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ELASTIC INVERTED PENDULUM UNDER HYSTERETIC NONLINEARITY IN SUSPENSION: STABILIZATION AND OPTIMAL CONTROL

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Abstract. *This paper is dedicated to investigation of the elastic inverted pendulum with hysteretic nonlinearity in the suspension point. We consider the problem of stabilization and optimal control of such a system. In the frame of the bionic model we present an algorithm which provides an effective procedure for finding of optimal control parameters together with application of this algorithm to the system under consideration. The results of numerical simulations, namely, the phase portraits and the dynamics of Lyapunov function are presented and discussed.*

1 INTRODUCTION

The problem of inverted pendulum plays a central role in the control theory [1, 2, 3, 4]. The model of inverted pendulum provides many challenging problems to control design. Because of their nonlinear nature pendulums have maintained their usefulness and they are now used to illustrate many of the ideas emerging in the field of nonlinear control [5].

Such a mechanical system can be found in various field of technical sciences, from robotics to cosmic technologies. The stabilization of inverted pendulum is considered in the problem of missile pointing because the engine of missile is placed lower than the center of mass and such a fact leads to aerodynamical unstability. Similar problem is solved in the self-balancing transport device (the so-called segway). Moreover, such a mechanical system can be be applied in various fields such as physics, applied mathematics, engineer sciences, neuroscience, economics etc.

First theoretical description of the inverted pendulum was carried out by Stephenson [6] and the first experimental investigation of the stabilization process for such a system (using oscillations of the suspension point) was considered in the works of Kapitza [7]. In general, the problem of inverted pendulum is of more than one hundred years history and it still relevant even in the present days (see, e.g., [8, 9, 10, 11] and related references).

Backlash in the suspension point is a kind of hysteretic nonlinearity. It should be pointed out that the hysteretic phenomenons are insufficiently known in our days. This fact leads to an interesting problem on the presence of a backlash in the suspension point of a pendulum. In this paper we consider an elastic inverted pendulum using the operator technique for hysteretic nonlinearities. Namely, we obtain the equation of motion of the elastic pendulum with a hysteretic nonlinearity in the suspension point. The numerical solution of the equations of motion can be obtained using the difference scheme. We analyze the problem of optimization for the system under consideration. The numerical realization of optimization procedure is made using the so-called bionic algorithm. Also we obtain the numerical results for such a system in the form of a phase portrait and dynamics of Lyapunov function.

2 ELASTIC INVERTED PENDULUM WITH BACKLASH IN SUSPENSION

Let us consider the model of stabilization of inverted pendulum in the vicinity of vertical position. The pendulum is considered as an elastic rod which is hingedly fixed on the cylinder. Motion of cylinder is excited by the horizontal motion of a piston (see the Fig. 1) Mathematical

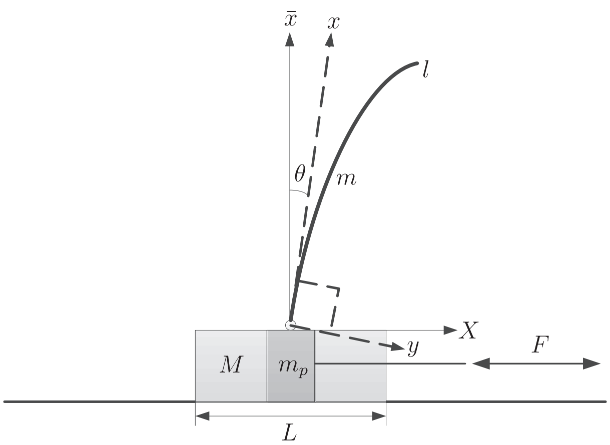

Figure 1: Model of elastic inverted pendulum: geometry of the problem.

model of a similar mechanical system (however without the hysteretic nonlinearity) was considered in [13].

Here (x, y) is a base of an elastic rod with mass m and density ρ; the Ox axis coincides with a tangent to rod's profile in the suspension point; θ is an angle of slope for the co-ordinates of a rod, I is a centroidal moment of inertia of the rod's section; $(X, \bar{x})$ is a co-ordinates of a considered mechanical system, M is a mass of a cylinder with length L, F is a force joined to a piston with mass m_p (such a force is treated as control).

The purpose of this paper is investigation of the possible stabilization (in a vicinity of vertical position) of elastic inverted pendulum in the presence of backlash in a suspension point together with investigation of various aspect of such a dynamical system.

In the following consideration we use the operator technique for hysteretic nonlinearities following the ideas of Krasnosel'skii and Pokrovskii [12]. Output of the backlash-inverter on the monotonic inputs can be described by the following expression:

$$X(t) = \Gamma[X_0, L]Y(t) = \begin{cases} 0, & |Y(t) - X_0| \leqslant \frac{L}{2}; \\ Y(t) - \frac{L}{2}, & Y(t) - X_0 > \frac{L}{2}; \\ Y(t) + \frac{L}{2}, & Y(t) - X_0 < -\frac{L}{2}; \end{cases}$$

Here $X(t)$ is a displacement of the cylinder's center, $Y(t)$ is a displacement of the piston in a horizontal plane.

It should be pointed out one more time that such a converter is considered on the monotonic inputs. On the piece-wise monotonic inputs this operator can be determined using the semi-group identity [12]

$$\Gamma[X(t_1), L]Y(t) = \Gamma\left[\Gamma[X_0, L]Y(t_1), L\right]Y(t).$$

And then, using the special limit construction such an operator can be redefined on all continuous functions.

2.1 Physical model

Let us assume that the deviation y and angle θ are small, i.e., $x \approx \bar{x}$ and the boundary conditions that determine the curvature of the pendulum are[1]:

$$\begin{cases} y(0,t) = y_{xx}(0,t) = 0, \\ y_{xx}(l,t) = y_{xxx}(l,t) = 0. \end{cases} \tag{1}$$

The function $X(\bar{x}, t)$ describes behavior of the pendulum's profile in time and shows deviation of the pendulum's points relative to vertical axis, $(X, \bar{x})$ are coordinates of the pendulum's profile, $X(0, t) = s(t)$ is a displacement of the suspension point in horizontal plane.

Coordinate system transformation in the matrix form is given by

$$\begin{pmatrix} X \\ \bar{x} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} X(0,t) \\ 0 \end{pmatrix}. \tag{2}$$

Let us construct the physical model of the considered mechanical system taking into account the backlash in the suspension point of an elastic rod. In order to do this we use the Lagrange formalism.

[1] In this paper we use the following notations: $a_x = \frac{\partial a}{\partial x}$, $a_t = \frac{\partial a}{\partial t}$.

Taking into account that y and θ are small the Lagrange function can be written as:

$$L(t) = \frac{Ms_t^2}{2} + \frac{1}{2}\int_0^l \left[\rho s_t^2 + \rho y_t^2 + \rho(x\theta_t)^2 + \rho(2s_t x\theta_t + 2x\theta_t y_t + 2s_t y_t) + 2\rho g y\theta - EIy_{xx}^2\right] \mathrm{d}x. \tag{3}$$

Taking s as the generalized coordinate in the Lagrange function we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial s_t} - \frac{\partial L}{\partial s} = f(t). \tag{4}$$

Here $f(t)$ is a force joined to the suspension point of a rod.

General peculiarity of the system under consideration is the presence of backlash in the suspension point. Due to the fact that the backlash can be considered as a hysteretic nonlinearity we can use the technique of hysteretic converters. According to classical patterns of Krasnosel'skii and Pokrovskii [12], the hysteretic operators are treated as converters in an appropriate function spaces. The dynamics of such converters are described by the relation of "input-state" and "state-output".

Thus, the force joined to suspension point, can be found from the relation:

$$f(t) = \Gamma\left[X(0,t), Y(t), L, F_0\right] F = \begin{cases} 0, & |X(0,t) - Y(t)| \leqslant L; \\ F, & |X(0,t) - Y(t)| > L, \end{cases} \tag{5}$$

where L is the length of a cylinder, F is a force (this force affects the piston) which can be treated as a control.

The equation of motion of a piston is:

$$m_p Y_{tt}(t) = F. \tag{6}$$

Here Y is a displacement of the piston in a horizontal plane.

Passing to coordinate system $(X, \bar{x})$ the system of equations which describes the physical model of the considered mechanical system has the following form[2]:

$$\begin{cases} X_{tt} + \dfrac{EI}{\rho} X_{xxxx} = gX_x(0,t), \\ MX_{tt}(0,t) + mgX_x(0,t) + EIX_{xxx}(0,t) = f(t), \\ (M+m)X_{tt}(0,t) + ml(X_{tt})_x(0,t) = f(t), \\ g(M+m)X(0,t) - \dfrac{MEI}{\rho} X_{xxx} = lf(t), \\ f(t) = \Gamma\left[X(0,t), Y(t), L, F_0\right] F, \\ m_p Y_{tt}(t) = F. \end{cases} \tag{7}$$

2.2 Stabilization

Let us consider the problem of control of the pendulum using the feedback principles, i.e., the force which affects the piston can be presented by the following equality:

$$F = k\,\mathrm{sign}(\alpha e_1 + e_2), \tag{8}$$

[2] In order to reduce the description we do not present all the steps of transformations because of their obviousness

where $\alpha > 0,\ k > 0$ and

$$e_1 = \int_0^l X_x \mathrm{d}l, \tag{9}$$

$$e_2 = \int_0^l (X_t)_x \mathrm{d}l. \tag{10}$$

Here e_1 is an average angle of rod's deviation, e_2 is an average angular velocity of the rod.

Thus, in order to solve the stabilization problem for the elastic inverted pendulum we should use the system of equation (7) together with the equalities (8)-(10):

$$\begin{cases} X_{tt} + \dfrac{EI}{\rho} X_{xxxx} = gX_x(0,t), \\ MX_{tt}(0,t) + mgX_x(0,t) + EIX_{xxx}(0,t) = f(t), \\ (M+m)X_{tt}(0,t) + ml(X_{tt})_x(0,t) = f(t), \\ g(M+m)X(0,t) - \dfrac{MEI}{\rho} X_{xxx} = lf(t), \\ f(t) = \Gamma\left[X(0,t), Y(t), L, F_0\right] F, \\ m_p Y_{tt}(t) = F, \\ F = k\,\mathrm{sign}(\alpha e_1 + e_2), \\ e_1 = \displaystyle\int_0^l X_x \mathrm{d}l, \\ e_2 = \displaystyle\int_0^l (X_t)_x \mathrm{d}l. \end{cases} \tag{11}$$

The solution of the posed problem on stabilization of elastic inverted pendulum in the vicinity of the upper position is consisted in search of the optimal values for coefficients α and k. Let us note that the numerical realization of the presented problem can be made using difference scheme.

3 OPTIMIZATION PROBLEM

As was mentioned above the solution of the problem on stabilization of elastic inverted pendulum in the vicinity of the upper position is consisted in search of the optimal values for coefficients α and k from the equality (8).

In many technical problems the question on stabilization has a general interest. However, together with the stabilization of the system there is the problem of optimization (this problem corresponds to asymptotically optimal characteristics of the system). In the system under consideration the problem of optimization corresponds to minimizing of the functional which determines the deviation of the pendulum from the vertical position. Let us consider a functional (the so-called objective functional) as follows:

$$\mathfrak{J} = \frac{1}{T} \int_0^T \left\{ \int_0^l (X_x)^2 \mathrm{d}l + \int_0^l [(X_t)_x]^2 \mathrm{d}l \right\} \mathrm{d}t. \tag{12}$$

Here T is the time interval in which we find an optimal control.

Solution of the equations (11) that describe the dynamics of the system under consideration should be obtained under conditions that provide the minimization of functional (12). Physically this means that the problem is equivalent to minimization of mean-square deviation of the pendulum relative to vertical position.

In order to solve the optimization problem in the system under consideration we use the bionic algorithms of adaptation because the hysteretic peculiarities in the considered pendulum's model lead to some difficulties in use of the classical optimization algorithms due to non-differentiability of the functions in the system of equations.

Such algorithms are a part of the line of investigation which can be called as "adaptive behavior". Main method of this line consists in the investigation of artificial organisms (in the form of computer program or a robot) that are called as animats (these animats can be adapted to environment). The behavior of animats emulates the behavior of animals.

One of the actual line of investigation in the frame of animat-approach is an emulation of searching behavior of animals. Let us consider the bionic model of adaptive searching behavior on the example of caddis flies larvae or *Chaetopteryx villosa*. Main schema of searching behavior can be characterized by two stages:

- Motion in a chosen direction (conservative tactics);
- Random change of the motion direction (stochastic searching tactics).

We consider this model for the simple case of maximum search for the function of two variables. Let we describe main stage of the considered model:

1. We consider an animat which is moved in the two-dimensional space x, y. Main purpose of animat is maximum search for the function $f(x, y)$.
2. Animat is functioned in discrete time $t = 0, 1, 2, \ldots$. Animat estimates the change of current value of $f(x, y)$ in comparison with the previous time $\Delta f(t) = f(t) - f(t-1)$.
3. Every time animat moves so its coordinates x and y change by $\Delta x(t)$ and $\Delta y(t)$ respectively.
4. Animat has two tactics of behavior: a) conservative tactics; b) stochastic searching tactics.

Displacement of animat in the next time $\Delta x(t+1)$, $\Delta y(t+1)$ for these tactics determines in a different ways. Switching between the cycles drives by $M(t)$. Time dependence of $M(t)$ can be determined using the equation:

$$M(t) = k_1 M(t-1) + \xi(t) + I(t), \tag{13}$$

where k_1 is a parameter which determines the switching persistence of tactics $(0 < k_1 < 1)$, $\xi(t)$ is a normal distributed variate with an average value equal to zero and mean-square deviation equal to σ, $I(t)$ is an intensity of irritant. For the value of $I(t)$ there are two possibilities:

$$I(t) = k_2 \Delta f(t) \tag{14}$$

and

$$I(t) = k_2 \frac{\Delta f(t)}{f(t-1)}, \tag{15}$$

where $k_2 > 0$. As follows from (14) and (15) the intensity is positive when the step leads to increasing of function, otherwise the intensity is negative. It should be noted also that the equation (15) can be applied in the case $f(t) > 0$.

We assume that at $M(t) > 0$ animat follows the tactics a) and at $M(t) < 0$ it follows tactics b). So, the value of $M(t)$ can be considered as a motivation to selection of tactics a).

Thus, the algorithm of maximum search can be considered as follows:

Tactics a): Animat moves in the chosen direction. The displacement of animat is determined by R_0

$$\Delta x(t+1) = R_0 \cos\varphi_0, \tag{16}$$

$$\Delta y(t+1) = R_0 \sin\varphi_0, \tag{17}$$

where the angle φ_0 defines the constant direction of motion of animat:

$$\cos\varphi_0 = \frac{\Delta x(t)}{\sqrt{\Delta x^2 + \Delta y^2}}, \tag{18}$$

$$\sin\varphi_0 = \frac{\Delta y(t)}{\sqrt{\Delta x^2 + \Delta y^2}}. \tag{19}$$

Tactics b): Animat makes an accidental turn. The displacement of animat is determined by r_0 but the direction of motion is accidentally varied

$$\Delta x(t+1) = r_0 \cos\varphi, \tag{20}$$

$$\Delta y(t+1) = r_0 \sin\varphi, \tag{21}$$

where $\varphi = \varphi_0 + w$, φ_0 is an angle which characterizes the direction of motion at current time t, w is a normal distributed variate (average value of w equal to zero and mean-square deviation equal to w_0), φ is an angle which characterizes the direction of motion at time $t+1$.

In that way we can use the proposed algorithm for searching the optimal control in the problem of stabilization of elastic inverted pendulum. Taking into account the reasoning presented above we can apply the presented algorithm to functional $\mathfrak{J}(\alpha, k)$ where the coefficients α and k determine the character of control of the mechanical system under consideration following the equation (8). Due to the fact that the presented bionic algorithm is used to maximum search of the function of two variables we will consider minimization of functional (12) as a procedure for finding the coefficients α and k that lead to realization of the condition

$$-\mathfrak{J}(\alpha, k) \to \max. \tag{22}$$

4 SIMULATION RESULTS

In this section we present a simulation of the behavior of elastic inverted pendulum with the backlash in suspension. We use the corresponding difference scheme and the bionic algorithm for finding of the optimal values of coefficients α and k.

The characteristics and initial conditions for the mechanical system under consideration are:

$$m = 1\,\mathrm{kg},\; M = 10\,\mathrm{kg},\; m_p = 1\,\mathrm{kg}\, l = 1\,\mathrm{m},\; \rho = 0.5,\; E = 10,\; I = 4,\; \theta_0 = 0.06^\circ.$$

In order to estimate the stability of the system under consideration we also use the Lyapunov criterion. Namely, we use the following Lyapunov function:

$$V = e_1^2 + e_2^2.$$

Using the bionic algorithm we have obtained the following optimal values of coefficients: $\alpha = 9$ and $k = 2$. The phase trajectory of such a system together with the dynamics of Lyapunov function in time (namely, in discrete time which corresponds to difference scheme) for different values of a control coefficients are presented in the Fig. 2. In this figure the integral angle e_1 and integral angular velocity e_2 correspond to equations (9) and (10) respectively. As we can see from presented figures the Lyapunov function satisfies the following condition

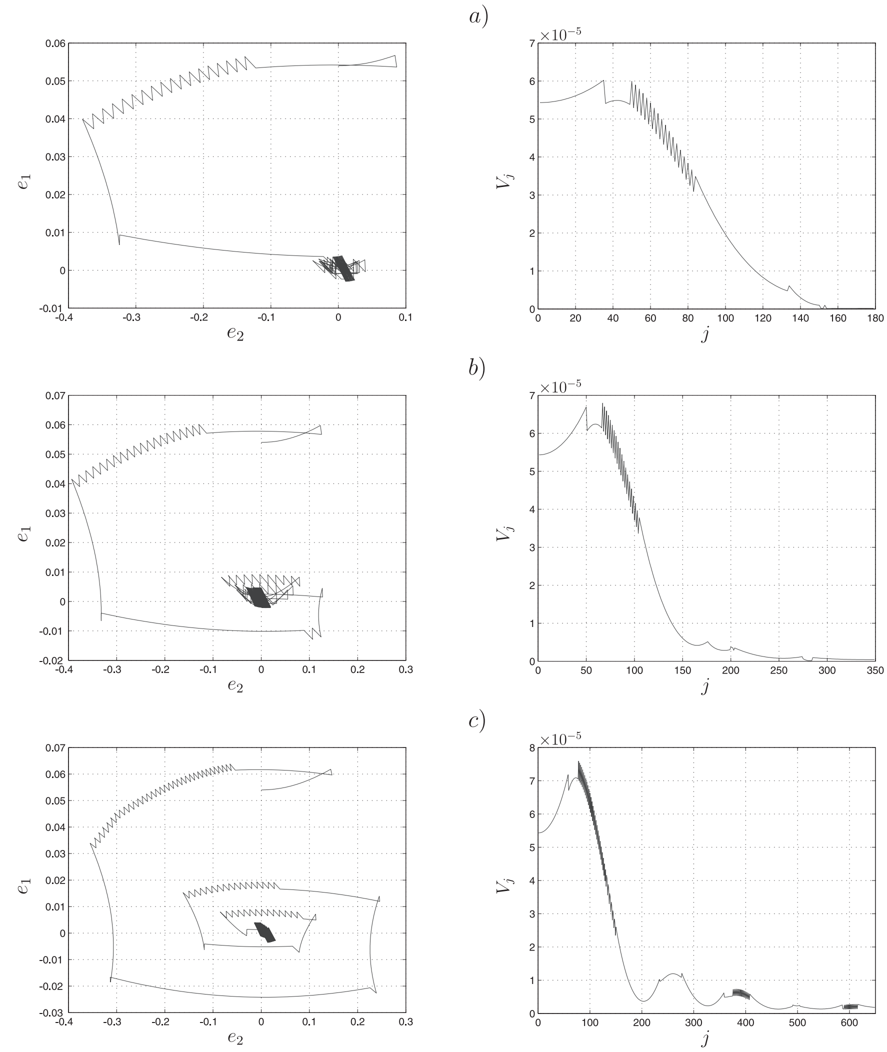

Figure 2: Phase trajectories (left panels) and dynamics of Lyapunov function (right panels) in the presence of a backlash in suspension point. Parameters of a backlash and control coefficients are: $a)$ – $L = 0.01\,\mathrm{m}$, $\alpha = 9$, $k = 2$; $b)$ – $L = 0.02\,\mathrm{m}$, $\alpha = 9$, $k = 2$; $c)$ – $L = 0.02\,\mathrm{m}$, $\alpha = 10.5$, $k = 1.5$.

(during all the considered time interval):

$$V_t(t) \leqslant 0.$$

This means that the considered inverted pendulum eventually tends to stable vertical position.

5 CONCLUSIONS

In this paper we have considered the stabilization problem of elastic inverted pendulum under hysteretic control in the form of backlash in suspension point. Also the problem of optimization for the system under consideration is analyzed. Main coefficients, namely α and k that provide the solution of optimization problem for the considered system are obtained using the so-called bionic algorithm. All the numerical results on stabilization of the system under consideration have obtained using the numerical method based on the difference scheme. The results of numerical simulations show that the considered system eventually tends to stable state. This fact is presented in the form of corresponding phase portraits for the considered system. Moreover, in order to estimate the stability of the elastic pendulum with the backlash in the suspension point we have used the Lyapunov criterion and the dynamics of corresponding Lyapunov function has also been presented.

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