasticity in powers of the relative shell thickness h_* are used. As a result in zeroth-order approximation the system of the 8th differential order is obtained. This approximation is based on the approximate elasticity relations $\boldsymbol{\sigma}_t = \boldsymbol{A}_* \cdot \boldsymbol{\varepsilon}_t$ which establish the dependence between the tangential stresses $\boldsymbol{\sigma}_t$ and strains $\boldsymbol{\varepsilon}_t$. These relations contain 6 effective elastic moduli while the initial elasticity relations contain 21 elastic moduli.

- The same governing system may be obtained also by two alternative ways. One of them consists of the direct using of the KL hypotheses in combination with the mentioned approximate elasticity relation, and then of the averaging the result in the thickness direction z. In the other way we introduce the approximate elastic energy density $\boldsymbol{\sigma}_t(z) \cdot \boldsymbol{\varepsilon}_t(z)/2$ based on the approximate elasticity relation. After calculating the variation of energy in u_1, u_2, w and after averaging in z we get the equilibrium equations and the boundary conditions.

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