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DYNAMIC ANALYSIS OF GRADIENT ELASTIC BEAMS BY FINITE ELEMENTS

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Abstract. *The dynamic stiffness matrix of a gradient elastic flexural Bernoulli-Euler beam finite element is analytically constructed with the aid of the basic and governing equations of motion in the frequency domain. The flexural element has one node at every end with three degrees of freedom per node, i.e., the displacement, the slope and the curvature. Use of this dynamic stiffness matrix for a plane system of beams enables one through a finite element analysis to determine its dynamic response to harmonically varying with time external load or the natural frequencies and modal shapes of that system. The response to transient loading is obtained with the aid of Laplace transform with respect to time and the numerical inversion of the transformed solution. Because the exact solution of the governing equation of motion in the frequency domain is used as the displacement function, the resulting dynamic stiffness matrices and the obtained structural responses or natural frequencies and modal shapes are also exact. Two examples are presented to illustrate the method.*

1. INTRODUCTION

Micro-electromechanical systems (MEMS) and nano-electromechanical systems (NEMS), usually modeled by liner elastic bars, beams, plates and shells, have extremely small dimensions and hence their mechanical behavior is significantly affected by their microstructure. Microstructural effects cannot be taken into account by the classical theory of elasticity and one should use generalized or higher-order elasticity theories. These theories are characterized by non-locality of stress and internal length parameters and can take into account microstructural effects in a macroscopic manner.

Among those theories, Mindlin's [1] general theory of elasticity with microstructure and in particular his form II theory associated with the second gradient of strain and consisting of just one constant (internal length) in addition to the other two classical elastic constants, has found many applications in structural analysis of microstructures. This simple theory, usually known as the gradient theory of elasticity, has been successfully used during the last 15 years or so to solve a variety of static and dynamic problem by analytical and numerical methods, as described, e.g., in the recent review article of Tsinoloulos et al [2].

Most of the works on static and dynamic analysis of gradient elastic structures have been devoted to beams. One can mention here the works of Chang Gao [3], Papargyri-Beskou et al [4, 5], Lam et al [6], Giannakopoulos and Stamoulis [7], Kong et al [8] and Papargyri-Beskou and Beskos [9] on gradient elastic Bernoulli- Euler beams and Papargyri- Beskou et al [10], Wang et al [11] Akgoz and Civalek [12], and Triantafyllou and Giannakopoulos [13] on gradient elastic Timoshenko beams. In all these works, the analysis was done by analytic methods and the beams were simple, statically determinate and under simple type of loading.

Problems of static and dynamic analysis of gradient elastic beams and beam structures involving statical indeterminancy, complex type of loading and variable beam cross-sections, cannot be practically solved by analytical methods. Only numerical methods of solution, such as the finite element method (FEM), can be efficiently used for the above type of problems.

Pegios et al [14] have very recently developed a FEM for the static and stability analyses of gradient elastic beams and beam structures by analytically constructing exact element stiffness matrices on the basis of displacement functions, which are the exact solutions of the governing equations of the static and stability problems, as described in Papargyri-Beskou et al [4]. Because use is made of exact stiffness matrices, the structural response is also exact and one can model every beam member by only one finite element, Asiminas and Koumousis [15] have recently constructed stiffness and consistent mass matrices for Bernoulli- Euler beam elements on the basis of the gradient elastic theory of Papargyri-Beskou et al [4] to study static and free vibration problems of simple beams. Their displacement function was the cubic polynomial solution of the classical theory and as a result of that their stiffness and mass matrices are approximate. Kahrobaiyan et al [16] and Zhang et al [17] have also recently constructed stiffness and consistent mass matrices for Bernoulli-Euler and

Timoshenko beam elements, respectively, on the basis of the gradient elastic theory of Lam et al [6]. The displacement function used by Kahrobaiyan et al [16] is the exact solution of the governing equation of equilibrium, while that by Zang et al [17] the classical cube polynomial. As a result of the above, Kahrobaiyan et al [16] have exact stiffness and approximate mass matrices, while Zang et al [17] have both approximate stiffness and mass matrices. Approximate stiffness and/or mass matrices in FEM imply discretization of every member of the structure into 2-4 finite elements for acceptable accuracy.

In this work, the dynamic stiffness matrix of a gradient elastic Bernoulli-Euler beam finite element is analytically / numerically constructed with the aid of the basic and governing equation of flexural motion of that element and its associated boundary conditions as described in Papargyri-Beskou et al [5] in frequency domain. Because the exact solution of the governing equation of motion in the frequency domain is used as the displacement function, the resulting dynamic stiffness matrices and hence the dynamic response to zero or harmonically varying with time forces is the exact one. When the external forces are general transient, the element dynamic stiffness is defined in the transformed with respect to time domain. The problem is then formulated and solved in the transformed domain and the time domain response is finally obtained by a numerical inversion of the transformed solution as described for classical beam structures in Beskos and Narayanam [18]. Thus, the present method is capable of providing the exact solution for both free and forced vibration problems involving gradient elastic beams, in contrast to all the existing methods, which can only provide approximate solutions. Of course, the expressions used in deriving dynamic stiffness matrices are more complicated here than in the other works. However, the discretization here involves only one finite element per physical member instead of 2-4 elements per physical member in all the other methods.

Two examples are presented to illustrate the method and demonstrate its advantages. These examples deal with free and forced vibrations of a gradient elastic cantilever beam.

2. GRADIENT ELASTIC BEAM THEORY

The basic and governing equations of a gradient elastic Bernoulli-Euler beam in bending under dynamic lateral loading as well as the associated classical and nonclassical boundary conditions derived in Papargyri-Beskou et al [5] are reproduced in this section for reasons of completeness and easy reference.

Consider a straight prismatic beam under a dynamic lateral load q(x,t) distributed along the longitudinal axis of the beam, as shown in Fig. 1, where t denotes time. Thus, the loading plane is x-y and the cross-section A of the beam is characterized by the two axes y and z with the former one being its axis of symmetry. Under the lateral load q(x,t), the beam experiences bending vibrations in the x-y plane measured by its lateral deflection u(x,t) along the x axis. Assuming gradient elastic material behavior, one has that the normal to the cross-section

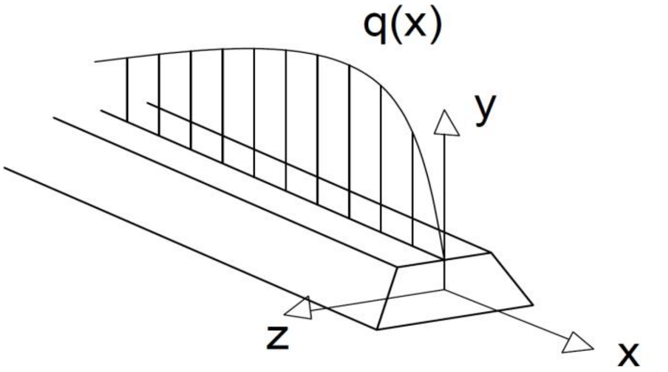

Fig. 1 Geometry and loading of a prismatic beam in bending

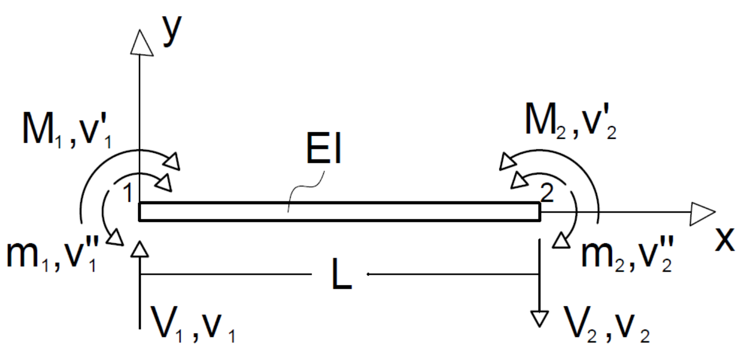

Fig. 2 Mechanics convention for generalized nodal forces and displacements of a gradient elastic flexural beam element in frequency domain.

bending stress σ_x has the form

$$\sigma_x = E\left(\varepsilon_x - g^2 \frac{\partial^2 \varepsilon_x}{\partial x^2}\right) \tag{1}$$

where E is the modulus of elasticity, g is the gradient coefficient with dimensions of length (internal length representing the microstructural effects macroscopically) and ε_x is the normal bending strain expressed as

$$\varepsilon_x = -y \frac{\partial^2 v}{\partial x^2} \tag{2}$$

Utilizing the kinematics of the Bernoulli-Euler theory, the constitutive relation (1) and the dynamic equilibrium of axial forces and bending moments, one can finally obtain the governing equations of dynamic equilibrium for the gradient elastic beam in terms of the lateral deflection u(x) as [5]

$$EI\left(\frac{\partial^4 v}{\partial x^4} - g^2 \frac{\partial^6 v}{\partial x^6}\right) + \mu \frac{\partial^2 v}{\partial t^2} = -q(x,t) \tag{3}$$

where I is the cross-sectional moment of inertia about the z axis and μ the mass per unit length of the beam. The above equation, which is of the sixth degree with respect to x, reduces to the classical one of the fourth degree for g=0.

Assuming that the lateral load q(x,t) varies harmonically with time in the form

$$q(\mathrm{x},\mathrm{t}) = \tilde{q}(\mathrm{x})\,\mathrm{e}^{i\omega t} \tag{4}$$

one has that the deflection $\upsilon(\mathrm{x},\mathrm{t})$ also varies harmonically with time in the form

$$\upsilon(\mathrm{x},\mathrm{t}) = v(\mathrm{x})\,\mathrm{e}^{i\omega t} \tag{5}$$

where $\tilde{q}(\mathrm{x})$ and v(x) represent amplitudes, ω is the circular vibration frequency and $i = \sqrt{-1}$. On account of Eqs (4) and (5), Eq.(3) takes the form

$$v^{IV} - g^2 v^{VI} - a^2 v = \tilde{q} / EI \tag{6}$$

Where primes and overdots indicate differentiation with respect to x and t, respectively and

$$a^2 = \mu\omega^2 / EI \tag{7}$$

Equation (6) represents the governing equation of lateral motion in the frequency domain.

If one considers a beam element of length L with its two ends defined by x=0 and x=L, as shown in Fig.2, and makes use of a variational statement, he can recover the governing equation (6) and all possible classical and non-classical boundary conditions so as to satisfy the following equations [5]:

$$[V(L) - EI[v'''(L) - g^2 v^{V}(L)]]\delta v(L) - [V(0) - EI[v'''(0) - g^2 v^{V}(0)]]\delta v(0) = 0$$

$$[M(L) - EI[v''(L) - g^2 v^{IV}(L)]]\delta v'(L) - [M(0) - EI[v''(0) - g^2 v^{IV}(0)]]\delta v'(0) = 0$$

$$[m(L) - EIv'''(L)]\delta v''(L) - [m(0) - EIg^2 v'''(0)]\delta v''(0) = 0 \tag{8}$$

In the above, V is the shear force, M is the bending moment and m is the double moment due to the microstructure, while primes denote derivatives with respect to x. The forces and moments are considered positive as shown in Fig. 2. It is observed that for g=0 Eqs (8) reduce to the corresponding ones for the classical case.

In view of Eqs (8) one can observe that, when dealing with the classical boundary conditions, either the deflection v or the shear forces $V=EI(v'''-g^2v^{V})$ and the strain (slope) v' or the bending moments $M=EI(v''-g^2v^{IV})$ at the boundary of the beam have to be specified. For the case of the non-classical boundary conditions, one has to specify either the boundary strain gradient (curvature) v'' or the boundary double moments $m=EIg^2v'''$.

2. GRADIENT ELASTIC DYNAMIC STIFFNESS MATRIX

This section deals with the development of the dynamic stiffness matrix in the frequency domain of a gradient elastic flexural beam element with two nodes 1 and 2 at its two ends, as shown in Fig.3. On the basis of Eqs (8) describing the boundary conditions of the problem, one concludes that there are three nodal generalized displacements (v=displacement, v'=slope, v''=curvature) and three nodal generalized forces (V, M, m) associated with those displacements at every node, as shown in Fig. 3 (matrix convention).

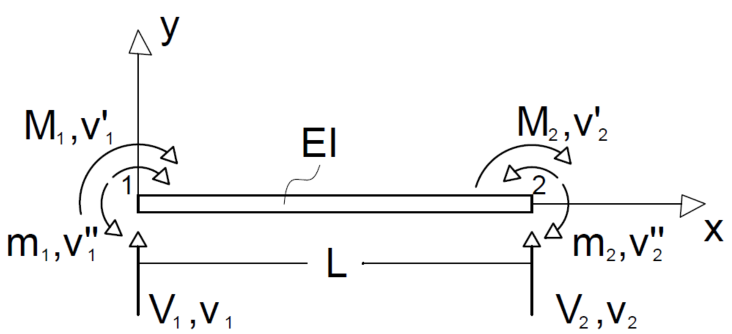

Fig. 3 Matrix convention for generalized nodal forces and displacements of a gradient elastic flexural beam element in frequency domain.

For the construction of the stiffness matrix of the finite element of Fig.3 one needs to select a displacement function and adopt a definition of that matrix. In this work, the displacement function is selected to be the exact solution of the homogeneous part of Eq. (6), which has the form [5]

$$v(x) = \sum_{i=1}^{6} C_i e^{\lambda_i x} \tag{9}$$

where the, in general complex, exponents λ_i are the 6 roots of the algebraic equation

$$\lambda^4 - g^2\lambda^6 - a^2 = 0 \tag{10}$$

and C_i are constants of integration to be determined. Because use is made of the exact solution of the governing equation of the problem as the displacement function, it is expected that the stiffness matrix to be constructed will be exact and hence the response of a beam structure to a harmonically varying with time loading analyzed on the basis of the finite element method with that element stiffness matrix for every physical member will be also exact.

The dynamic stiffness matrix of the finite element of Fig. 3 will be constructed here on the basis of the displacement function (9) and the basic definition of any coefficient of that matrix. Thus, for the dynamic stiffness matrix [k] connecting the vector of the amplitudes of the generalized nodal forces {f} with the vector of the amplitudes of the corresponding nodal displacements {u} as

$$\{f\} = [k]\ \{u\} \tag{11}$$

The stiffness coefficient k_{ij} in the frequency domain is defined as the nodal generalized force at the degree of freedom ı due to unit nodal generalized displacement of the degree of freedom j, while all the other displacements are zero. It is obvious that since v(x) depends on the frequency ω, so does the dynamic stiffness matrix [k]. In this work, since every node has 3 dof, the finite element of Fig. 3 has 2x3=6 of and hence the size of the stiffness matrix [k] will be 6x6, while the indices ı, j will take the values 1, 2, …, 6. Thus, the dynamic stiffness matrix will be constructed here column by column on the basis of six generalized displacements states, the displacement function (9) and the expressions for *V*, *M*, *m* in terms of that displacement and its derivatives as defined in Eqs (8).

Thus the derivatives of v=v(x) of Eq. (9) are evaluated and listed as

$$v^{(n)}(x)=\sum_{i=1}^{6} C_i \lambda_i^n e^{\lambda_i x} \tag{12}$$

where the superscript (n) indicates the n[th] (n=1,2,..,5) order derivative with respect to x, while the generalized forces V, M and m are expressed in terms of the above derivatives with the aid of Eqs (8) as

$$\begin{aligned} V(x) &= EI(v''-g^2v^V) \\ M(x) &= EI(v''-g^2v^{IV}) \\ m(x) &= EIg^2v''' \end{aligned} \tag{13}$$

Consider the first displacement state defined as

$$\begin{aligned} &v(0)=v_1=1, \quad v'(0)=v_1'=0, \quad v''(0)=v_1''=0. \\ &v(L)=v_2=0, \quad v'(L)=v_2'=0, \quad v''(L)=v_2''=0. \end{aligned} \tag{14}$$

where $v_1, v_2', v_1'', v_2, v_2', v_2''$ are the generalized nodal displacements of the finite element of Fig. 3. The above Eqs (14) in view of the expressions (9) and (12) can be written in the form

$$\begin{aligned} &\sum_{i=1}^{6} C_i=1, \quad \sum_{i=1}^{6} C_i\lambda_i=0, \quad \sum_{i=1}^{6} C_i\lambda_i^2=0, \\ &\sum_{i=1}^{6} C_i e^{\lambda_i L}=0, \quad \sum_{i=1}^{6} C_i\lambda_i e^{\lambda_i L}=0, \quad \sum_{i=1}^{6} C_i\lambda_i^2 e^{\lambda_i L}=0, \end{aligned} \tag{15}$$

The above Eqs (15) can be thought of as a linear system of six equations with six unknowns, the constants C_i (i=1,2,…,6) and easily solved. Using the definition of the dynamic stiffness coefficients k_{ij} and the different sign convention of Fig.2 (mechanics convention) and Fig.3 (matrix convention), one can obtain the dynamic stiffness coefficients for the first column of [k] corresponding to the displacement state (14) in the form)

$$k_{11} = V(0), \quad k_{21} = M(0), \quad k_{31} = m(0)$$

$$k_{41} = -V(L), \quad k_{51} = -M(L), \quad k_{61} = -m(L) \tag{16}$$

where the right-hand sides of Eqs (16) can be computed by using Eqs (13), (9) and (12) with values of the constants C_i (i=1,2,…,6) those obtained from the solution of Eqs (15).

Because Eq. (10) has to be solved numerically, the whole abovementioned procedure for the computation of the k_{ij} of Eq. (16) is done numerically with the aid of Mathematica [19].

After repeating the above procedure five more times, the remaining four columns of the matrix [k] can be also determined. Due to the symmetry of the matrix [k], the satisfaction of the relation $k_{ij} = k_{ji}$ with the aid of Mathematica [19] serves as a verification of the exactness of these expressions for k_{ij} . Thus, the stiffness equation (11) connecting the vectors {f} and {u} through the dynamic stiffness matrix [k] for the finite element of Fig.3 can be explicitly written down with

$$\{f\}=\{V_1, M_1, m_1, V_2, M_2, m_2\}^T$$

$$\{u\}=\}=\{v_1, v_1', v_1'', v_2, v_2', v_2''\}^T \tag{17}$$

and the various elements k_{ij} of the dynamic matrix [k] determined with the aid of Mathematica [19] and not shown explicitly fere due to their complexity.

4. FREE AND FORCED FLEXURAL VIBRATIONS

Consider a beam structure experiencing flexural vibrations under dynamic loading varying harmonically with time. Following standard procedures (e.g., Martin [20]), one is able to formulate this problem into a FEM form in the frequency domain reading as

$$\{F\}=[K(\omega)]\{U\} \tag{18}$$

In the above, [K(ω)] is the total structural dynamic stiffness matrix depending on frequency ω and obtained as an appropriate superposition of the dynamic stiffness matrices [k] of the various finite elements comprising the structure as given by (11) and {U} and {F} are the vectors of the amplitudes of the generalized nodal displacements and external forces of the whole structure, respectively. After application of the boundary conditions of the problem in terms of the generalized nodal displacements, one can solve Eq. (18) for the known operational frequency ω and determine the unknown amplitudes of the generalized nodal displacements.

In case of free vibrations, one has {F}={0} and in order for the resulting equation in (18) after application of the boundary conditions to have non-zero solutions for {U}, the condition

$$\mathrm{Det}[K(\omega)]=0 \tag{19}$$

Should be satisfied. Equation (19) is the frequency equation with roots ω_i (i=1,2,...∞) the natural frequencies of the beam structure. Once the natural frequencies have been obtained, the modal shapes {U} can be also determined from (18) with {F}={0} in the standard way. Because Eq.(19) is too complicated, it can only be solved numerically by determining the value of the det[K(ω)] for a sequence of values of ω and recording those values for which (19) is satisfied. However, since the whole procedure involves complex arithmetic, one computes the real valued function D(ω)=ln|det[K(ω)]| versus ω and identifies the natural frequencies as the local minima of D(ω) by following Kitahara [21].

When the external loading is transient, the whole problem is solved in the Laplace transformed with respect to the time domain and the time domain response is obtained by a numerical inversion of the transformed solution (Beskos and Narayanan [18]). Consider the Laplace transform defined for a function Φ(x,t) given by

$$\bar{\Phi}(x,s) = L\{\Phi(\mathrm{x},\mathrm{t})\} = \int_0^\infty \Phi(x,t)e^{-st}dt \tag{20}$$

where s is the, in general complex, Laplace transformed parameter. Application of the above Laplace transform onto Eq. (3) under zero initial conditions results in

$$\bar{\upsilon}^{IV} - g^2\bar{\upsilon}^{VI} + a^2\bar{\upsilon} = \bar{q}\,/\,EI \tag{21}$$

where

$$a^2 = \mu s^2 / EI \tag{22}$$

and indicates that one can go from the frequency into the Laplace transform domain by simply replaced ω by -is. Hence, Eq. (18) can also be used in the Laplace transform domain in the form

$$\{\bar{F}\} = [K(s)]\{\bar{U}\} \tag{23}$$

where [K(s)] is [K(ω)] with ω=-is and overbars in vectors $\{\bar{U}\}$ and $\{\bar{F}\}$ indicate transformed quantities. Equation (23) is solved numerically for $\{\bar{U}\}$ for a sequence of values of s and the transformed solution is inverted numerically to obtain the time domain response. This inversion is done with the highly accurate algorithm of Durbin [22] as explained in Beskos and Narayanan [18].

5. NUMERICAL EXAMPLES

Consider a gradient elastic cantilever beam of length L and flexural rigidity EI subjected to a vertical lateral concentrated load P_0 suddenly applied at its free end, as shown in Fig.4. For this structure the first five natural frequencies and its dynamic response to the load P_0 are determined by the FEM.

The whole beam is considered to be a single finite element with nodes 1 and 2 at the fixed and free ends, respectively, as shown in Fig.4. The dynamic behavior of that beam element 1-2 is described by the stiffness equation (11) in the frequency domain with [k] being of the order of 6X6. Application of the boundary conditions

$$v_1, = v_1' = v_1'' = 0 \tag{24}$$

with the first two conditions in (24) being the classical and the third condition the non-classical one, reduces Eq.(11) to the form

$$\begin{Bmatrix} P \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} K_{44} & K_{45} & K_{46} \\ K_{54} & K_{55} & K_{56} \\ K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} v_2 \\ v_2' \\ v_2'' \end{Bmatrix} \tag{25}$$

where $K_{ij} = K_{ij}(\omega)$ and P_0, v_2, v_2' and v_2'' stand for amplitudes. The frequency equation (19) with [K(ω)] being explicitly given in (25), is solved as described in section 4 and the first five normalized natural frequencies $\omega_n/\omega_n^{\,c}$ (n=1,2,…,5) are shown in

Table 1 for various values of the normalized gradient coefficient g/L. The normalizing factors ω_n^c are the classical natural frequencies $\omega_n^c = [(2n-1)^2 \pi^2 / 4L^2)]\sqrt{EI/\mu}$ [23].

One can observe from Table 1 that frequencies increase for increasing values of the gradient coefficient g, as expected due to the stiffening effect of gradient elasticity. Furthermore, one can observe from Table 1 that for g/L=0.0001→0, one recovers the values of the natural frequencies of the classical case.

The forced vibration problem is described by Eq.(25) considered to be in the Laplace transformed domain. In that case P is replaced by P_0/s , $K_{ij}(\omega)$ become $K_{ij}(s)$ by simply replacing ω by –is and v_2, v_2' and v_2'' become $\overline{v}_2$, $\overline{v}_2'$ and $\overline{v}_2''$,respectively. Thus, according to Section 4, Eq.(25) in the Laplace transformed domain is solved for a sequence of values of s and the time domain response is obtained by a numerical inversion of the transformed solution using the algorithm of Durbin [22]. Figure 5 provides the free end deflection of the cantilever beam versus time for various values of g/L on the basis of the data in Papargyri-Beskou et al [5] reading as L=1.219m, I=5993.73 cm^4 , E=20.685X10^4MPa, μ=60.709 Kg/m and P_0 =35584N. The present results are identical with those in [5] obtained by an analytic method. One can observe from Fig.5 that increasing values of g/L decrease the maximum values of the deflection and shift them to smaller time values of occurrence. Furthermore, one can observe that for g→0 , one can recover the response of the classical case [24], as expected.

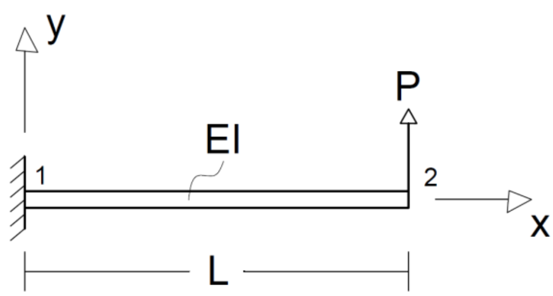

Fig.4. Geometry and loading of a cantilever gradient elastic flexural beam.

Table 1. First five normalized natural frequencies of gradient elastic cantilever beam for various values of g/L.

g/L	0.0001	0.005	0.01	0.050	0.100
ω_1^g/ω_1^c	1.0000	1.01018	1.02037	1.10658	1.22523
ω_2^g/ω_2^c	1.0000	1.01028	1.02102	1.12476	1.29145
ω_3^g/ω_3^c	1.0000	1.01059	1.02258	1.16470	1.42304
ω_4^g/ω_4^c	1.0000	1.01124	1.02525	1.22553	1.60883
ω_5^g/ω_5^c	1.0000	1.01227	1.02903	1.30263	1.82896

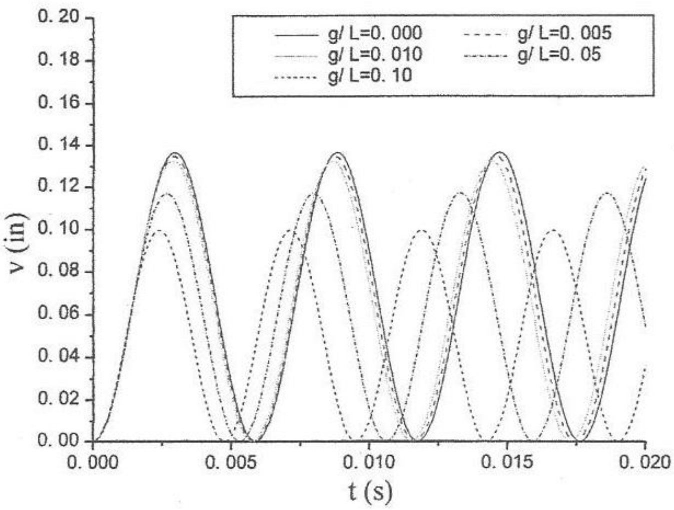

Fig. 5 Free end deflection of gradient elastic cantilever beam versus time for various values of g/L

6. CONCLUSIONS

One the basis of the previous developments, the following conclusions can be stated:

1) The dynamic stiffness matrix for a gradient elastic Bernoulli-Euler finite beam element has been constructed in the frequency domain. The displacement function used is the exact solution of the governing equation of motion in the frequency domain and this results to an exact dynamic stiffness matrix, exact dynamic response to harmonically varying with time load and exact natural frequencies.

2) When the external loading is transient, the response is obtained with the aid of the Laplace transform with respect to time and a numerical inversion of the transformed solution. The Laplace transformed domain formulation is obtained from the frequency domain one by simply replacing the frequency ω by -is, where s is the Laplace transform parameter .

3) The proposed FEM formulates the problem in a static –like form in the frequency or the Laplace transform domain with obvious conceptual and computational gains. Since the present FEM is associated with exact stiffness matrices, the discretization is restricted to one finite element per physical member,

which leads to matrices of much smaller size. Of course, the stiffness coefficients are more complicated than in the conventional FEM that employs approximate displacement functions in polynomial form. The advantages of the proposed FEM over analytic methods for dynamic analysis of gradient elastic beam structures are generality, versatility and efficiency, especially for complicated geometries and loading.

4) It was observed, at least in the examples considered here, a decrease of deflections and an increase of the natural frequencies with increasing values of the gradient coefficient as a result of the stiffening effect of gradient elasticity theory.

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