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CALCULATING THE FREQUENCY OF AN ACTUAL STRUCTURE USING THE RAYLEIGH TECHNIQUE

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Abstract. *Beams and columns constitute a continuous system with infinite degrees of freedom. To study the behavior of these systems the use of discretization techniques is required. However, one can associate them to a system with a single degree of freedom, restricting the form in which the system will deform and write their properties as a function of the generalized coordinate. This technique is the called Rayleigh Method and is a resource largely used to study the vibration of elastic systems. However, actual structures are systems more complex than simple beams and columns, because they have properties varying along its length. In such cases, the use of the Rayleigh Method should be done by parts and its integrals resolved within the limits established for each interval, being the generalized properties calculated for each segment of the structure. The actual structure selected for this study is a metallic high slenderness pole for which the frequency of the first vibration mode was calculated analytically, and as well as by a FEM - Finite Element Method model based computer for comparative purposes, getting a very good result.*

1 INTRODUCTION

The vibration analysis response of framed structures modeled as beams and columns have been studied by many researchers and continue to be treated extensively in the literature. Beams and columns constitute a continuous system with infinite degrees of freedom. To study the behavior of these systems the use of discretization techniques is required, in which the structure is transformed into subsystems defined by points called joints. However, one can associate them to a system with a single degree of freedom, restricting the form in which the system will deform and write their properties as a function of the generalized coordinate. This technique was used by Rayleigh [1] in the study of vibration of elastic systems, being its equations valid in the whole domain of the problem. Leissa [2] claims that the precision obtained through this method depends entirely on the functional form that is used to represent the free vibration mode.

However, actual structures are systems more complex than simple beams and columns, because their properties vary along its length. In such cases, the use of the Rayleigh method should be done by parts and its integrals resolved within the limits established for each interval, being the generalized properties calculated for each segment of the structure, paying special attention to the geometric stiffness because each parcel must take into account the normal force distributed in its range and efforts that operate in the upper segments. The actual structure selected for this study is a metallic high slenderness pole for which the frequency of the first vibration mode was calculated as described above, and a computer model was developed using the Finite Element Method (FEM) for comparative purposes, getting a very good result, hence its importance can be realized independently of sophisticated computational programs.

2 A QUICK OVERVIEW ABOUT THE MATHEMATICS CONSIDERATIONS

The analytical formulation developed in this paper is based on the principle of virtual work combined with a technique similar to that of Rayleigh [1]. Rayleigh assumed that a system containing infinite degrees of freedom can be replaced by a finite SDOF (single degree of freedom) system that approximates their frequency. Applications of the Rayleigh technique to mechanical systems with vibration problems are found in a wide range of scientific studies. It is important to observe that the technique developed by Rayleigh and presented in his first book was only used to calculate the fundamental frequency. Leissa [2] claims that the precision obtained through this method depends entirely on the functional form that is used to represent the free vibration mode. If the exact shape were assumed, the exact corresponding frequency would be generated by this method.

The basic concept behind the Rayleigh method is the principle of conservation of energy in mechanical systems, therefore it is applicable to linear and nonlinear structures, according to Clough [3]. According to Temple [4], the fundamental principles developed by Rayleigh are applied both to systems with finite degrees of freedom and to continuous systems. The purpose is to determine the fundamental period of vibration and to analyze the stability of the elastic systems with the precision required for engineering problems. To do this, the Virtual Works Principles must be described by adequate chosen of generalized coordinates and by a functional form that describes the first mode of vibration. At the end of the calculation, the movement equation is written in terms of the generalized coordinate, from which one can extract the generalized elastic and geometric properties of the system.

Assuming that the well-known trigonometric function

$$\phi(x) = 1 - \cos\left(\frac{\pi x}{2L}\right) \quad (1)$$

one finds the elastic stiffness,

$$K_0 = \int_0^L EI\left(\frac{d^2\phi(x)}{dx^2}\right)^2 dx \tag{2}$$

with *E* being the elasticity modulus and *I* the inertia of the section. The geometrical stiffness is

$$K_g = \int_0^L N(x)\left(\frac{d\phi(x)}{dx}\right)^2 dx \tag{3}$$

where *N(x)* is a function of normal force. The total generalized mass, for their turn is

$$M = m_0 + m \tag{4}$$

where m_0 is the lumped mass at the element joint and

$$m = \int_0^L \bar{m}\left(\phi(x)\right)^2 \tag{5}$$

with $\bar{m}$ the mass per unit length.

Finally, the natural cyclical frequency should be calculated doing

$$\omega = \sqrt{\frac{K}{\Pi}} \tag{6}$$

taking into account that

$$K = K_0 - K_g \tag{7}$$

For a more detailed information about the development this analytical procedure Wahrhaftig *et all* [5] can be consulted.

3 ACTUAL STRUCTURE CHARACTERISTICS

The actual structure selected for this study is a metallic high slenderness pole, showed in Figure 1 and Figure 2. The geometric details also can be seen in Figure 2, where *t* means the thickness of the wall of each segment of the structure.

This is a metal pole for supporting the radiating system of cellular mobile telephone signal. The structure has 48 meters of height and has hollow circular section of external diameter (ϕ_{ext}) and thickness (*t*) variables, having slenderness equal to 310.

Table 1 presents the structure proprieties and model discretization. Table 2 shows the structural parameters and the existent devices on structure.

Figure 1: Actual structure.

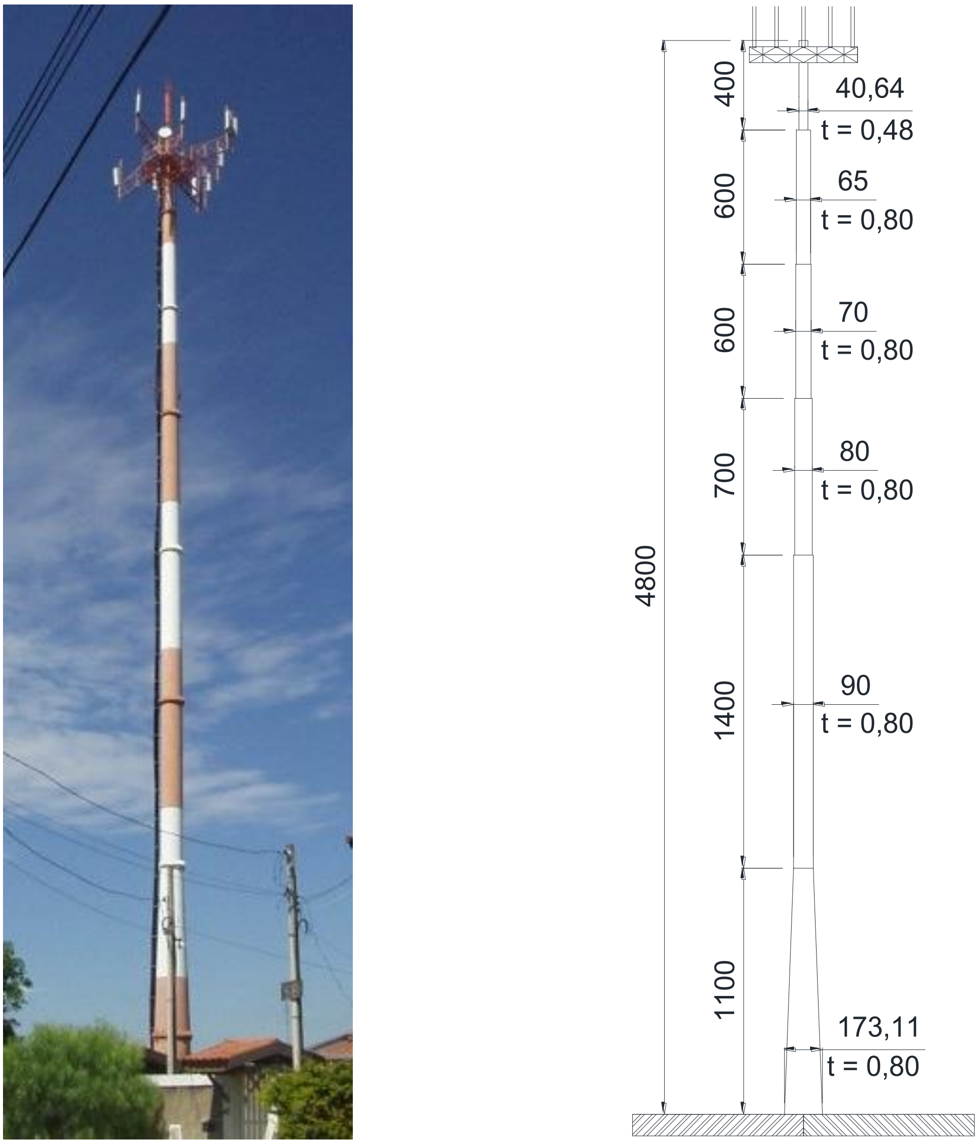


Figure 2: A slender metallic pole – geometric details.

Height (m)	ϕ_{ext} (cm)	t (cm)	Height (m)	ϕ_{ext} (cm)	t (cm)	Height (m)	ϕ_{ext} (cm)	t (cm)	Height (m)	ϕ_{ext} (cm)	t (cm)
48.00	40.64	0.48	30.00	80.00	0,80	20.00	90.00	0.80	10.00	97.56	0.80
46.00	40.64	0.48	29.00	80.00	0,80	19.00	90.00	0.80	9.00	105.11	0.80
44.00	40.64	0.48	28.00	80.00	0,80	18.00	90.00	0.80	8.00	112.67	0.80
42.00	65.00	0.80	27.00	80.00	0,80	17.00	90.00	0.80	7.00	120.22	0.80
40.00	65.00	0.80	26.00	80.00	0,80	16.00	90.00	0.80	6.00	127.78	0.80
38.00	65.00	0.80	25.00	80.00	0,80	15.00	90.00	0.80	5.00	135.33	0.80
36.00	70.00	0.80	24.00	90.00	0,80	14.00	90.00	0.80	4.00	142.89	0.80
34.00	70.00	0.80	23.00	90.00	0,80	13.00	90.00	0.80	3.00	150.44	0.80
32.00	70.00	0.80	22.00	90.00	0,80	12.00	90.00	0.80	2.00	158.00	0.80
31.00	80.00	0.80	21.00	90.00	0,80	11.00	90.00	0.80	1.00	165.56	0.80
									0.00	173.11	0.80

Table 1: Structural geometry and discretization of the FEM model.

Device	Height	Weight and distributed weight
Pole	from 0 to 48 m	7850 kN/m^3
Stair	from 0 to 48 m	0.15 kN/m
Cables	from 0 to 48 m	0.25 kN/m
Antenna and Supports	48 m	3.36 kN

Table 2: Devices and weighs on structure.

4 SIMULATIONS AND RESULTS

4.1 Analytical procedure

4.1.1. Geometric definitions and parameters adopted

The parameters used for application of analytical procedure using the Rayleigh Method are the follows:

- elasticity modulus of steel: E = 205 GPa,
- density of steel: ρ = 7850 kg/m^3,
- lumped mass on the top: m_0 = 342.40 kg,
- distributed mass per unity height: m_e = 40 kg/m,
- acceleration of gravity: g = 9.806650 m/s^2.

The corresponding order the heights in the structure and the geometric properties of the cross sections of the respective segments are given by:

On the base to $x = 0$, one has: $D_1 = 173.11\text{cm}$, $t_1 = 0.80\text{cm}$, $d_1 = D_1 - 2t_1$, $A_1 = \frac{\pi}{4}\left(D_1^2 - d_1^2\right)$, $I_1 = \frac{\pi}{64}\left(D_1^4 - d_1^4\right)$.

To the follow segment, of variable proprieties, one has: $D(x) = \frac{D_2 - D_1}{L_1} x + D_1$, $d(x) = D(x) - 2t_1$, $A(x) = \frac{\pi}{4}\left(D(x)^2 - d_1(x)^2\right)$, $I(x) = \frac{\pi}{64}\left(D(x)^4 - d(x)^4\right)$.

To the ordinate $L_1 = 11\text{m}$, one defines: $D_2 = 90.00\text{ cm}$, $t_2 = 0.80\text{cm}$, $d_2 = D_2 - 2t_2$, $A_2 = \frac{\pi}{4}\left(D_2^2 - d_2^2\right)$, $I_2 = \frac{\pi}{64}\left(D_2^4 - d_2^4\right)$.

To $L_2 = 25.00\text{m}$, one has: $D_3 = 80.00\text{ cm}$, $t_3 = 0.80\text{cm}$, $d_3 = D_3 - 2t_3$, $A_3 = \frac{\pi}{4}\left(D_3^2 - d_3^2\right)$, $I_3 = \frac{\pi}{64}\left(D_3^4 - d_3^4\right)$.

To $L_3 = 32.00\text{m}$, one has: $D_4 = 70.00\text{ cm}$, $t_4 = 0.80\text{cm}$, $d_4 = D_4 - 2t_4$, $A_4 = \frac{\pi}{4}\left(D_4^2 - d_4^2\right)$, $I_4 = \frac{\pi}{64}\left(D_4^4 - d_4^4\right)$.

To $L_4 = 38.00\text{m}$, one has: $D_5 = 65.00\text{ cm}$, $t_5 = 0.80\text{cm}$, $d_5 = D_5 - 2t_5$, $A_5 = \frac{\pi}{4}\left(D_5^2 - d_5^2\right)$, $I_5 = \frac{\pi}{64}\left(D_5^4 - d_5^4\right)$.

To $L_5 = 44.00\text{m}$ and $L_6 = 48.00\text{m}$, one has: $D_6 = 40.64\text{cm}$, $t_6 = 0.48\text{cm}$, $d_6 = D_6 - 2t_6$, $A_6 = \frac{\pi}{4}\left(D_6^2 - d_6^2\right)$, $I_6 = \frac{\pi}{64}\left(D_5^4 - d_5^4\right)$.

4.1.2. The calculation of generalized mass

The sub-indices in Roman numerals, introduced from this point on, aim to prevent redundancy of notation in expressions. The generalized mass was measured by the following integral arranged.

To the first segment

$$m_1 = \int_0^{L_1} m_I(x)\phi(x)^2 dx \text{, with } m_I(x) = A(x)\rho + m_e \tag{8}$$

To the second segment

$$m_2 = \int_{L_1}^{L_2} m_{II}\phi(x)^2 dx \text{, with } m_{II} = A_2\rho + m_e \tag{9}$$

Similarly, to the other segments, can be written in the general form

$$m_i = \int_{L_{i-1}}^{L_i} m_i\phi(x)^2 dx \text{, with } m_i = A_i\rho + m_e \tag{10}$$

The generalized distributed mass is obtained by

$$m_R = \sum_{i=1}^{6} m_i \tag{11}$$

and the total generalized mass by:

$$M = m_0 + m_R \tag{12}$$

4.1.3. The calculation of generalized geometric stiffness

To compute the generalized geometric stiffness is necessary to determine the normal efforts for the parts defined in geometry. From top to bottom the structure of the axial forces are

$$F_0 = m_0 g \tag{13}$$

$$F_6 = \int_{L_5}^{L} m_{VI} g dx \tag{14}$$

$$F_5 = \int_{L_4}^{L_5} m_V g dx \tag{15}$$

and so on, or better

$$F_i = \int_{L_{i-1}}^{L_i} m_i g dx \tag{16}$$

The axial force on the first segment, which is linearly variable, was obtained by the following expression

$$F_1 = \int_{0}^{L_1} m_I(x) g dx \tag{17}$$

where *g* is de gravity acceleration. The generalized axial force *F* is so

$$F = \sum_{i=0}^{6} F_i \tag{18}$$

and the geometric stiffness is calculated following the expressions

$$\begin{aligned}
K_{g6} &= \int_{L_5}^{L} \left[F_0 + m_{VI}(L_6 - x) g \left(\frac{d}{dx} \phi(x) \right)^2 \right], \\
K_{g5} &= \int_{L_4}^{L_5} \left[F_0 + F_6 + m_V(L_5 - x) g \left(\frac{d}{dx} \phi(x) \right)^2 \right], \\
K_{g4} &= \int_{L_3}^{L_4} \left[F_0 + F_6 + F_5 + m_{IV}(L_4 - x) g \left(\frac{d}{dx} \phi(x) \right)^2 \right], \\
K_{g3} &= \int_{L_2}^{L_3} \left[F_0 + F_6 + F_5 + F_4 + m_{III}(L_3 - x) g \left(\frac{d}{dx} \phi(x) \right)^2 \right], \\
K_{g2} &= \int_{L_1}^{L_2} \left[F_0 + F_6 + F_5 + F_4 + F_3 + m_{II}(L_2 - x) g \left(\frac{d}{dx} \phi(x) \right)^2 \right], \\
K_{g1} &= \int_{0}^{L_1} \left[F_0 + F_6 + F_5 + F_4 + F_3 + F_2 + m_I(x)(L_1 - x) g \left(\frac{d}{dx} \phi(x) \right)^2 \right].
\end{aligned} \tag{19}$$

So that, the generalized geometric stiffness K_g of structure is obtained by

$$K_g = \sum_{i=1}^{6} K_{gi} \tag{20}$$

4.1.4. The calculation of generalized elastic stiffness

The elastic geometric parcels are

$$K_{01} = \int_{0}^{L_1} EI(x) \left(\frac{d^2}{dx^2} \phi(x) \right)^2 dx \tag{21}$$

$$K_{02} = \int_{L_1}^{L_2} EI_2 \left(\frac{d^2}{dx^2} \phi(x) \right)^2 dx \tag{22}$$

which in the same way, to the others segments, can be written as

$$K_{0i} = \int_{L_{i-1}}^{L_i} EI_i \left(\frac{d^2}{dx^2} \phi(x) \right)^2 dx \tag{23}$$

and the generalized elastic stiffness, is given by the sum of their parcels, so

$$K_0 = \sum_{i=1}^{6} K_{0i} \tag{24}$$

4.1.5. The calculation of frequency

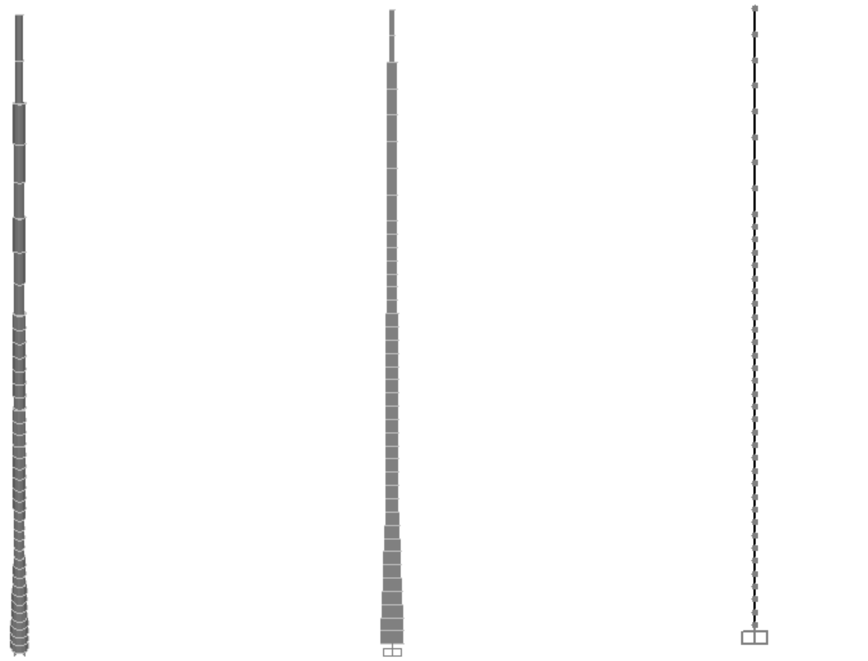
The frequency of the first mode calculated using Eq. (6), transformed to Hertz, is 0.569646 Hz.

4.2 Finite Element Method

To compare the results development above was elaborated a computational modeling by Finite Element Method (FEM). The vibration mode and frequency obtained for that method is listed in Figure 5.

The structure analyzed was modeled using bar elements with constant and variable cross-sections, as appropriate. The model was awarded the forces described in Table 2, with the corresponding masses. Figure 3 shows the three-dimensional model available by the program and the discretization of the built structure with 40 bar elements.

The frequency by FEM was calculated taking into account the effect of normal force, using the correspondent formulation for them, the named geometric nonlinearity (GNL). For more details about the specific formulations in Finite Element Method Clough [3], Cook [6], Bucalem [7] and Zienkiewicz [8] can be consulted.



(a) 3D (b) Lateral view (c) Discretization

Figure 3: Model by Finite Element Method (FEM).

The form and frequency to the first mode vibration by Finite Element Method is presented in Figure 4.

Finite Element Method
(FEM)
with geometric nonlinearity (GNL)

Frequency of the 1st Mode:
0.489339 Hz

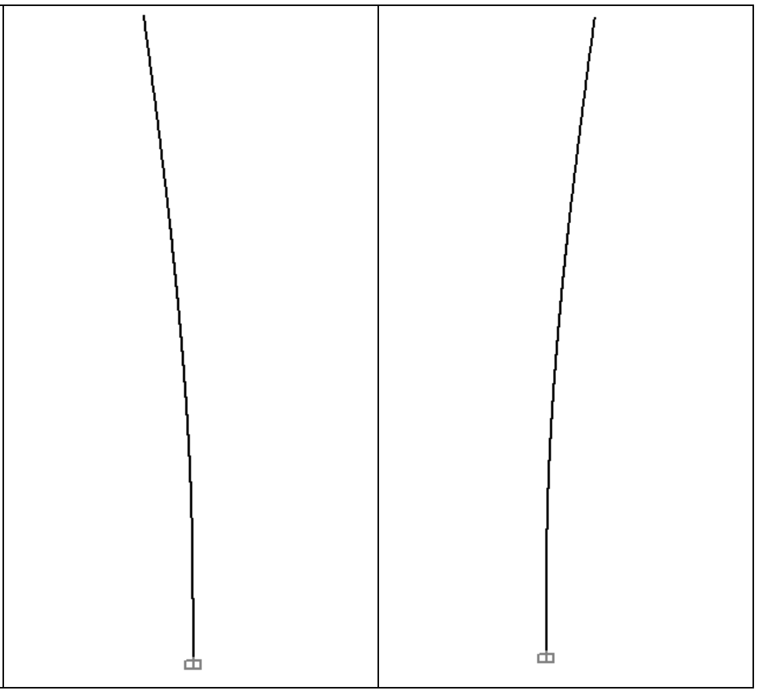


Figure 4: Natural mode and frequencies by Finite Element Method.

A comparison which assesses the quality of analytical results of the development of this work is the appreciation of normal force on each segment of the structure, as shown in Table 3.

L (m)	Analytical procedure (kN)	FEM (kN)	Differences	
			Absolute	(%)
48.00	3.355520	3.355520	0.0000	0.000000
44.00	6.786842	6.786842	0.0000	-0.000001
38.00	16.585633	16.585633	0.0000	-0.000001
32.00	26.964392	26.964392	0.0000	-0.000001
25.00	40.426203	40.426204	0.0000	-0.000002
11.00	70.056344	70.056345	0.0000	-0.000002
0.00	102.174047	102.174049	0.0000	-0.000002

Table 3: Normal effort on structure.

5 CONCLUSIONS

- The formulation for calculating the geometric stiffness of the present study was reviewed by the Finite Element Method (FEM), for comparing the results of axial forces on each segment of the structure (Table 3), without difference between them.
- The difference in relation to the vibration frequency of the fundamental mode, calculated under geometric nonlinearity, proposed for this paper, of 0.569646 Hz and the obtained by the Finite Element Method, of 0.489339 Hz, is different to 16 % upper.
- This paper does not terminate the studies for the application of analytical procedure based in the Rayleigh Method to actual structures. That method can allow the calculation of frequency without the use of sophisticated computer resources, in a correct way and within the limits required by engineering.

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