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NEW REDUCED ORDER FINITE ELEMENT FORMULATIONS FOR VIBRATION ATTENUATION USING PIEZOELECTRIC SHUNT DAMPING

W. Larbi, L. P. da Silva and J.-F. Deü

Structural Mechanics and Coupled Systems Laboratory,
Conservatoire National des Arts et Métiers,
292 rue Saint-Martin, 75141 Paris Cedex 03, France
e-mail: walid.larbi@cnam.fr

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Abstract. *In this work, various reduced order finite element models for the prediction of passive reduction of structural vibration by means of shunted piezoelectric system are presented. The problem consists of an elastic structure with surface-mounted piezoelectric patches. The piezoelectric elements are connected with resonant shunt circuits in order to damp specific resonant frequencies of the structure. An efficient electromechanical finite element formulation for the dynamic analysis of the problem is first recalled. Different strategies to solve the problem using a modal reduction approach are then proposed. These strategies are based of the use of two original projection bases: (i) the combined basis formed by both the short-circuited and open-circuited modes, and (ii) the coupled basis formed by the electromechanical modes that take into account the effect of the inductances of the electrical shunts circuit. Various results are presented in order to validate and illustrate the efficiency of the proposed new finite element reduced order formulations.*

1 INTRODUCTION

Piezoelectric materials are widely used in vibration damping and noise reduction. These materials, which enable the transformation of mechanical energy into electrical energy (direct piezoelectric effect) and vice-versa (inverse piezoelectric effect), allow direct connection with an input/output electrical signal and make them well adapted to distributed sensing and actuation. This work is focused on passive control strategies using piezoelectric elements for vibration damping. As compared to the active control techniques, passive techniques have the advantage of being simple to implement, always stable and do not require digital signal processors and bulky power amplifiers. Examples of active control systems can be found for example in [1, 2].

In this paper, the specific application of passive vibration reduction by means of shunted piezoelectric patches is addressed. In this technology, the elastic structure is equipped with piezoelectric patches that are connected to a passive electrical circuit, called a shunt. The piezoelectric patches transform mechanical energy of the vibrating structure into electrical energy, which is then dissipated into the form of Joule heat to provide electrical damping to the structure by the shunt circuits. Several shunt circuits can be considered: the classical R- and RL-shunts, proposed by Hagood and Von Flotow [3], improvements of those techniques, by the use of several piezoelectric elements [4, 5, 6], active fiber composites [7] or adaptive shunts [8], and recently semi-passive techniques, commonly known as switch techniques [9, 10, 11]. Since those techniques are passive (or semi-passive if some electronic components have to be powered), a critical issue is that their performances, in terms of damping efficiency, directly depend on the electromechanical coupling between the host structure and the piezoelectric elements, which has to be maximized.

The present work concerns the numerical modeling of vibration reduction of elastic structures in the low frequency range by using shunted piezoelectric elements. The aim is to propose efficient reduced order finite element models able to predict the shunt damping. An electromechanical finite element formulation for the dynamic analysis of smart structures with piezoelectric elements is first proposed. In this formulation, the electrical state is fully described by very few global discrete unknowns [12]: (i) the electric charge contained in the electrodes and (ii) the voltage between the electrodes. This formulation is well adapted to practical applications since realistic electrical boundary conditions, such that equipotentiality on the electrodes and prescribed global electric charges, naturally appear. The charge/voltage global variables are also intrinsically adapted to include any external electrical circuit into the electromechanical problem and to simulate the effect of shunt damping techniques. We present then different strategies to solve the problem using a modal reduction approach. In this technique, the electromechanical coupled system is solved by projecting the mechanical displacement unknown on a truncated basis composed by the first structural normal modes while the few initial electrical unknowns are kept in the reduced system. The projection bases widely used in the literature are obtained using the short-circuited or open-circuited eigenvalue problems [13, 14]. We propose in this work two new modal projection bases: (i) the combined basis formed by both the short-circuited and open-circuited modes, and (ii) the coupled basis formed by the electromechanical modes that take into account the effect of the inductances of the electrical shunts circuit. Numerical examples are finally presented in order to evaluate the effectiveness of the proposed new strategies of modal projection compared to the classic ones in terms of prediction of the vibration attenuation using piezoelectric shunt systems.

2 FINITE ELEMENT FORMULATION

We briefly recall in this section the finite element formulation of an elastic structure with surface mounted shunted piezoelectric patches.

An elastic structure occupying the domain Ω_E is equipped with P piezoelectric patches (figure 1). Each piezoelectric patch is covered on its upper and lower surfaces with a very thin electrode. The pth patch, $p \in \{1, \cdots, P\}$, occupies a domain $\Omega^{(p)}$ such that $(\Omega_E, \Omega^{(1)}, \cdots, \Omega^{(P)})$ is a partition of the all structure domain Ω_S. In order to reduce the vibration amplitudes of the problem, a resonant shunt circuit made up of a resistance $R^{(p)}$ and an inductance $L^{(p)}$ in series is connected to each patch [3, 11].

Moreover, the structure is clamped on a part Γ_u and subjected to a given surface force density F_i^d on the complementary part Γ_σ of its external boundary.

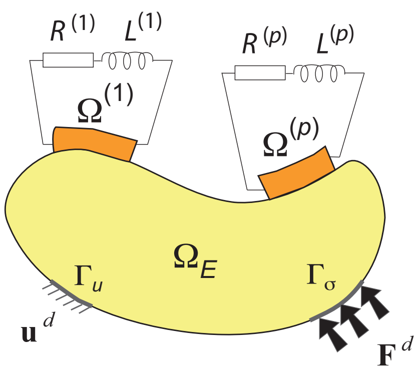

Figure 1: Vibrating structure connected to RL shunt circuits.

For each piezoelectric patch, a set of hypotheses, which can be applied to a wide spectrum of practical applications, are formulated:

- The piezoelectric patches are thin, with a constant thickness, denoted $h^{(p)}$ for the pth patch;
- The thickness of the electrodes is much smaller than $h^{(p)}$ and is thus neglected;
- The piezoelectric patches are polarized in their transverse direction (i.e. the direction normal to the electrodes).

Under those assumptions, the electric field vector, of components $E_k^{(p)}$, can be considered normal to the electrodes and uniform in the piezoelectric patch [12], so that for all $p \in \{1, \cdots, P\}$:

$$E_k^{(p)} = -\frac{V^{(p)}}{h^{(p)}} n_k \quad \text{in} \quad \Omega^{(p)} \tag{1}$$

where E_i is the electric field, $V^{(p)}$ is the potential difference between the upper and the lower electrode surfaces of the pth patch which is constant over $\Omega^{(p)}$ and n_k is the kth component of the normal unit vector to the surface of the electrodes.

After variational formulation and finite element discretization, we obtain the following matrix system in frequency domain:

$$\begin{bmatrix} \mathbf{K}_u & \mathbf{C}_{uV} \\ -\mathbf{C}_{uV} & \mathbf{K}_V \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{Q} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M}_u & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{Q} \end{bmatrix} \tag{2}$$

where $\mathbf{U}$ is the column vector of nodal values of mechanical displacement of length N_s (N_s is the number of mechanical degrees of freedom); $\mathbf{M}_u$ and $\mathbf{K}_u$ are the mass and stiffness matrices of the structure (elastic structure and piezoelectric patches) of size $N_s \times N_s$ and $\mathbf{F}$ is the applied mechanical force vector of length N_s. Moreover, $\mathbf{Q} = (Q^{(1)}\, Q^{(2)} \cdots Q^{(P)})^T$ and $\mathbf{V} = (V^{(1)}\, V^{(2)} \cdots V^{(P)})^T$ are the column vectors of electric charges and potential differences; $\mathbf{C}_{uV}$ is the electric mechanical coupled stiffness matrix of size $N_s \times P$; $\mathbf{K}_V = \text{diag}\left(C^{(1)}\, C^{(2)} \cdots C^{(P)}\right)$ is a diagonal matrix filled with the P capacitances of the piezoelectric patches where $C^{(p)} = \epsilon_{33} S^{(p)}/h^{(p)}$, ϵ_{33} being the piezoelectric permittivity in the direction normal to the electrodes and $S^{(p)}$ the area of the patch electrodes surfaces.

The above discretized formulation equation is adapted to any elastic structure with surface-mounted piezoelectric patches. Its originality lies in the fact that the system electrical state is fully described by very few global discrete unknowns: only a couple of variables per piezoelectric patch, namely (1) the electric charge contained in the electrodes and (2) the voltage between the electrodes. Once the electrical part of the problem is fully discretized at the weak formulation step, by introducing the above cited voltage/charge variables, without any restriction on the mechanical part of the problem, any standard FE formulation can be easily modified to include the piezoelectric patches and thus the effect of an external electrical action. A second advantage of this formulation is that since global electrical variables are used, realistic electrical boundary conditions are naturally introduced. First, the equipotentiality in any of the patches electrodes is exactly satisfied when introducing the potential difference variable. Second, the use of the global charge contained in the electrodes, as the second electrical variable, is realistic since plugging an external electrical circuit to the electrodes of the patches imposes only the global charge contained in the electrodes and not a local charge surface density. Another advantage of using the global charge voltage variables is that they are intrinsically adapted to include any external electrical circuit into the electromechanical problem and to simulate the effect of shunt damping techniques. In this case, neither $\mathbf{V}$ nor $\mathbf{Q}$ is prescribed by the electrical network but the latter imposes only a relation between them [3]. For the case of a resonant shunt composed of a resistor R and an inductor L in series, connected to the pth patch, the relation writes

$$-\omega^2 L^{(p)} Q^{(p)} + i\omega R^{(p)} Q^{(p)} + V^{(p)} = 0 \tag{3}$$

Combining equations (2) and (3), we finally obtain the general FE formulation of the electromechanical spectral problem when the piezoelectric patches are shunted

$$-\omega^2 \begin{bmatrix} \mathbf{M}_u & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{Q} \end{bmatrix} + i\omega \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{Q} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_u + \mathbf{C}_{uV}\mathbf{K}_V^{-1}\mathbf{C}_{uV}^T & \mathbf{C}_{uV}\mathbf{K}_V^{-1} \\ \mathbf{K}_V^{-1}\mathbf{C}_{uV}^T & \mathbf{K}_V^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix} \tag{4}$$

where $\mathbf{R} = \text{diag}\left(R^{(1)}\, R^{(2)} \cdots R^{(P)}\right)$ and $\mathbf{L} = \text{diag}\left(L^{(1)}\, L^{(2)} \cdots L^{(P)}\right)$ are the diagonal matrices filled with the electrical resistances and the electrical inductances of the shunt circuits. Note that since $\mathbf{K}_V$ is diagonal, the evaluation of $\mathbf{K}_V^{-1}$ is straightforward.

3 REDUCED ORDER MODELS

In order to evaluate the electrical and mechanical frequency response functions of an elastic structure with shunted piezoelectric patches and subjected to a considered harmonic mechanical excitation, the full finite element model of Eq. (4) is applicable only to a small model and low frequency band. To overcome these limitations, a model reduction approach based on a normal mode expansion and truncation of high-frequency modes is proposed in this section. We present first a review of the existing reduced order models. These models, widely used in the literature, are obtained using the short-circuited or open-circuited basis. In this paper, two original reduced order models are proposed to solve the problem at lower cost: (i) the combined basis formed by both the short-circuited and open-circuited modes, and (ii) the coupled basis formed by the electromechanical modes taking into account the effect of the inductances of the electrical shunt circuits. A comparative study of the effectiveness of these reduced order models in terms of prediction of the vibration attenuation using piezoelectric shunt systems is proposed in next section.

3.1 Projection on the short-circuited basis

The first M_s eigenmodes of the elastic structure with all patches short-circuited are obtained from the following equation:

$$\left[\mathbf{K}_u - \omega_i^2 \mathbf{M}_u\right] \mathbf{\Phi}_i = \mathbf{0} \quad \text{for } i \in \{1, \cdots, M_s\} \tag{5}$$

where $(\omega_i, \mathbf{\Phi}_i)$ are the natural frequency and eigenvector for the ith structural mode. These modes verify the following orthogonality properties

$$\mathbf{\Phi}_i^T \mathbf{M}_u \mathbf{\Phi}_j = \delta_{ij} \quad \text{and} \quad \mathbf{\Phi}_i^T \mathbf{K}_u \mathbf{\Phi}_j = \omega_i^2 \delta_{ij} \tag{6}$$

where δ_{ij} is the Kronecker symbol and $\mathbf{\Phi}_i$ have been normalized with respect to the structure mass matrix. Note that the structure is fixed on Γ_u, which eliminates any rigid body motion.

By introducing the matrix $\mathbf{\Phi} = [\mathbf{\Phi}_1 \cdots \mathbf{\Phi}_{M_s}]$ of size $(N_s \times M_s)$ corresponding to the basis previously defined (N_s is the total number of degrees of freedom in the finite elements model associated to the structure), the displacement $\mathbf{U}$ is sought as

$$\mathbf{U} = \mathbf{\Phi} \mathbf{q}(t) \tag{7}$$

where the vector $\mathbf{q} = [q_1 \cdots q_{M_s}]^T$ is the unknown modal amplitudes.

By applying the Ritz-Galerkin projection method, which consists of substituting Eq. (7) into Eq. (4) and premultiplying the first row by $\mathbf{\Phi}^T$, we obtain the reduced matrix system

$$-\omega^2 \begin{bmatrix} \mathbf{\Phi}^T \mathbf{M}_u \mathbf{\Phi} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{Q} \end{bmatrix} + i\omega \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{Q} \end{bmatrix} + \begin{bmatrix} \mathbf{\Phi}^T \left(\mathbf{K}_u + \mathbf{C}_{uV} \mathbf{K}_V^{-1} \mathbf{C}_{uV}^T\right) \mathbf{\Phi} & \mathbf{\Phi}^T \mathbf{C}_{uV} \mathbf{K}_V^{-1} \\ \mathbf{K}_V^{-1} \mathbf{C}_{uV}^T \mathbf{\Phi} & \mathbf{K}_V^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_s^T \mathbf{F} \\ \mathbf{0} \end{bmatrix} \tag{8}$$

This matrix equation represents the reduced-order model using the short-circuited basis of the vibration reduction problem with piezoelectric shunt-damping treatments. If only few modes are kept for the projection, the size of this reduced-order model $(Ms + P)$ is much smaller than the initial one $(Ns + P)$. Eq. (8) can be also written in the following form of coupled differential equations:

- M_s mechanical oscillators

$$-\omega^2 q_i + 2i\omega\xi_i\omega_i q_i + \omega_i^2 q_i + \sum_{p=1}^{P} \sum_{k=1}^{N_s} \frac{\gamma_i^{(p)} \gamma_k^{(p)}}{C^{(p)}} q_i + \sum_{p=1}^{P} \frac{\gamma_i^{(p)}}{C^{(p)}} Q^{(p)} = F_i \tag{9}$$

- P electrical equations

$$-\omega^2 L^{(p)} Q^{(p)} + i\omega R^{(p)} Q^{(p)} + \frac{Q^{(p)}}{C^{(p)}} + \sum_{i=1}^{N_s} \frac{\gamma_i}{C^{(p)}} q_{si} = 0 \tag{10}$$

where $F_i = \mathbf{\Phi}_i^T \mathbf{F}$ is the mechanical excitation of the ith mode and $\gamma_i = \mathbf{\Phi}_i^T \mathbf{C}_{uV}$ is the electromechanical coupling factors.
Note that the modal damping coefficients ξ_i have been added in Eq. (9) in order to take into account the structural damping, which can be measured experimentally. This is mandatory in order to quantify the attenuation due to the shunt at the resonance (of course, without damping, the amplitude of the response at the resonance is theoretically infinite).

The major interest of choosing the short-circuit eigenmodes as the expansion basis is that its can be computed with a classical elastic mechanical problem. This operation can thus be done by any standard finite elements code.

3.2 Projection on the open-circuited basis

The first M_s natural frequencies $\widehat{\omega}_i$ and eigenvectors $\widehat{\mathbf{\Phi}}_i$ of the elastic structure with all patches open-circuited are obtained from the following equation:

$$\left[(\mathbf{K}_u + \mathbf{C}_{uV}\mathbf{K}_V^{-1}\mathbf{C}_{uV}^T) - \widehat{\omega}_i^2 \mathbf{M}_u\right] \widehat{\mathbf{\Phi}}_i = \mathbf{0} \quad \text{for } i \in \{1, \cdots, M_s\} \tag{11}$$

These modes verify the following orthogonality properties

$$\widehat{\mathbf{\Phi}}_i^T \mathbf{M}_u \widehat{\mathbf{\Phi}}_j = \delta_{ij} \quad \text{and} \quad \widehat{\mathbf{\Phi}}_i^T (\mathbf{K}_u + \mathbf{C}_{uV}\mathbf{K}_V^{-1}\mathbf{C}_{uV}^T) \widehat{\mathbf{\Phi}}_j = \widehat{\omega}_i^2 \delta_{ij} \tag{12}$$

Thus, the displacement $\mathbf{U}$ is sought as

$$\mathbf{U} = \widehat{\mathbf{\Phi}} \widehat{\mathbf{q}}(t) \tag{13}$$

where the matrix $\widehat{\mathbf{\Phi}} = \left[\widehat{\mathbf{\Phi}}_1 \cdots \widehat{\mathbf{\Phi}}_{M_s}\right]$ of size $(N_s \times M_s)$ corresponds to the projection basis constituted by the first M_s eigenmodes of the structure with all patches open-circuited and where the vector $\widehat{\mathbf{q}} = [\widehat{q}_1 \cdots \widehat{q}_{M_s}]^T$ is the unknown modal amplitudes.

By applying the Ritz-Galerkin projection method using the procedure described in section 3.1, we obtain the following coupled differential equations:

- M_s mechanical oscillators

$$-\omega^2 \widehat{q}_i + 2i\omega\xi_i\widehat{\omega}_i\widehat{q}_i + \widehat{\omega}_i{}^2 \widehat{q}_i + \sum_{p=1}^{P} \frac{\widehat{\gamma}_i{}^{(p)}}{C^{(p)}} Q^{(p)} = \widehat{F}_i \tag{14}$$

- P electrical equations

$$-\omega^2 L^{(p)} Q^{(p)} + i\omega R^{(p)} Q^{(p)} + \frac{Q^{(p)}}{C^{(p)}} + \sum_{i=1}^{N_s} \frac{\widehat{\gamma}_i}{C^{(p)}} \widehat{q}_i = 0 \tag{15}$$

where $\widehat{F}_i = \widehat{\mathbf{\Phi}}_i^T \mathbf{F}$ and $\widehat{\gamma}_i = \widehat{\mathbf{\Phi}}_i^T \mathbf{C}_{uV}$ are the mechanical excitation and the electromechanical coupling factor of the ith mode respectively.

3.3 Projection on the combined basis

This method consists in building a projection basis formed by both the short-circuited and open-circuited modes:

$$\mathbf{\Phi}_c = \left[\mathbf{\Phi}\ \widehat{\mathbf{\Phi}}\right] = \left[\mathbf{\Phi}_1 \cdots \mathbf{\Phi}_{M_s}\ \widehat{\mathbf{\Phi}}_1 \cdots \widehat{\mathbf{\Phi}}_{M_s}\right] \tag{16}$$

Thus, the displacement $\mathbf{U}$ is sought as

$$\mathbf{U} = \mathbf{\Phi}_c \mathbf{q}_c(t) = \mathbf{\Phi}\mathbf{q}(t) + \widehat{\mathbf{\Phi}}\widehat{\mathbf{q}}(t) \tag{17}$$

By applying the Ritz-Galerkin projection method, which consists of substituting Eq. (17) into Eq. (4) and premultiplying the first row by $\mathbf{\Phi}_c^T$, we obtain the reduced matrix system,

$$-\omega^2 \begin{bmatrix} \mathbf{\Phi}_c^T\mathbf{M}_u\mathbf{\Phi}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{q}_c \\ \mathbf{Q} \end{bmatrix} + i\omega \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{q}_c \\ \mathbf{Q} \end{bmatrix} + \begin{bmatrix} \mathbf{\Phi}_c^T\left(\mathbf{K}_u + \mathbf{C}_{uV}\mathbf{K}_V^{-1}\mathbf{C}_{uV}^T\right)\mathbf{\Phi}_c & \mathbf{\Phi}_c^T\mathbf{C}_{uV}\mathbf{K}_V^{-1} \\ \mathbf{K}_V^{-1}\mathbf{C}_{uV}^T\mathbf{\Phi}_c & \mathbf{K}_V^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{q}_c \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_c^T\mathbf{F} \\ \mathbf{0} \end{bmatrix} \tag{18}$$

For this technique, two drawbacks exist: (i) two modal analysis are required and (ii) the short-circuited and open-circuited eigenvectors are not mutually orthogonal with respect to the mass and stiffness matrices of the problem. Thus, a Gram-Schmidt algorithm is applied in this work to orthogonalize these eigenvectors and build well-conditioned matrices.

3.4 Projection on the coupled basis

In the case of a resonant shunt, this technique consists in computing the eigenmodes of the real conservative system associated to Eq. (4). Thus, the inductances of the shunt circuits are taken into account in the mass matrix, whereas the effect of the resistances and mechanical damping is not considered. The eigenmode $(\tilde{\omega}_i, \tilde{\mathbf{\Phi}}_i)$ of this problem is solution of the following equation:

$$\left(-\tilde{\omega}_i^2 \begin{bmatrix} \mathbf{M}_u & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_u + \mathbf{C}_{uV}\mathbf{K}_V^{-1}\mathbf{C}_{uV}^T & \mathbf{C}_{uV}\mathbf{K}_V^{-1} \\ \mathbf{K}_V^{-1}\mathbf{C}_{uV}^T & \mathbf{K}_V^{-1} \end{bmatrix}\right) \tilde{\mathbf{\Phi}}_i = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{19}$$

where the eigenvector $\tilde{\mathbf{\Phi}}_i^T = \begin{bmatrix} \tilde{\mathbf{\Phi}}_{si}^T & \tilde{\mathbf{\Phi}}_{qi}^T \end{bmatrix}$ consists of the mechanical degrees-of-freedom $\tilde{\mathbf{\Phi}}_{si}$ of size N_s followed by the electric degrees-of-freedom $\tilde{\mathbf{\Phi}}_{qi}$ of size P.

By introducing the modal basis $\tilde{\mathbf{\Phi}} = \left[\tilde{\mathbf{\Phi}}_{s1} \cdots \tilde{\mathbf{\Phi}}_{sM_s}\right]$ of size $(N_s \times M_s)$, the displacement $\mathbf{U}$ is sought as

$$\mathbf{U} = \tilde{\mathbf{\Phi}}\tilde{\mathbf{q}}(t) \tag{20}$$

By applying the Ritz-Galerkin projection method, which consists of substituting Eq. (20) into Eq. (4) and premultiplying the first row by $\tilde{\mathbf{\Phi}}^T$, we obtain the reduced matrix system,

$$-\omega^2 \begin{bmatrix} \tilde{\mathbf{\Phi}}^T\mathbf{M}_u\tilde{\mathbf{\Phi}} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}} \\ \mathbf{Q} \end{bmatrix} + i\omega \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}} \\ \mathbf{Q} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{\Phi}}^T\left(\mathbf{K}_u + \mathbf{C}_{uV}\mathbf{K}_V^{-1}\mathbf{C}_{uV}^T\right)\tilde{\mathbf{\Phi}} & \tilde{\mathbf{\Phi}}^T\mathbf{C}_{uV}\mathbf{K}_V^{-1} \\ \mathbf{K}_V^{-1}\mathbf{C}_{uV}^T\tilde{\mathbf{\Phi}} & \mathbf{K}_V^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}} \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{\Phi}}^T\mathbf{F} \\ \mathbf{0} \end{bmatrix} \tag{21}$$

Remarks:

- This method requires the knowledge of the optimal values of the electric inductances before the computation of the eigenmodes.

- The projection for the various methods concerns concerns only the mechanical variables $\mathbf{U}$. The electrical unknown field $\mathbf{Q}$ is not concerned by the reduction because the dimension of this vector corresponds to the number of piezo-patches and therefore is very small compared to the mechanical finite element degrees-of-freedom (displacement in the host structure and the piezo-patches).

- Substituting $\tilde{\mathbf{\Phi}}_{qi}$ in terms of $\tilde{\mathbf{\Phi}}_{si}$ from the second line of Eq. (19), the modal problem amounts to solve:

$$\Big(-\tilde{\omega}_i^2\mathbf{M}_u + \Big[\mathbf{K}_u + \mathbf{C}_{uV}\mathbf{K}_V^{-1}\mathbf{C}_{uV}^T - \mathbf{C}_{uV}\mathbf{K}_V^{-1}\left(\mathbf{K}_V^{-1} - \tilde{\omega}_i^2\mathbf{L}\right)^{-1}\mathbf{K}_V^{-1}\mathbf{C}_{uV}^T\Big]\Big)\tilde{\mathbf{\Phi}}_{si} = \mathbf{0} \quad (22)$$

Two cases can be distinguished, depending on the value of the inductance:

  – short-circuit case (inductances $L^{(i)} = \mathbf{0}$)

$$\left[\mathbf{K}_u - \tilde{\omega}_i^2\mathbf{M}_u\right]\tilde{\mathbf{\Phi}}_{si} = \mathbf{0} \quad (23)$$

  – open-circuit case (inductances $L^{(i)} = \infty$)

$$\left[\left(\mathbf{K}_u + \mathbf{C}_{uV}\mathbf{K}_V^{-1}\mathbf{C}_{uV}^T\right) - \tilde{\omega}_i^2\mathbf{M}_u\right]\tilde{\mathbf{\Phi}}_{si} = \mathbf{0} \quad (24)$$

4 NUMERICAL EXAMPLES

4.1 Cantilever beam with shunted piezoelectric patch

In this first example, we propose to compare the proposed modal reduction techniques in terms of prediction of vibration attenuation with piezoelectric shunt on a simple system. The system under study consists of a cantilever beam with one piezoelectric device, as sketched in Figure 2. The beam is made of aluminum (E = 74 GPa, ν = 0.33, and ρ = 2700 kg/m^3) with

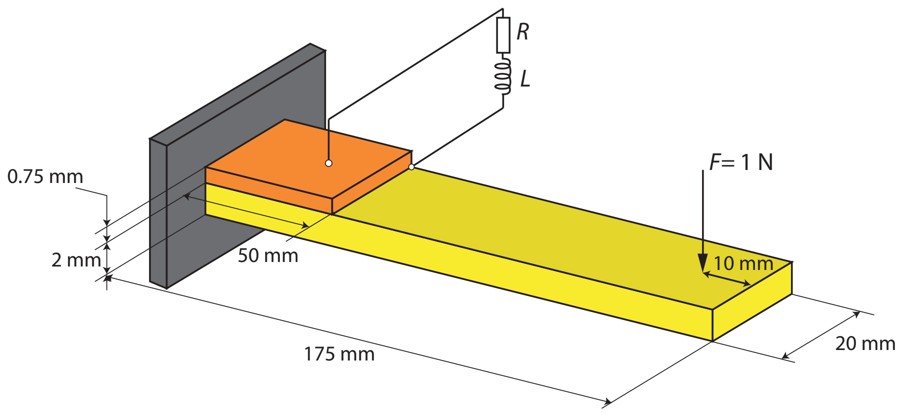

Figure 2: Cantilever beam partially covered with collocated piezoelectric element: geometrical data.

L_b=175 mm (length), w_b=10 mm (width), and h_b=2 mm (thickness). The piezoelectric element in PIC-151 is assumed to be perfectly bounded to the beam and has the same width. For the mechanical and electrical characteristics of the piezoelectric material PIC-151, the reader can be referred to [15]. Concerning the finite element discretization, the beam is modelled in 2D using 672 four nodes plate elements (QUAD4). The portion of the plate covered by the piezoelectric patch and the patch itself is modeled according to laminated theory [16].

Table 1 presents the first five eigenfrequencies of the beam (with short-circuit or open-circuit patches) and the corresponding effective electromechanical modal coupling factor (EEMCF), characterizing the energy exchanges between the mechanical structure and the piezoelectric patches and defined by:

$$k_{eff,i}^2 = \frac{\hat{\omega}_{si}^2 - \omega_{si}^2}{\omega_{si}^2} \tag{25}$$

The first, second and fourth frequencies are associated with the first three bending modes lower than 1100 Hz, while the third one corresponds to the first bending mode in the plane and the fifth frequency to the first torsional vibration mode. This classification is found from the mode shapes which are not shown here for the sake of brevity. Moreover, as expected for bending modes, the natural frequencies of the open-circuit modes are higher than those obtained in the short-circuited case due to the electrical effect of the patch. The bending mode in the plane and the torsional mode do not present any electromechanical coupling due to the transverse polarization of the patch.

Type	Short circuit frequencies	Open circuit frequencies	k_{eff}
F	71.89	73.48	21.15
F	379.49	383.97	15.42
F_p	587.02	587.02	0
F	969.11	970.05	4.41
T	1048.71	1048.71	0

F: Bending mode, F_p Bending mode in the plane, T: Torsional mode

Table 1: Computed frequencies (Hz) of the beam and the electromechanical coupling coefficient.

The beam is now excited by a harmonic transverse load (see Figure 2). In addition, no mechanical damping was introduced in this example. The vibration output is detected at the excitation point. Figure 3 presents the frequency response of the system in terms of the transverse displacement computed with a direct method in which the displacement and the charge vectors are calculated at each frequency step.

In order to achieve maximum vibration dissipation of the first mode, the patch is tuned to an RL shunt circuit. The resistance R and the inductance L can be adjusted and properly chosen to maximize the damping effect of a particular mode. The optimal resistance and inductance of the ith mode for a series resonant shunt are given by [3]:

$$R^{\text{opt}} = \frac{\sqrt{2k_{\text{eff},i}^2}}{C\omega_i(1 + k_{\text{eff},i}^2)} \tag{26a}$$

$$L^{\text{opt}} = \frac{1}{C\omega_i^2(1 + k_{\text{eff},i}^2)} \tag{26b}$$

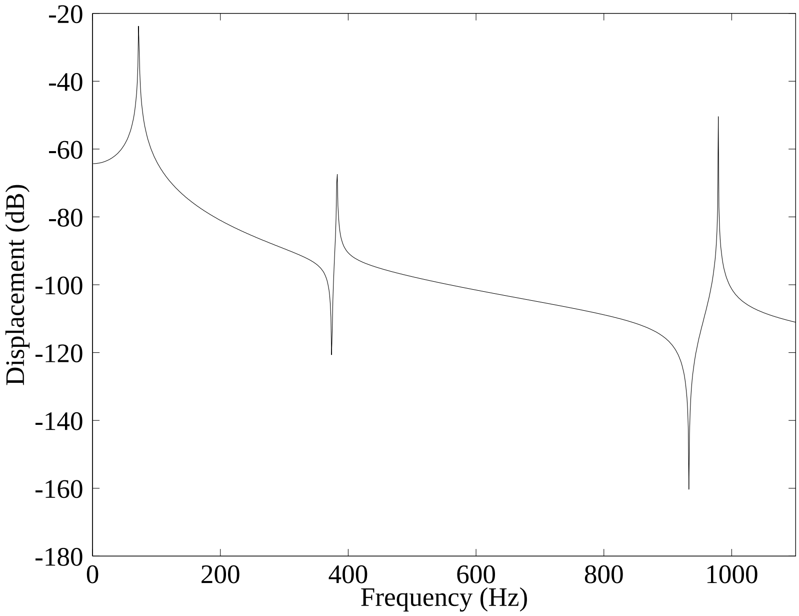

Figure 3: Frequency response function of the cantilever beam with short circuited patch and subjected to a harmonic transverse load.

where ω_i is the short circuit natural frequency of the ith mode, C is the capacitance of the piezoelectric patch and $k_{\text{eff},i}$ is the effective electromechanical coupling coefficient given in Eq. (25).

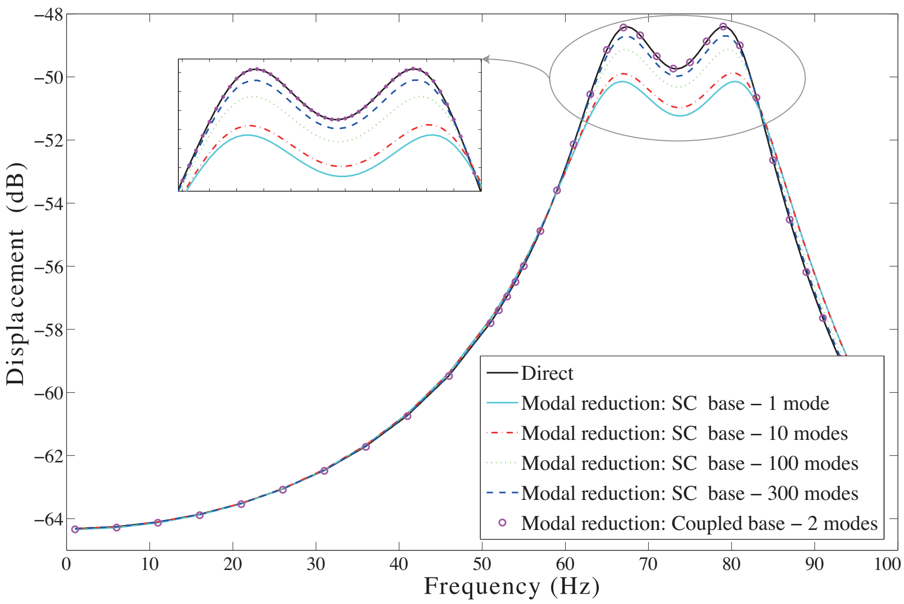

Figure 4: Frequency responses of the cantilever beam with shunted piezoelectric patch: comparison between (i) the direct approach, (ii) modal reduction using short-circuited basis and (iii) modal reduction using coupled basis.

Figures 4 and 5 show a comparison between the frequency responses of the cantilever beam when the shunt circuit is tuned to damp the first vibration mode. We compare all the proposed reduction techniques to the direct approach. The following observations can be made:

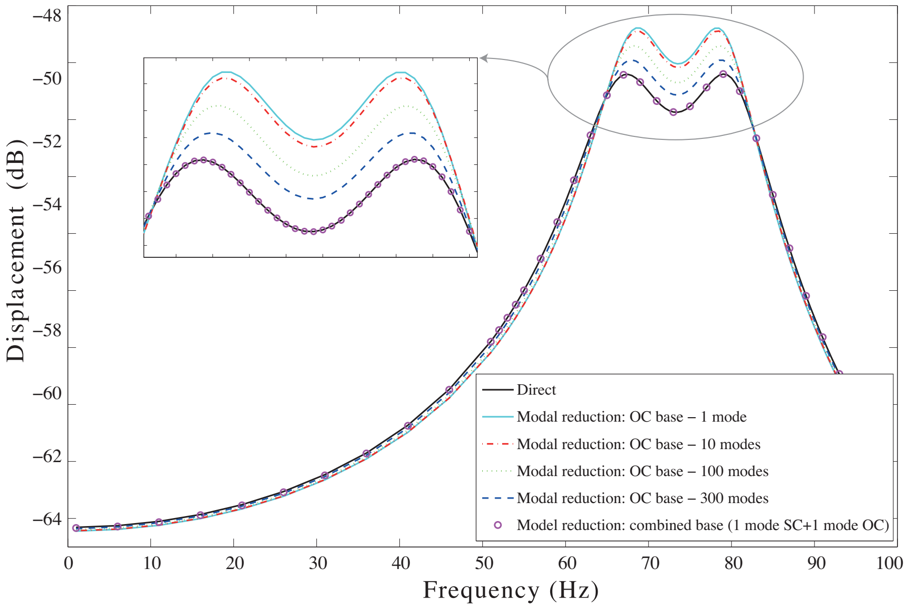

Figure 5: Frequency responses of the cantilever beam with shunted piezoelectric patch: comparison between (i) the direct approach, (ii) modal reduction using open-circuited basis and (iii) modal reduction using combined basis.

- The reduced order models built from the short-circuit or the open circuit basis converge with a large number of eigenvectors. With a small number of eigenvectors, the short-circuit basis overestimates the attenuation while the open-circuit basis underestimates it.
- The reduced order model computed from the combined basis gives an excellent results with only one short-circuited mode and one open-circuited mode. This efficiency is related to the introduction of additional electromechanical coupling terms in the reduced system which allows better modeling of the effects of the piezo-patches and the shunt-circuit on the response of the system.
- The reduced order model built from the coupled basis is extremely effective. The model response converges when projecting the unknown mechanical displacements only on the first two eigenvectors. The attenuation of a resonance mode with the shunt technique is characterized by a decrease of the amplitudes of the peaks around the targeted frequency. In this example, these two peaks correspond to the first two modes of the coupled basis with mode shapes very close. This leads us to conclude that this basis formed by the real modes of the coupled system (elastic structure with shunted piezoelectric patches), is very effective in the prediction of vibration attenuation.

4.2 Multimode shunt control of a vibrating plate

In this last example, a clamped rectangular plate with two rectangular piezoelectric patches perfectly bonded on its surface is considered. The plate of 2-mm-thick is made in aluminum and the patches of 0.5-mm-thick are in PIC 151 (we used the same properties as the previous example). The geometrical data of the problem are given in Figure 6. Concerning the finite element discretization, the plate is modelled with 672 QUAD4. The portion of the plate covered by the piezoelectric patches and the patch itself are modeled according to laminated theory. The plate is excited by a normal unit sinusoidal force as sketched in Figure 6. The vibration output is detected at the excitation point, where the displacement reaches a maximum. Figure 7 presents

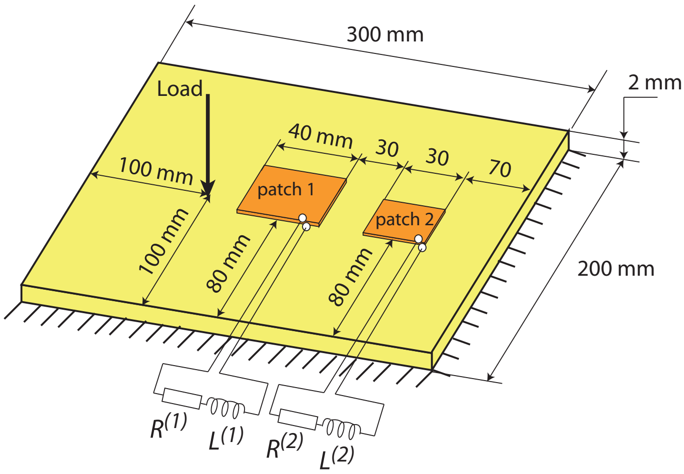

Figure 6: Clamped plate partially covered with two collocated piezoelectric elements: geometrical data.

the frequency response of the system in terms of the transverse displacement computed with direct nodal method.

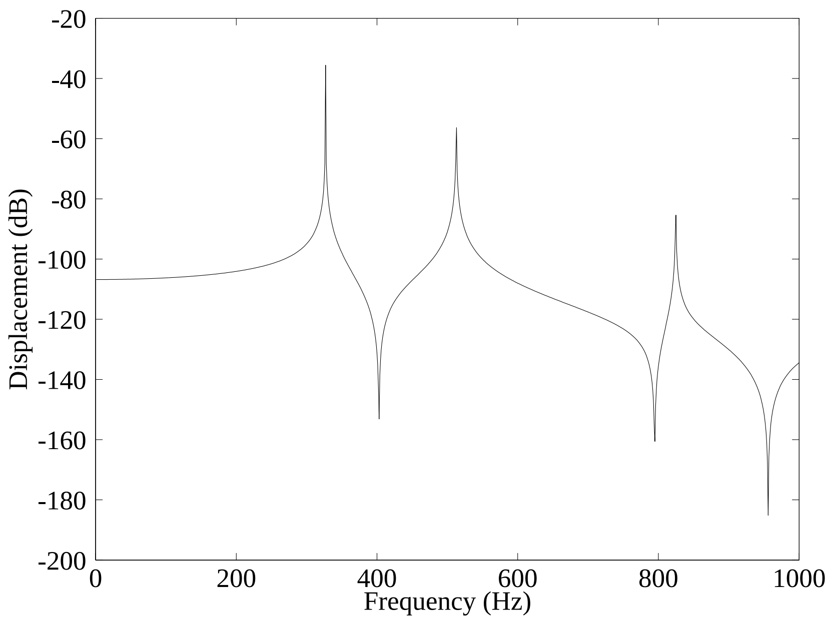

Figure 7: Frequency response function of the plate with short circuited patches and subjected to a harmonic load.

Table 2 presents the first ten eigenfrequencies of (i) the clamped plate with all patches short-circuited, (ii) the clamped plate with all patches open-circuited and (iii) the coupled problem given by Eq. (18). For the third case, the two patches are tuned to the two first vibration modes respectively and the used optimal inductances are computed from Eqs. 26b. For each frequency of the open-circuit case, except for modes without electromechanical coupling, corresponds two frequencies in the coupled case (one lower and the other higher to this frequency). The mode shapes associated to these frequencies are very similar to the mode of the original one. These modes are not shown here for the sake of brevity.

Figures 8 and 9 show a comparison between the frequency responses of the clamped plate

Short circuit frequencies	Open circuit frenquencies	Coupled frenquencies
326.98	327.85	306.68
512.76	516.10	348.79
824.66	827.05	486.10
829.68	829.68	544.23
1005.77	1005.77	829.68
1265.51	1265.61	836.79
1291.48	1291.48	1005.77
1600.37	1604.25	1265.63
1727.30	1727.30	1291.48
1751.45	1763.06	1617.75

Table 2: Computed frequencies (Hz) of the plate with piezoelectric patches in: (i) short-circuit case, (ii) open-circuited case and (iii) coupled case.

when the shunt circuits are tuned to damp the first two vibration modes. We compare all the proposed reduction techniques to the direct approach. The results confirm the findings obtained with the previous example:

- The reduced order models built from the short-circuited or open-circuited bases does not fully converge with a reduced number of eigenvectors.
- Using the combined bases, excellent results are obtained in the resonance zones when the unknown mechanical displacements are projected on the basis formed by the first two short-circuited modes and the first two open-circuited. Using the first 38 short-circuited (respectively open-circuited) modes and the first two open-circuited (respectively short-circuited) modes, the response of the model converges in areas of resonance and also outside these areas.
- The reduced order model built from the coupled basis gives excellent results compared to the results of the direct method. This convergence is achieved by using the first 40 modes. We note that with only four modes (modes resulting from the resonant shunt technique), the convergence is limited to resonant frequencies and is not assured in the other areas.

These Figures show also that the modal resonant magnitude for each considered mode have been significantly reduced simultaneously. Indeed, the strain energy present in the piezoelectric material is converted into electrical energy and hence dissipated into heat using the RL shunt device.

From these two examples, we can confirm the good performances of the combined and coupled projection bases compared to the results of the direct method with a reduced number of modes. However, the reduced order model using the coupled modes may be applied only in the case of a resonant shunt and requires knowledge of the optimal values of inductors before projection. For the combined basis, two drawbacks remain: (i) the need to carry two modal calculations (short-circuit and open-circuit) and (ii) the loss of orthogonality after projection of the mass and stiffness matrices.

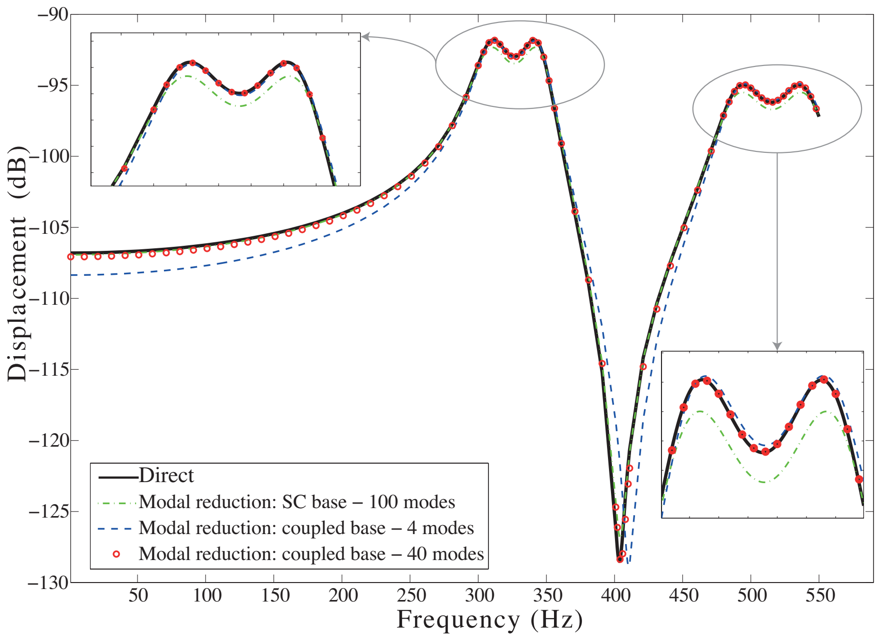

Figure 8: Frequency responses of the plate with shunted piezoelectric patches: comparison between (i) the direct approach, (ii) modal reduction using short-circuited basis and (iii) modal reduction using coupled basis.

5 CONCLUSIONS

In this paper, reduced order finite element formulations for passive reduction of structural vibration by means of shunted piezoelectric system are presented. Two new reduced-order models, based on a normal mode expansion, are then developed: (i) the combined basis, and (ii) the coupled basis. Despite their reduced size, these model are proved to be very efficient with a reduced number of modes compared to classical bases for simulations of vibration attenuation of selected frequency resonances. However, some drawbacks remain and concerns the necessity of two modal calculations for the combined basis and the knowledge of the optimal values of inductors before projection for the coupled basis. Despite these disadvantages, these new techniques can be recommanded for the resolution of electromechanical problems with shunt devices.

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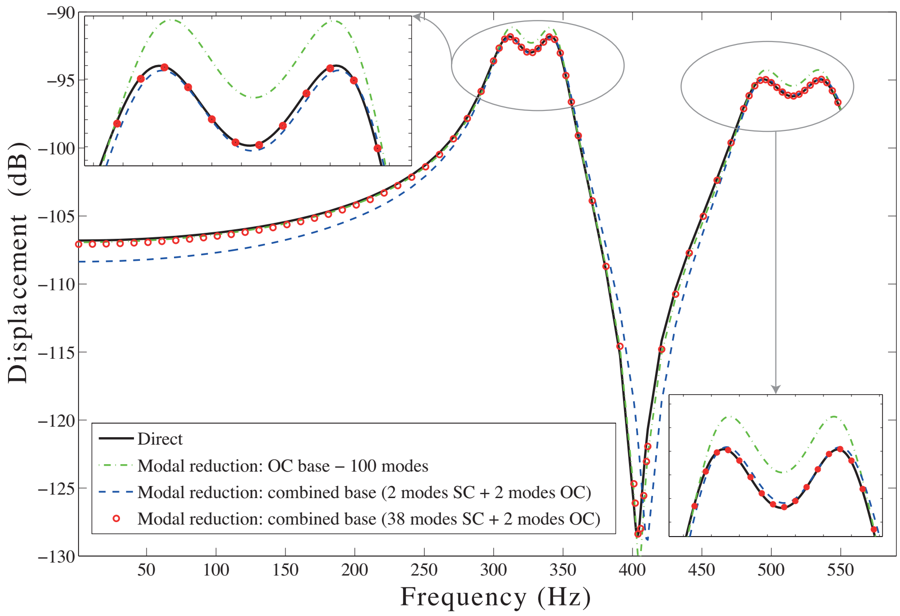

Figure 9: Frequency responses of the plate with shunted piezoelectric patches: comparison between (i) the direct approach, (ii) modal reduction using open-circuited basis and (iii) modal reduction using combined basis.

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