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COMPARISON OF NUMERICAL AND SEMI-ANALYTICAL SOLUTION OF A SIMPLE NON-LINEAR SYSTEM IN STATE OF THE STOCHASTIC RESONANCE

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Abstract. *Stochastic resonance (SR) is a phenomenon which can be observed at some non-linear dynamic systems under combined excitation including deterministic harmonic force and random noise. The Duffing single degree of freedom oscillator is treated and the Gaussian white noise as the random excitation component is considered. Mathematical basis of this phenomenon follows from properties of the Duffing system with negative linear part of the stiffness. Under certain combinations of the system and excitation parameters the SR can emerge. It manifests by stable periodic hopping between two nearly constant limits perturbed by random noises. SR is observed and practically used in a number of disciplines in physics, biophysics, chemistry, etc. However it seems to be promising as a theoretical model of several aeroelastic post-critical effects arising at a prismatic beam in a cross flow.*

Three independent theoretical solution methods have been addressed and tested in order to compare results of the system response. The first kind is semi-analytic dealing with the relevant Fokker-Planck Equation (FP). It is solved by means of the stochastic moment procedure. The multiharmonic non-stationary solution of the Probability Density Function (PDF) is expected. The Galerkin approach is adopted. The second way is based on the FEM solution of the FP equation. It is analyzed in an original evolutionary form which enables an analysis of transition effects starting the Dirac type initial conditions. The last procedure represents simulation regarding the original Duffing or relevant Ito stochastic system. Comparison of results provided by the above three methods has revealed appropriate domains of their application to particular problems regarding a preliminary analysis or careful detailed inspection in specific small domains in final stage of an engineering system design.

1 INTRODUCTION

Stochastic resonance (SR) is a phenomenon which can be observed at some non-linear dynamic systems under combined excitation including deterministic harmonic force and random noise, see for instance papers [1], [2], monograph [3] and many other resources. This phenomenon manifests by stable periodic hopping between two nearly constant limits perturbed by random noises. SR is observed in a number of disciplines in physics, life sciences and industry (optics, plasma physics, chemistry, biophysics, etc.). The occurrence of this phenomenon depends on certain combinations of input parameters, which can be determined theoretically and verified experimentally. The long-term experience shows that SR can be assumed either as a dangerous effect of a post-critical system response which should be avoided (plasma physics, aeroelasticity, etc.) or oppositely as a requested process representing functionality of the system itself (optics, special excitation devices, etc.).

With reference to wind channel observations in a wind channel, it seems that SR is promising as a theoretical model inherent for several aeroelastic post-critical effects arising at a prismatic beam in a cross flow. In particular the divergence or buffeting of a bridge deck can be modeled as a post-critical process of the SR type at an SDOF or TDOF system. Dealing with a particular project these post-critical effects should carefully investigated in order to eliminate any danger of the bridge deck collapse due to aeroelastic effects.

The mathematical basis of SR follows in the most simple case from properties of the Duffing equation with negative linear part of the stiffness. Therefore in this study the Duffing SDOF oscillator is treated under excitation by a Gaussian white noise together with a deterministic harmonic force with a fixed frequency. It should be noticed that also other types of nonlinearity can produce a significant periodic part of the response when excited by additive random noise. See for instance [4], where influence of bistable nonlinear damping is discussed.

Let us assume the nonlinear mass-unity oscillator with one degree of freedom:

$$\begin{aligned} \dot{u} &= v; \\ \dot{v} &= -2\omega_b \cdot v - V'(u) + P(t) + \xi(t). \end{aligned} \tag{1}$$

$V(u)$ - potential energy being introduced in a form corresponding with the Duffing equation:

$$V(u) = -\frac{\omega_0^2}{2}u^2 + \frac{\gamma^4}{4}u^4 \qquad \Rightarrow \qquad V'(u) = dV(u)/du = -\omega_0^2 \cdot u + \gamma^4 \cdot u^3 \tag{2}$$

$\xi(t)$ - Gaussian white noise of intensity $2\sigma^2$ respecting conditions:

$$\mathbf{E}\{\xi(t)\} = 0 \; ; \qquad \mathbf{E}\{\xi(t)\xi(t')\} = 2\sigma^2 \cdot \delta(t - t') \tag{3}$$

$P(t) = P_o \exp(\mathrm{i}\Omega t)$ - external harmonic force with frequency Ω. Amplitude P_o should be understood per unit mass.

Symbols ω_0 and ω_b have a usual meaning of the circular eigen-frequency and circular damping frequency of the associated linear system. The linear part of the $V'(u)$ is negative making the system metastable in the origin, while the cubic part acts as stabilizing factor beyond a certain interval of displacement u. The system is drafted in the Fig. 1 in two versions: (a) system with symmetric potential typical by an equivalent energy needed for hopping from the left into the right potential well and in opposite direction (b) system with asymmetric potential due to the supplementary linear string which could be able (when rising its stiffness) to bring the oscillator to monostable type.

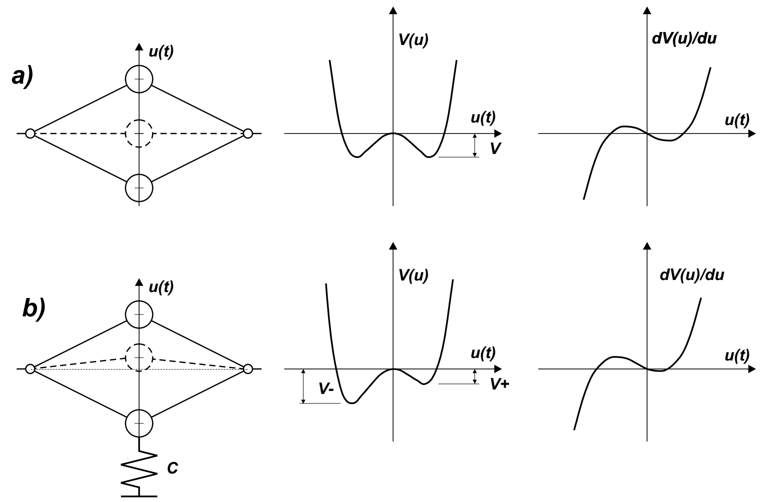

Figure 1: Bistable nonlinear system: a) Symmetric potential; b) Non-symmetric potential.

Response of the system defined by Eq. (1) can be significantly enhanced introducing an optimal amount of additive random noise, which results in the phenomenon called stochastic resonance. The traditional interpretation of stochastic resonance involves the concept of photon inter-well hopping in nonlinear optics, see e.g. [1]. For the symmetric potential ($\Delta V- = \Delta V+$) and in the absence of periodic forcing the approximate frequency of escape from one well into the second (bistable system of type (2)) is given by the following estimate published in the comprehensive study [5]:

$$\omega_e = \sqrt{2} \cdot \exp(-\Delta V/T) \tag{4}$$

where ΔV is height of the barrier separating potential minima and T absolute temperature. It is obvious that T is proportional to σ^2 introduced in Eq. (3). The formula (4) is a result of theoretical and empirical investigation motivated by problems of nonlinear optics. However, it is widely used and works very well. During the last decade a number of areas of optics, quantum mechanics, chemistry, neurophysiology, etc. investigated this formula attempting to use the phenomenon of stochastic resonance for description of various effects arising in their branches using both experimental and theoretical ways of investigation, see e.g. [2, 6].

The Duffing equation under above excitation is treated addressing three independent theoretical solution methods in order to compare results of the system response in general and mainly looking for system and excitation parameters leading to the SR phenomenon.

2 FOKKER-PLANCK EQUATION – SEMI-ANALYTIC SOLUTION

The first procedure considered is based on a manipulation with the response probability density function of the system (1). It is commonly supposed that input and output can be regarded as Markov type processes. In such a case the most effective analytic tool of investigation offers the Fokker-Planck Equation. We try to formulate a semi-analytic solution using the stochastic moment procedure with non-Gaussian closure. The multiharmonic solution is suggested and therefore the solution in a periodic form is fixed. The transition effects due to initial conditions are omitted and only quasi-stationary part is investigated.

Taking into account that random noise in Eq. (1) has an additive character, no Wong-Zakai correction terms emerge, see e.g. [7, 8, 9]. Then the relevant FP equation, e.g. [10] can be easily written out:

$$\kappa_u = v \; ; \quad \kappa_v = -2\omega_b \cdot v - V'(u) + P(t) \; ; \quad \kappa_{vv} = 2\sigma^2 \tag{5}$$

$$\frac{\partial p(u,v,t)}{\partial t} = -v\frac{\partial p(u,v,t)}{\partial u} + \frac{\partial}{\partial v}[2\omega_b \cdot v + V'(u) - P(t)]p(u,v,t) + \sigma^2 \frac{\partial^2 p(u,v,t)}{\partial v^2} \tag{6}$$

together with boundary and initial conditions:

$$\lim_{u,v\to\pm\infty} p(u,v,t) = 0 \;\; \text{(a)} \qquad p(u,v,0) = \delta(u,v) \;\; \text{(b)} \tag{7}$$

Provided $P_0 = 0$ and the external excitation is limited to random component only, the right side of Eq. (6) becomes time independent and it is meaningful to investigate the FP equation with canceled left side. In such a case it admits the closed form time independent solution of the Boltzmann type. For details, see e.g. [11, 12, 13], and other papers and monographs. The appropriate solution has the form:

$$p_o(u,v) = D \cdot \exp\left(-\frac{2\omega_b}{\sigma^2} H(u,v)\right) \tag{8}$$

where D is the normalizing constant and $H(u,v)$ represents the Hamiltonian function of the basic system. In particular:

$$H(u,v) = \frac{1}{2}v^2 + V(u) = \frac{1}{2}v^2 - \frac{1}{2}\omega_o^2 u^2 + \frac{1}{4}\gamma^4 u^4 \tag{9}$$

which implicates:

$$\begin{gathered} p_o(u,v) = p_u(u) \cdot p_v(v), \\ p_u(u) = D_u \cdot \exp(-\frac{2\omega_b}{\sigma^2}(\frac{1}{2}\omega_o^2 u^2 + \frac{1}{4}\gamma^4 u^4)), \quad p_v(v) = D_v \cdot \exp(-\frac{2\omega_b}{\sigma^2}\frac{1}{2}v^2) \end{gathered} \tag{10}$$

indicating that u, v are stochastically independent processes.

It is obvious that solution of type Eq. (8) or (10) type can be provided for any arbitrary symmetric/non-symmetric potential including cases leading the system into monostable type. PDF of the response has significantly non-Gaussian form in displacement u (see Fig. 2) and Gaussian form in velocity v. The final shape is influenced by quotient $2\omega_b/\sigma^2$, i.e. approaching zero as quickly as the damping ω_b rises and white noise intensity σ^2 decreases.

When excitation force consists of both component ($P_0 > 0$), time independent solution of Eq. (6) no more exists. In order to approximate its solution, the formula (8) can be used as a basic part which should be multiplied by a space and time dependent series. The PDF can be expected periodic or cyclic-stationary in a certain meaning of the term in time coordinate for $t \to \infty$ and the Floquet theorem or the maximum entropy principle, e.g. [11] could also be alternatively used as a basic tool for the solution.

With respect to the linearity of the FP equation the periodicity of the PDF should be relevant to the frequency of the deterministic excitation component Ω and its integer powers. Therefore the series can be written in the following form:

$$p(u,v,t) = p_o(u,v)\sum_{j=0}^{J} q_j(u,v) \cdot \exp(\mathrm{i}j\Omega t) \tag{11}$$

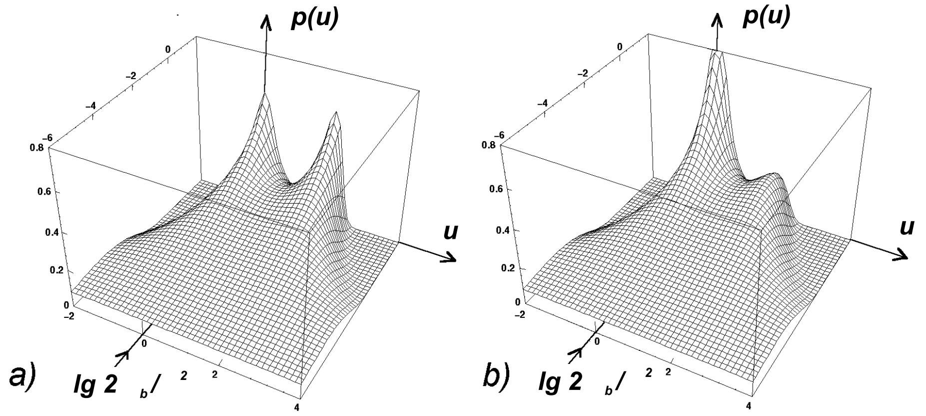

Figure 2: Response PDF of the system excited by white noise only: (a) Symmetric potential; (b) Non-symmetric potential.

where Ω is the harmonic component excitation frequency. The series Eq. (11) represents an approach of a weak solution of FP equation, which repeats in the period $T = 2\pi/\Omega$. It gives a true picture of solution within one period, but cannot express any influence of initial conditions.

The unknown functions $q_j(u,v)$ in Eq. (11) can be searched for using the generalized method of stochastic moments as it can be found e.g. in [10]. Using the Galerkin method approach, the expression (11) should be substituted into Eq. (6) and the whole equation should be multiplied by the testing functions $\alpha(u,v)$.

The testing functions $\alpha(u,v)$ and unknown function $q_j(u,v)$ are assumed to have a following advantageous form:

$$\alpha(u,v) = \alpha_{r,s}(u,v) = u^r \cdot H_s(\beta v) \; ; \quad r = 0,...,R \; ; \; s = 0,...,S \tag{12}$$

$$q_j(u,v) = \sum_{k,l=0}^{R,S} q_{j,kl} u^k \cdot H_l(\beta v) \tag{13}$$

where $H_s(\beta v)$ are l'Hermite polynomials and $\beta = \sqrt{\omega_b/\sigma^2}$. After applying the mathematical mean operator with respect to probability density function $p_o(u,v)$, see (8), and employing orthogonality of l'Hermite polynomials a linear algebraic system for unknown coefficients $q_{j;k,l}$ arises ($q_o(u,v) = 1, \; q_{-1}(u,v) \equiv 0$):

$$2\beta(\mathrm{i}j\Omega + 2\omega_b s)\mathbf{A}\mathbf{q}_{j,s} - 2(s+1)\mathbf{C}\mathbf{q}_{j,s+1} + \mathbf{B}\mathbf{q}_{j,s-1} = 2\beta^2 P_o \mathbf{A}\mathbf{q}_{j-1,s-1} \tag{14}$$

$\mathbf{q}_{j,s} = [q_{j,0s}, q_{j,1s}, ..., q_{j,Rs}]^T$ - column vector ($R+1$ components) and $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{(R+1)\times(R+1)}$ - square arrays containing moments:

$$A_{r,k} = \int_{-\infty}^{\infty} u^{r+k}\Phi(u)du \; ; \quad B_{r,k} = \int_{-\infty}^{\infty} k u^{r+k-1}\Phi(u)du \; ; \quad C_{r,k} = \int_{-\infty}^{\infty} r u^{r+k-1}\Phi(u)du$$
$$\Phi(u) = \exp(\beta\omega_o^2 u^2 - \tfrac{1}{2}\gamma^4 u^4) \tag{15}$$

Function $\Phi(u)$ is symmetric with respect to zero and therefore $A_{r,k} = 0$ for odd $r+k$, while $B_{r,k}, C_{r,k}$ vanish for even $r+k$.

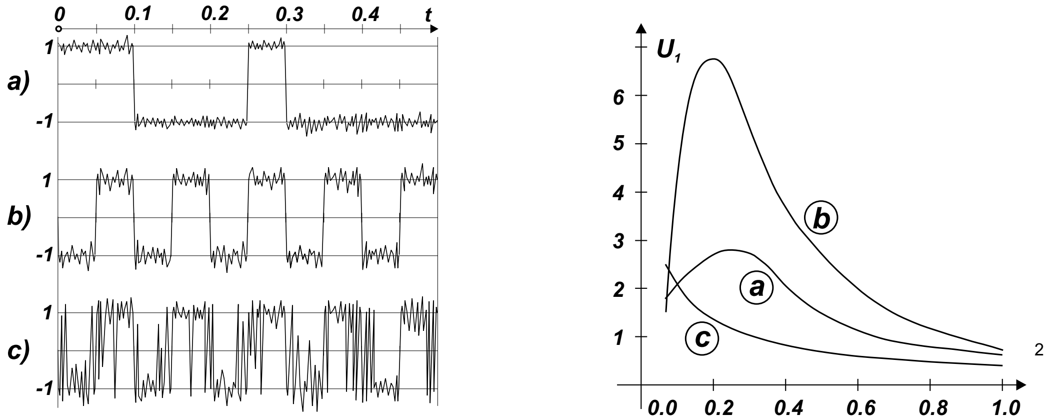

Figure 3: Typical response of a system with combined excitation - white noise and harmonic (left). The spectral amplification of the system response due to stochastic resonance effect (right). Three levels of noise are assumed: a) low noise, b) optimal noise – stochastic resonance occurs, c) high noise .

For each j the three-term recurrence formula (14) forms an algebraic system of size $(S+1)(R+1) \times (S+1)(R+1)$ for all unknown coefficients $q_{j,rs}$. The block diagonal of the system matrix consists from scaled regular matrices $\mathbf{A}$, see (15), and thus it is invertible.

Let us deal with the main idea of the solution method from other point of view. The Eq. (6) represents a linear partial differential equation of the second order with non-symmetric operator defined on an infinite domain, with homogeneous boundary conditions and parametric "excitation". It can be shown, that the weak solution exists and can be found using variational methods. Nevertheless some properties of the operator (non-symmetry, etc.) are limiting in a choice of methods. For instance Ritz method cannot be used due to operator non-symmetry. The modified Galerkin method seems to be acceptable. The presented solution process corresponds in principle to this method. The basic series employed is given by the formula (11) together with Eq. (13). Because $q_j(u,v)$ are polynomials and $p_o(u,v)$ can be interpreted as a weight function, every term of the series (11) separately fulfills the boundary conditions requested for the unknown $p(u,v,t)$ and its derivatives in u, v. The series (11) with respect to Eqs (8,9) is then substituted in fact into the Eq. (6) and then the orthogonality to the function $\alpha(u,v)$ for any arbitrary r,s is requested. The proces of orthogonalization corresponds to the application of mathematical mean value operator with respect to the PDF defined by Eq. (8). Equivalence appeared ascertains the convergence of the approximate formula (13), see e.g. [14]. Therefore the PDF can be reached with any prescribed accuracy. Due to structure of Eq. (11) the same holds also for stochastic moments which are to be used studying various physical effects related with PDF.

In order to validate the approaches presented in the this paper, the differential system (1) will be examined using several numerical methods. The following values of the individual parameters are adopted: $\omega_b = 0.5\,\mathrm{rad.\,s^{-1}}, \omega_0^2 = 1\,\mathrm{rad.\,s^{-2}}, \gamma^4 = 1\,\mathrm{rad.\,m^{-2}\,s^{-2}}, P_o = 0.4\,\mathrm{m.s^{-2}}, \Omega = 0.0942\,\mathrm{rad.s^{-1}}$. The stochastic resonance occurs for noise level $\kappa_{vv} = 2\sigma^2 = 0.25$, where the signal-to-noise ratio curve reaches its maxima, cf. the case b) in Fig. 3.

To approximate the time dependent probability density function according to relation (11) the coefficient matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ has been computed according to relation (15) and the block tridiagonal matrix was set up following the three-term recurrence formula (14). The values of R, S has been selected to $R = S = 8$ because for higher polynomial degrees the system matrix became badly conditioned and serious numerical errors were introduced in the solution.

The resulting probability density function varies in time with periodicity, which correspond

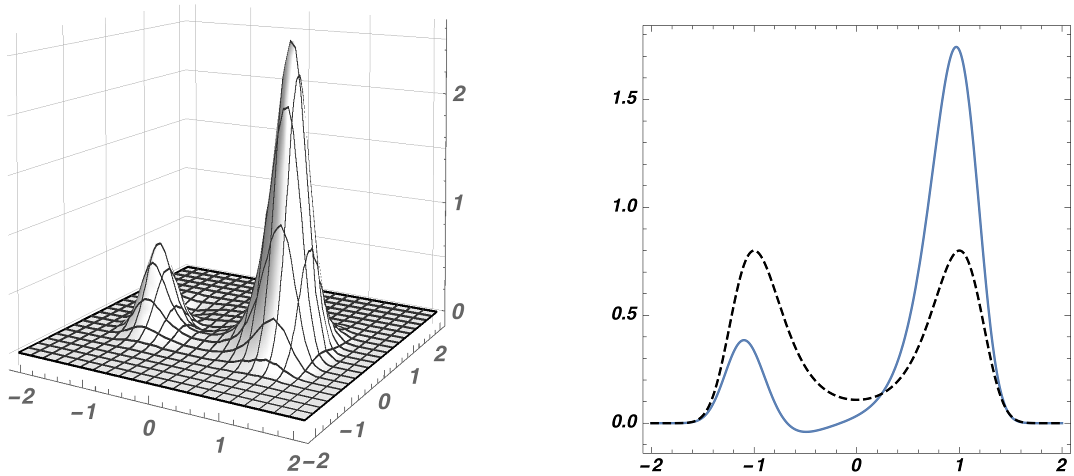

Figure 4: Left: the probability density function according to approximate relation (11) for $t = 30$. Right: the corresponding planar section for $v = 0$ (solid line) and stationary solution of the Boltzmann type (dashed).

to the frequency of external loading Ω. The individual peaks alternate but the lower peak never vanishes completely, see Fig. 4. The computed joint probability density is shown in the left figure, the corresponding curve for the displacement variable u (section for $v = 0$) is in the right graph. The solid line shows the computed time-dependent probability density for $t = 30$, the dashed line corresponds to the stationary solution of the Boltzmann type $p_o(u, v)$, see Eq. (8). Only $J = 5$ terms of expansion Eq. (11) was used in this case, the shape of the curves was not influenced by higher terms of Fourier expansion.

Despite of good theoretical properties of the polynomial ansatz Eqs (12-13) the current model does not fit well the expected flat (low probability) areas. Moreover, there is no theoretical reason preventing subtle overshoots into the negative values. Occurrence of these overshoots is understandable from numerical point of view, however, it can cause problems in subsequent analysis of the approximate probability density function. On the other hand, the presented approach brings results which can provide insight into the problem and mediate a good qualitative analysis of the stochastic resonance effect.

3 FOKKER-PLANCK EQUATION – FEM NUMERICAL SOLUTION

The second solution procedure is based on the FEM solution of the FP. The FP is analyzed in an original evolutionary form which enables an analysis of transition effects starting the (nearly) Dirac type initial conditions. The FEM efficiency when solving FP which follows from the Duffing stochastic differential equation without external harmonic forces was already studied by the authors in [15]. With the periodic force taken into account, certain difficulties arise due to the time inhomogeneity of the corresponding stochastic process.

The method is based on the approximaltion solution of Eq. (6) in the Galerin-Petrov meaning on the piecewise smoothly bounded domain $\Psi \in u \times v$, in $\mathbb{R}^d, d = 2$. The initial conditions at $t = 0\,$s for PDF are considered in a form of the Gauss distribution function with an initial system position at the point $u_0 = 0, v_0 = 0$. For a small values of standard deviation it approaches to the Dirac function as it is primarily requested.

After a spatial discretizating of Ψ onto the rectangular finite elements using the bilinear approximation functions and implying boundary condition $p(\partial\Psi, t) = 0$, the system of ordinary differential equations emerge with global matrices $\mathbf{M}$, $\mathbf{K}(t)$ and vector of PD values $\mathbf{p}(t)$ in

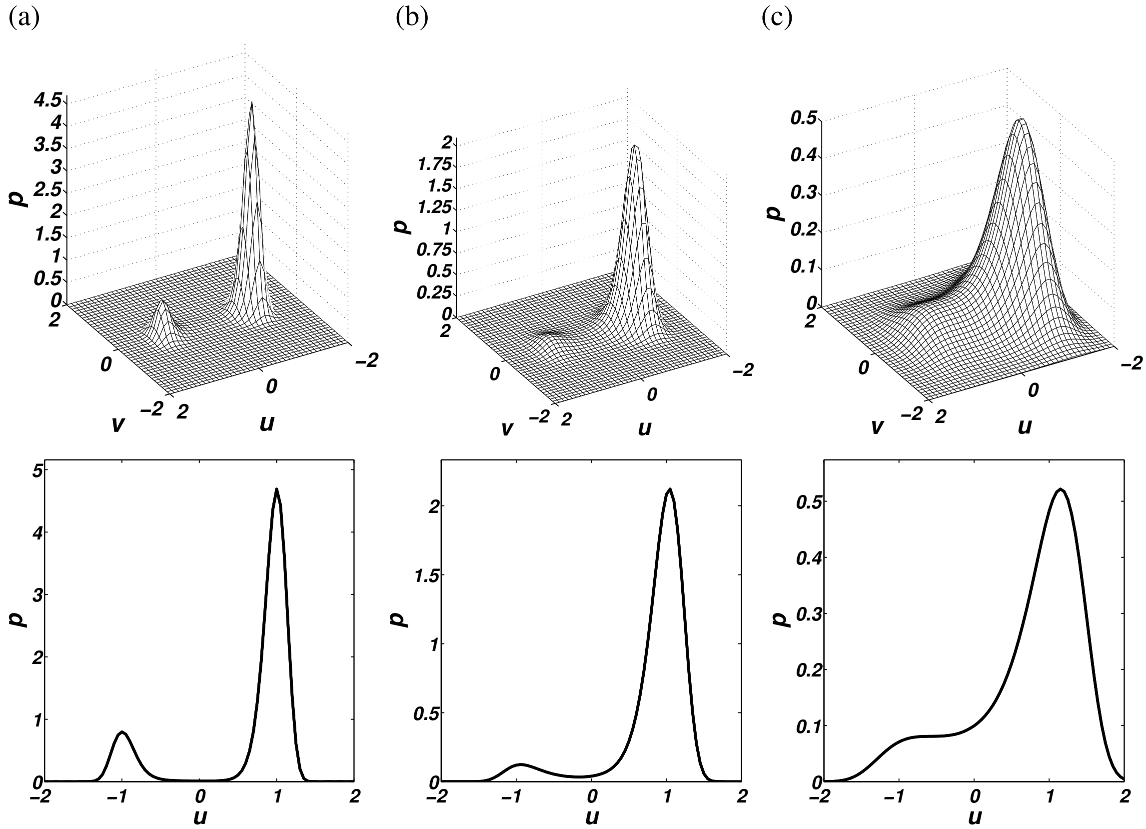

Figure 5: Axonometric and sectional display of the PDF at the highest value of probability of residing in selected potential well: a) $\kappa_{vv} = 0.10$; b) $\kappa_{vv} = 0.25$ - Stochastic resonance c) $\kappa_{vv} = 1.0$.

nodes of the mesh:

$$\mathbf{M}\dot{\mathbf{p}}(t) = \mathbf{K}(t)\mathbf{p}(t). \tag{16}$$

The matrix $\mathbf{K}(t)$ is time-dependent due to the periodic perturbation entering the drift term of FP and, in the result, the solution oscillates periodically between the potential wells. In the regime of SR, the switchings are in phase with the external periodic signal $P(t)$ and the mean residence time is most close to half the signal period $2\pi/\Omega$.

Since convection indicators of SR such as Spectral Power Amplification (SPA), the Signal-to-Noise Ratio (SNR) or the residence-time distribution density are not applicable in the present approach, other way of treating the stochastic response in SR regime is adopted. As a specific measure of optimally selected noise intensity, the integral value of PD distribution over one potential well, $u \in (0, \infty)$, for various κ_{vv} is introduced as:

$$P_N(t;\kappa_{vv}) = \int_{-\infty}^{\infty} \left(\int_0^{\infty} p(u,v,t;\kappa_{vv})\mathrm{d}u \right) \mathrm{d}v \tag{17}$$

that is periodic in time with the amplitude $A_p(\kappa_{vv})$. The stochastic resonance occurs for such κ_{vv} for which the amplitude $A_p(\kappa_{vv})$ assumes its maximum.

Let us solve Eq. (6) with the identical parameters as introduced in the previous section. The computational domain in size of $u, v \in \langle -10, 10 \rangle$ is adopted and split onto the structured mesh grid with an element edge size of $0.1\,\mathrm{m}$. Selected results of the FEM solution are depicted in Fig. 5. It contains a response PDF in the axonometric view and vertical section under various noise intensities. The figures represent the solution taken in the time instant corresponding

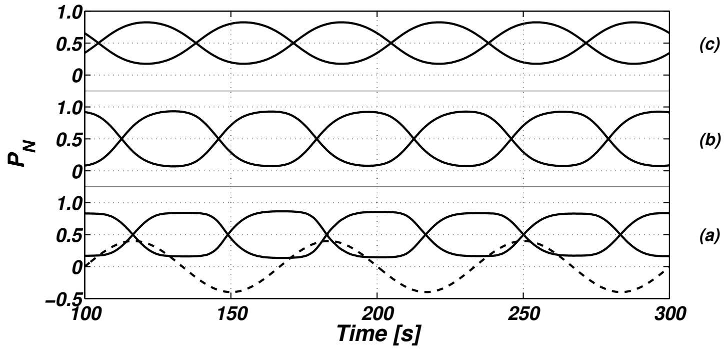

Figure 6: Time history of integral value of PDF over positive and negative domains $P_{N,\langle -10,0\rangle}$ and $P_{N,\langle 0,10\rangle}$ for a) $\kappa_{vv} = 0.10$; b) $\kappa_{vv} = 0.25$ - Stochastic resonance c) $\kappa_{vv} = 1.0$.

to the highest value of probability of residing in one selected potential well. In the temporal evolution case, the peaks periodically oscillate between the relevant potentials with a period close to the excitation frequency. Note that the role of transient effect following from the initial condition has subsided.

Fig. 5(a) shows the approximation PDF at a low noise intensity $\kappa_{vv} = 0.10$ with regards to the stochastic resonance. The solution forms a bimodal shape with two strong peaks around the stable points at $(u, v) = (\pm 1, 0)$ separated by a deep saddle point at $(u, v) = 0$. As the noise intensity increases to the optimal value for the stochastic resonance $\kappa_{vv} = 0.25$, the PDF becomes wider over Ψ and consequently, its peak values drop while still maintaining the evident bistable character, see Fig. 5(b). For higher excitation densities the bistable shape together with the saddle point disappear as pointed out for $\kappa_{vv} = 1.0$ in Fig. 5(c).

The results given by Eq. (17) are illustrated in Fig. 6 that depicts the time history of integral value $P_N(t; \kappa_{vv})$ for three sets of noise intensities. Each set consists of one couple of the integral value determined for both the positive and negative integral domain $u \in (\langle -10, 0\rangle, \langle 0, 10\rangle)$. Starting at low density of noise with $\kappa_{vv} = 0.1$, plot (a), the flattened sine-like functions emerge with the amplitude of $A_p(0.1) = 0.864$. As the intensity grows, the curves change the shape towards a typical sine form and the phase shift between the external excitation and integral value function decreases. At optimal dose of noise in sense of stochastic resonance, the system reaches maximum amplitude $A_p(0.25) = 0.927$, plot (b). Beyond this point the amplitude decreases again, as presented in plot (c) for which yields $A_p(1.0) = 0.825$.

4 STOCHASTIC SIMULATION

The last procedure represents simulation regarding the original Duffing or relevant Ito stochastic system. This way is used, among other things because it allows an effective procedure of searching domains of adequate parameters leading to SR phenomenon.

Introducing the potential energy $V(u)$ Eq. (2) into the general oscillator equation (1), the detailed Duffing system reads:

$$\begin{aligned} \dot{u} &= v; \\ \dot{v} &= -2\omega_b \cdot v + \omega_0^2 u - \gamma^4 u^3 + P_o \exp(\mathrm{i}\Omega t) + \xi(t). \end{aligned} \tag{18}$$

The equation Eq. (18) is solved numerically with the same parameters as used before. The time period taken into account was $T = 1000\,\mathrm{s}$ with the time step $\delta_t = 0.05\,\mathrm{s}$. The white noise was simulated as a finite sum of harmonic functions with uniformly distributed random frequencies $\omega_i \in (0, \omega_{max}\rangle$ and phases $\varphi_i \in (0, 2\pi\rangle$, cf. Eq. (19). The parameter ω_{max} represents the noise cut-off frequency, value $\omega_{max} = 10\,\text{rad.s}^{-1}$ was used in the simulation.

$$\xi(t) = \sqrt{2}\sigma \sum_{i=1}^{N} cos(\omega_i t + \varphi_i) \tag{19}$$

The results of the stochastic resonance analysis are illustrated starting by Fig. 7 which presents the signal-to-noise ratio as the function of the noise intensity expressed by $2\sigma^2 = \kappa_{vv}$ and the results (Fourier spectra) of the numerical simulations using the basic system Eqs (18). In the individual spectral lines it can be seen the influence of rising white noise intensity, which acts together with a harmonic force onto the system. For very low level of the noise the harmonic component is hardly able to overcome the inter-well barrier and therefore only seldom irregular jumps between stable points occur.

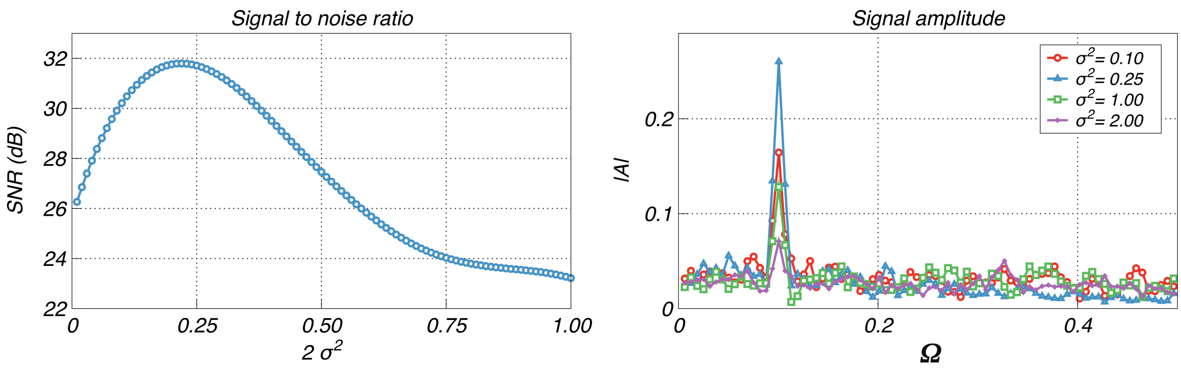

Figure 7: The spectral amplification of the system response due to stochastic resonance effect. The signal-to-noise ratio (SNR) as the function of various noise strengths (σ^2) due to stochastic resonance. Right - Fourier spectra of the response obtained by numerical solution of Eqs (18).

In local regimes the system response is relatively small and nearly linear. Optimal ratio of the noise intensity and the amplitude of the harmonic force results for its certain frequency in the system response containing a visible spectral peaks (amplification) corresponding with the frequency of the external harmonic modulation. The single peak (in the case of coloured noise more peaks may appear) and thus the "optimal" noise strength can be identified.

5 CONCLUSIONS

The response of a dynamic non-linear single degree of freedom system with cubic characteristic to a combination of additive random noise and external deterministic periodic force were studied. Three different approaches have been addressed in order to compare results obtained using various definitions of the problem.

The effect of stochastic resonance has been described firstly theoretically using the FP equation and method of generalized stochastic moments for its approximate solution. The convergence of this method is expected with respect to theoretical properties of the solution process being similar to Galerkin method applied to the equations with non-symmetric operator. The convergence has also been observed when numerical evaluation of formula and algorithms obtained has been done. The non-stationary character of the response PDF with the basic and the higher harmonics has been verified and used for periodic component detection in the input signal containing both random and deterministic parts. Therefore adding noise to a system can improve its ability of information transfer.

The FEM solution of the FP equation exhibits with a high accuracy the effect of stochastic resonance. Concurrently acting additive deterministic input and random excitation lead to the periodically driven differential system with the time-dependent stiffness matrix. In order to identify the effect of stochastic resonance, the transition response of such a noise-controlled stochastic system was examined using integral formula applied over one potential well. This treatment fully detected the optimal dose of noise for the stochastic resonance regime.

The paper describes also the numerical simulation of the Duffing system. It shows, that under certain ”optimal” value of the parameters, the signal-to-noise ratio of the response increases and the resonant-like peak occurs in the amplitude spectra. This makes an optimistic perspective for the experimental analysis, which together with the analytical and numerical ones should continue to obtain better insight into the general tendencies when individual parameters of the system and the input signal are changed.

Possibilities of other analytical solution procedures should be also investigated (Floquet theorem, maximum entropy principle, etc.). Also other types of non-linearities in the system stiffness (especially non-symmetric) should be carefully studied. Other application areas like post-critical processes in aero-elasticity or vehicle dynamics should be thoroughly examined from the point of view of the adverse effects of the response amplification due to stochastic resonance effects.

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