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ADVANCED MODELS FOR THE LIMIT ANALYSIS OF MASONRY STRUCTURES

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Abstract. *We formulate a novel procedure for the limit analysis of masonry structures subject to horizontal loads. The proposed approach works in the framework of free discontinuity methods, on examining collapse mechanisms that exhibit free crack opening discontinuities. Numerical examples illustrate the practical application of the proposed procedure to the limit analysis of masonry structures subject to seismic actions, within a conventional static approach.*

1 INTRODUCTION

In the present work we describe a procedure for determining the ground acceleration that activates the in-plane mechanism of a wall panel with openings. The method generalizes and extends a numerical model proposed in [1], based on the (crude) assumption that the material is unilateral (namely a No-Tension material in the sense of Heyman [2]), leans on the kinematic theorem of limit analysis.

The unilateral model for masonry was first rationally introduced by Heyman in [2]. Several authors of the Italian school of Structural Mechanics has divulgated and extended such model [3–17]. The analysis of masonry-like materials can be conducted within the frame of limit analysis, specifically by applying the static and kinematic theorems on the basis of admissible stress and strain fields (see [18]). The first tool that can be introduced for applying the unilateral No-Tension model to masonry structures is the systematic use of singular stress and strain fields, within the framework defined by the two theorems of Limit Analysis [19, 20]. The use of singular equilibrated stresses for approximating plane equilibrium problems can be traced in [17, 21–23]. In particular, in the application we present, we focus on the kinematic approach. By working with the kinematic theorem, we admit singular strains representing concentrated fractures; in other words we allow for strong discontinuities in the displacements. A recent computational model for fracture nucleation and propagation in 2d brittle solids, based on variational fracture is proposed in [24–26]. Essentially the analysis is based on the variational formulation of Griffith–type fracture [27], the main difference being the fact that we rely on local rather than on global minimization. Nucleation and propagation of fracture is obtained by minimizing in a step by step process a form of energy that is the sum of bulk and interface terms. In the present paper we adopt a similar, though simplified, strategy to explore compatible mechanisms having free discontinuities. The practical implementation of the proposed approach is illustrated through a collection of numerical examples dealing with masonry structures subject to vertical and horizontal loads.

2 THE BOUNDARY VALUE PROBLEM FOR RIGID NO-TENSION MATERIALS

2.1 Constitutive restrictions and equilibrium problem

We consider a body $\Omega \in \mathbb{R}^n$ (here $n = 2$), loaded by the given tractions $\underline{\mathbf{s}}$ on the part $\partial\Omega_{\mathrm{N}}$ of the boundary, and subject to given displacements $\underline{\mathbf{u}}$ on the complementary, constrained part of the boundary $\partial\Omega_{\mathrm{D}}$, is in equilibrium under the action of such given surface displacements and tractions, besides body loads $\mathbf{b}$ and distortions $\underline{\mathbf{E}}$ (the set of data being denoted: $(\underline{\mathbf{u}}, \underline{\mathbf{E}}; \underline{\mathbf{s}}, \mathbf{b})$), and undergoes small displacements $\mathbf{u}$ and strains $\mathbf{E}(\mathbf{u})$[1].

The body Ω is composed of Rigid No-Tension material, that is the stress $\mathbf{T}$ is negative semidefinite

$$\mathbf{T} \in \mathrm{Sym}^- \ , \tag{1}$$

the effective strain $\mathbf{E}^* = \mathbf{E}(\mathbf{u}) - \underline{\mathbf{E}}$ is positive semidefinite

$$\mathbf{E}^* \in \mathrm{Sym}^+ \ , \tag{2}$$

and the stress $\mathbf{T}$ does no work for the corresponding effective strain $\mathbf{E}^*$

$$\mathbf{T} \cdot \mathbf{E}^* = 0 \ . \tag{3}$$

[1]When eigenstrains are considered, under the small strain assumption, the total strain $\mathbf{E}(\mathbf{u})$ is decomposed additively as follows: $\mathbf{E}(\mathbf{u}) = \mathbf{E}^* + \underline{\mathbf{E}}$, $\mathbf{E}^*$ being the *effective* strain of the material.

In order to avoid trivial incompatible loads ($\underline{\mathbf{s}}, \mathbf{b}$), it is assumed that the tractions $\underline{\mathbf{s}}$ satisfy the condition

$$\underline{\mathbf{s}} \cdot \mathbf{n} < 0 \ , \ \text{or } \underline{\mathbf{s}} = \mathbf{0}, \ \forall \mathbf{x} \in \partial\Omega_{\text{N}} \ . \tag{4}$$

2.2 Admissible fields

For RNT materials is natural to define the sets of statically admissible stress fields $\mathcal{H}$ and kinematically admissible displacement fields $\mathcal{K}$, as follows

$$\mathcal{H} = \left\{ \mathbf{T} \in \mathcal{S}(\Omega) \text{ s.t. div}\mathbf{T} + \mathbf{b} = \mathbf{0} \ , \ \mathbf{Tn} = \underline{\mathbf{s}} \text{ on } \partial\Omega_{\text{N}} \ , \ \mathbf{T} \in \text{Sym}^{-} \right\} \ , \tag{5}$$

$$\mathcal{K} = \left\{ \mathbf{u} \in \mathcal{T}(\Omega) \text{ s.t. } \mathbf{u} = \underline{\mathbf{u}} \text{ on } \partial\Omega_{\text{D}} \ , \ (\mathbf{E}(\mathbf{u}) - \underline{\mathbf{E}}) \in \text{Sym}^{+} \right\} \ , \tag{6}$$

where a convenient choice for the function spaces $\mathcal{S}(\Omega)$ and $\mathcal{T}(\Omega)$ is

$$\mathcal{S}(\Omega) = SMF(\Omega) \ , \qquad \mathcal{T}(\Omega) = \{\mathbf{u} \ , \text{s.t. grad}\, \mathbf{u} \in SMF^{*}(\Omega)\} \ , \tag{7}$$

$SMF(\Omega)$ being the set of Special Measures (that is measures with null Cantor part) whose jump set is finite, in the sense that the support of their singular part consists of a finite number of regular $(n-1)$d arcs. With $SMF^{*}(\Omega)$ we denote the subset of $SMF(\Omega)$ for which the support of the singular part is restricted to a finite number of $(n-1)$d segments.

3 COMPATIBILITY CONDITIONS

3.1 Compatibility and incompatibility of loads and distortions

The data of a general BVP for a RNT body can be split into two parts

$$\ell \ \leftrightarrow \ (\underline{\mathbf{s}}, \mathbf{b}) \ \approx \ \text{loads} \ , \qquad \ell^{*} \ \leftrightarrow \ (\underline{\mathbf{u}}, \underline{\mathbf{E}}) \ \approx \ \text{distortions} \ . \tag{8}$$

The *equilibrium problem* and the *kinematical problem* for RNT materials, namely the search of admissible stress or displacement fields for given data, are essentially independent, in the sense that they are uncoupled but for condition (3).

It has to be pointed out that, for RNT bodies, there are non-trivial compatibility conditions, both on the loads and on the distortions; that is the existence of statically admissible stress fields for given loads, and the existence of kinematically admissible displacement fields for given distortions, is submitted to special conditions on the data (for a thorough study of compatibility conditions on the loads see [7, 28]).

The definition of compatible loads and distortions is rather straightforward:

$$\{\ell \text{ is compatible}\} \ \Leftrightarrow \{\mathcal{H} \neq \emptyset\} \ , \qquad \{\ell^{*} \text{ is compatible}\} \ \Leftrightarrow \{\mathcal{K} \neq \emptyset\} \ . \tag{9}$$

Therefore the more direct way to prove compatibility, both for loads and distortions, is to construct a s.a. stress field or a k.a. displacement field, as done in the previous examples.

To prove the existence of a solution to the BVP for a No-Tension body, the compatibility of ℓ and ℓ^{*} is necessary but not sufficient, since the further condition

$$\mathbf{T} \cdot \mathbf{E}^{*}(\mathbf{u}) = 0 \ , \tag{10}$$

must be satisfied (this is the material restriction (3)). Then one can say that a possible solution to the BVP is given, if there exist a s.a. stress field and and a k.a. displacement field, which are reconcilable in the sense of condition (3).

The way to verify the incompatibility of the data is less straightforward [20].

$$\{\ell \text{ incompatible}\} \Leftarrow \left\{\exists \mathbf{u}^0 \in \mathcal{K}^0 \text{ s.t. } \left\langle \ell, \mathbf{u}^0 \right\rangle > 0\right\} \ , \tag{11}$$

$$\{\ell^* \text{ incompatible}\} \Leftarrow \left\{\exists \mathbf{T}^0 \in \mathcal{H}^0 \text{ s.t. } \left\langle \ell^*, \mathbf{T}^0 \right\rangle > 0\right\} \ , \tag{12}$$

where $\mathcal{H}^0$ and $\mathcal{K}^0$ are the same as defined in (5) and(6) but with set of data $(\underline{\mathbf{u}}, \underline{\mathbf{E}}; \underline{\mathbf{s}}, \mathbf{b}) = (\mathbf{0}, \mathbf{0}; \mathbf{0}, \mathbf{0})$, and $\langle \ell, \mathbf{u}^0 \rangle$, $\langle \ell^*, \mathbf{T}^0 \rangle$ represent the work of the loads and distortions for $\mathbf{u}^0$, $\mathbf{T}^0$, respectively.

4 LIMIT ANALYSIS

We concentrate on necessary or sufficient conditions for the compatibility of a given set of loads $(\underline{\mathbf{s}}, \mathbf{b})$, restricting to the case of zero kinematical data $(\underline{\mathbf{u}}, \underline{\mathbf{E}})$.

4.1 Theorems of Limit Analysis.

Strictly admissible stress fields and load classification. On denoting $\langle \ell, \mathbf{u} \rangle$ the work of the load $\ell = (\underline{\mathbf{s}}, \mathbf{b})$ for the displacement $\mathbf{u}$, the load can be classified as follows:

$$(\ell \text{ is a collapse load}) \Leftrightarrow (\exists \mathbf{u}^* \in \mathcal{K} \text{ s.t. } \langle \ell, \mathbf{u}^* \rangle > 0) \tag{13}$$

$$(\ell \text{ is a limit load}) \Leftrightarrow (\langle \ell, \mathbf{u} \rangle \leq 0, \ \forall \mathbf{u} \in \mathcal{K} \text{ and } \exists \mathbf{u}^* \in \mathcal{K} - \mathcal{K}^{00} \text{ s.t. } \langle \ell, \mathbf{u}^* \rangle = 0) \tag{14}$$

$$(\ell \text{ is a safe load}) \Leftrightarrow (\langle \ell, \mathbf{u} \rangle < 0 \ , \ \forall \mathbf{u} \in \mathcal{K}) \tag{15}$$

where $\mathcal{H}^{00}$ and $\mathcal{K}^{00}$ are the sets $\mathcal{H}^0$ and $\mathcal{K}^0$ corresponding to null stress and strain fields, depending on the geometry of the boundary, of the loads and of the constraints. A stress field $\mathbf{T} \in \mathcal{H}$ such that $\text{tr}\mathbf{T} < 0$ and $\det\mathbf{T} > 0$, $\forall \mathbf{x} \in \Omega$, is said to be *strictly admissible*.

Notice that, if $\mathbf{T}$ is strictly admissible, then at each point of Ω (that is the open set $\dot{\Omega}$ to which the fixed part of the boundary $\partial\Omega_{\text{D}}$ is added) it results: $\sigma_1 < 0$, $\sigma_2 < 0$, $\sigma_1, \ \sigma_2$ being the eigenvalues of $\mathbf{T}$ at the point $\mathbf{x}$.

Kinematic Theorem. If ℓ is a collapse load (in the sense of item (1) above) then $\mathcal{H}$ is void.

Static Theorem. If a strictly admissible stress field $\mathbf{T}$ exists, then the load ℓ is safe (in the sense of item (3) above).

Limit Theorem. If $\mathcal{H}$ is not void and there exists $\mathbf{u}^* \in \mathcal{K} - \mathcal{K}^{00}$ such that $\langle \ell, \mathbf{u}^* \rangle = 0$, then the load ℓ is limit (in the sense of item (2) above).

For the proof of these theorems we refer to the paper [18]. The reader must be warned that the proofs given by Del Piero refer to a similar function space for the displacement but to a different functional setting for the stress (namely $L^2(\Omega)$). In the present paper we assume that these theorem are still valid in the present larger setting for the stress [20].

4.2 Formalization of the kinematical problem

We focus on the formalization of the kinematical problem under the effect of kinematical data (such as settlements and distortions). By restricting to displacement fields characterized by strain fields that are, purely, line Dirac deltas with support on a finite number of segments, the body Ω can be divided into a finite number, say n, of domains Ω_i (forming a partition of Ω) each exhibiting a rigid body motion. Under the assumption of small strains, for each element Ω_i, such rigid body motion, that is the displacement $\mathbf{u}^i$ of any point of Ω_i, can be described in terms of three displacement parameters $u^i, \ v^i, \ \phi^i$, as follows

$$u_1^i(x_1, x_2) = u^i - \phi^i x_2, \qquad u_2^i(x_1, x_2) = v^i + \phi^i x_1. \tag{16}$$

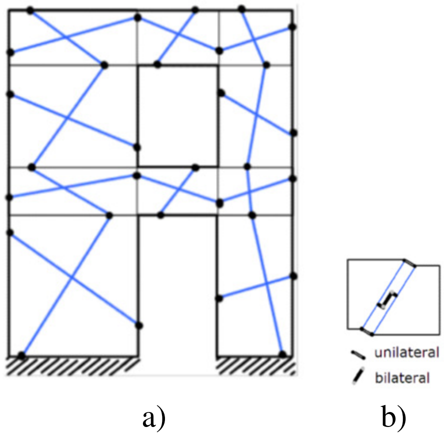


Figure 1: The Rigid–Block Masonry Model with Fixed and Floating Interface

Under these restrictive assumptions the generalized displacement of the structure, denoted $\mathbf{\hat{u}}$ is a vector of $3n$ components, namely the three displacement parameters per each element Ω_i:

$$\mathbf{\hat{u}} = \{u^1, v^1, \phi^1, \ldots, u^i, v^i, \phi^i, \ldots, u^n, v^n, \phi^n\} . \tag{17}$$

5 A METHOD FOR KINEMATIC COLLAPSE ANALYSIS

In our method, the three basic assumptions of Heymans model for masonry are adopted, i.e. *i*) Tensile stresses (forces) are forbidden and, therefore, the material can separate freely (with zero energy expended); *ii*) the material can withstand stresses of infinite intensity (infinite strength to compression, i.e. no possibility of crushing); and *iii*) friction coefficient is infinite: no sliding on separation lines. We, also, restrict to deformations represented by line Dirac deltas on a finite number of segments. In Fig.1 the main idea is depicted for simple plane wall with regular openings: thick solid lines represent the boundary; thin solid lines represent fixed interfaces; grey lines represent moving interfaces.

The movement of the interfaces is controlled by the movement of their end nodes constrained along fixed interfaces and boundary segments. In 1b the constraints that are assumed along the interfaces are reported. The unilateral constraint considered along the interfaces, incorporates the incompenetrability condition. The normality condition (2), (3) (that is the non-sliding assumption along the interface) is enforced by the bilateral pendulum (see 1b). The kinematical and statical problems for such a structure are coupled, in the sense that, given the assumption of zero dissipation on any interface (Assumption *i*)), the work of the reactions for the displacements both at the internal and at the boundary interfaces must be zero. In general there will be infinite elements $\hat{u} \in \mathcal{K}$ and infinite elements $\hat{R} \in \mathcal{H}$, and the no–work assumption gives a criterion to select (may be not uniquely), among them, a couple $(\hat{u}^0, \hat{R}^0)$, that is called: solution of the kinematical and statical problem. (Notice that, restricting to a finite number of rigid blocks, having fixed or moving interfaces, the sets $\mathcal{H}$ and $\mathcal{K}$ become finite dimensional). There is a way to select variationally such a couple. The idea is to introduce the potential energy of the structure, that is minus the potential energy of the loads, i.e. the scalar product of the loads and couples applied at the centroids of the pieces, collected in a generalized force vector $\hat{f}$, for the generalized displacement $\hat{u}$ collecting the parameters of translation and rotation of each piece of S). Call it $E(\hat{u})$, and minimize $E(\hat{u})$ over the set $\mathcal{K}$. Can write

$$E(\hat{u}) = -\left\langle \hat{f}, \hat{u} \right\rangle \ , \tag{18}$$

a linear function of the generalized displacement u of the structure, and

$$E(\hat{u}^0) == \min_{\hat{u}\in\mathcal{K}} E(\hat{u}) \ . \tag{19}$$

This is a linearly constrained minimization problem for a linear function if the interfaces are not moving, that is the pieces are fixed. In such a simplified case the problem can be solved by using the Simplex Method. If both the load data and the distorsion data are fixed, the minimum criterion selects, among all the kinematically admissible displacements $\hat{u}$ the displacement $\hat{u}^0$ that is more convenient on an energetical ground. If the load is assigned with a load parameter λ (say: the vertical component of the load is fixed and the horizontal component is gradually increased with λ), at each stage of the loading program (that is at any given value of λ) the minimal displacement can be calculated through the minimum condition. The limit value λ^0 of the load parameter is obtained when a mechanism (that is an indefinite increase of the displacement) for which the loads perform zero work is detected.

6 NUMERICAL RESULTS

6.1 A simple case study

As a first case study, we analyze the masonry wall in Fig. 2 under the action of the self-weight and a settlement of the central base panel. On assuming that the adopted mesh is fixed, we regard the analyzed structure a set S of $n = 8$ rigid bodies connected by unilateral and bilateral constraints (Fig. 2a). An arbitrary generalized displacement of the wall is given by

$$\hat{u} = \{u(1), v(1), \phi(1), , u(n), v(n), \phi(n)\}. \tag{20}$$

with the displacement parameters being referred to the centroid of mesh elements. The corresponding generalized dual force is as follows

$$\hat{F} = \{H(1), V(1), M(1), , H(n), V(n), M(n)\}. \tag{21}$$

Assuming pure dead loads due to the self-weight of the wall, we have: $H(i) = 0,\ V(i) = P(i),\ M(i) = 0$, for any $i = 1, 2, ..., n$, where $P(i)$ is the self–weight of the panel i. The bilateral and unilateral constraints depicted in Fig.1b are considered to be active on all the mesh interfaces, which leads us to the following system of equality and inequality constraints:

$$\Gamma'\hat{u} = 0 \ , \qquad \Gamma''\hat{u} \leq \delta \ , \tag{22}$$

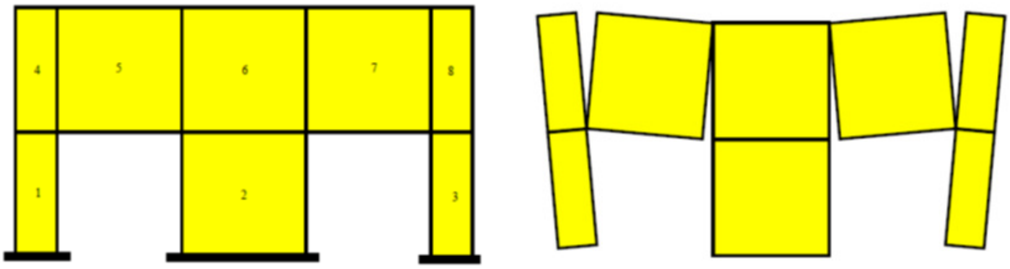
δ being the vector of the applied settlements. In the present case, the latter accounts for a translational vertical settlement the center base panel, which we assume equal to $-0.06L$. The set of kinematically admissible generalized displacement is given by

$$\mathcal{K} = \{\hat{u} : \Gamma'\hat{u} = 0 \ , \Gamma''\hat{u} \leq \delta\}, \tag{23}$$

and the collapse mechanism $\hat{u}^0$ is found via a simplex algorithm, by minimizing the following objective function

$$E(\hat{u}) = -\hat{F} \cdot \hat{u} \ , \tag{24}$$

over $\mathcal{K}$ (Fig.2b).

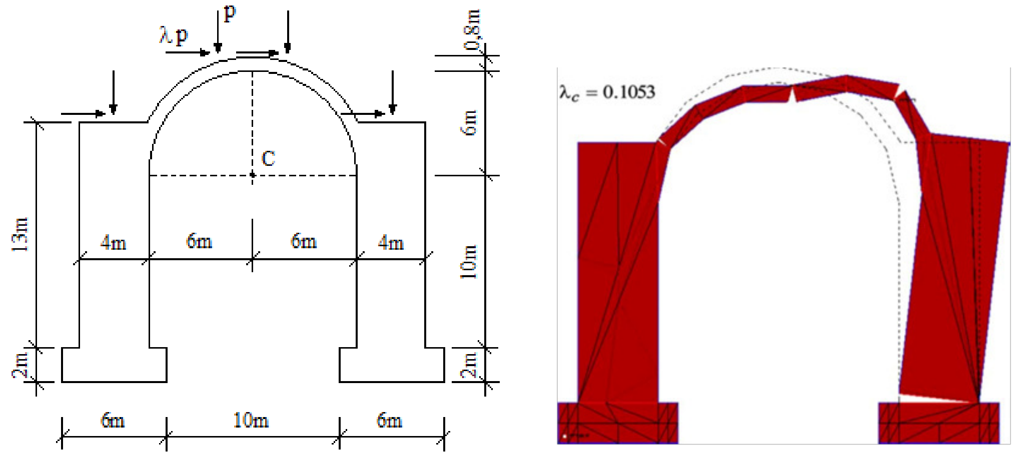


a) b)

Figure 2: A simple case study: a) Geometry and notation, b) collapse mechanism

6.2 Collapse mechanism of a barrel vault subject to seismic loading

A second example deals with the collapse analysis of the monumental masonry structure shown in Fig.4-left, consisting of a barrel vault resting on thick buttresses [34]. The loading condition is represented by the self–weight p of the structure (masonry unit weight equal to $17KN/m^3$), and horizontal forcesλp (static seismic loading, cf. Fig.3a). We estimate the collapse multiplier λ^0 of the horizontal forces by analyzing a $1.0m$ long slice of the structure. Fig.3b shows the collapse mechanism obtained for the present example through the self-adaptive mesh depicted in the same figure ($\lambda^0 = \lambda_c = 0.1053$, cf. [34]). Such a mechanism shows four opening hinges (cracks) in the masonry, one of which is located in a buttress and the other three in the vault (semi-global mechanism).

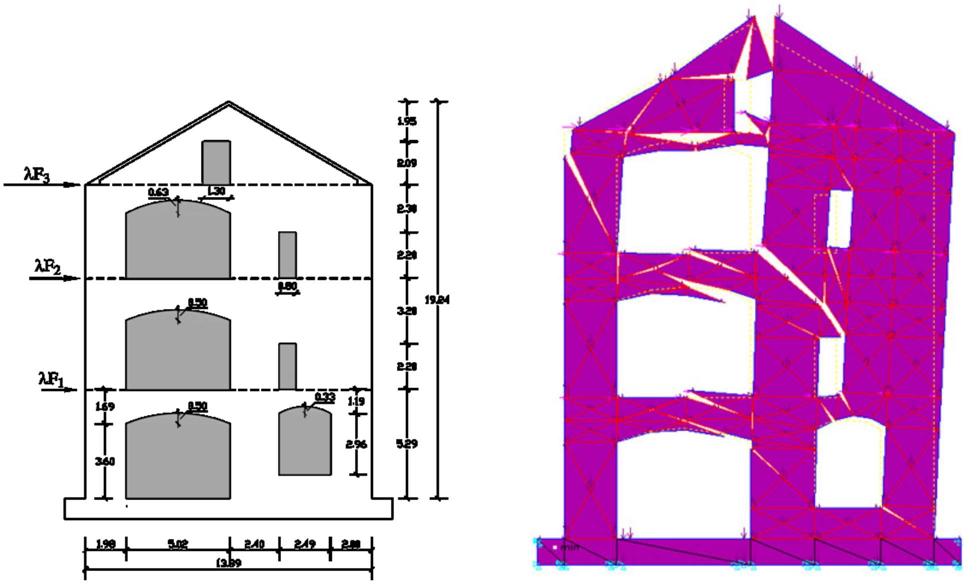


a) b)

Figure 3: Masonry structure covered with a barrel vault a) and corresponding collapse mechanism b)

6.3 Collapse mechanism of a multi storey masonry wall subject to seismic loading

Our final example is concerned with a three storey masonry wall subjected to fixed vertical loads and variable horizontal forces (Fig.4a). The wall is made up of tufe stones with 1.0 m thickness (constant over the height) and $18kN/m^3$ unit weight. A permanent loading of $7.5kN/m$ is acting in correspondence with each storey level. The base values of horizontal forces are: $F_1 = 33.82kN$; $F_2 = 45.58kN$; $F_3 = 70.31kN$ [34]. Fig.4b shows the employed mesh and the corresponding collapse mechanism ($\lambda^0 = 3.9827$).



a) b)

Figure 4: Masonry wall with openings subjected to fixed vertical loads and variable horizontal forces a) and corresponding collapse mechanism b).

7 CONCLUDING REMARKS

The free-discontinuity model presented in this work can be used to predict the ultimate load carrying capacity of masonry structures under vertical and horizontal loads, with the latter describing seismic excitations within a simplified static approach. We address a generalization of the present theory dealing with the limit behavior of vaults, membranes, and structural networks; and the inclusion of different free-discontinuity approaches to the variational theory of fracture, crushing and fracturing of masonry, as well as interface debonding and ripping (FRP-reinforced structures) to future work [22, 25, 29–34].

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