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ON THE IMPLEMENTATION OF BOUNDARY CONDITIONS IN RBF-PS METHOD APPLIED TO VIBRATION ANALYSIS OF KIRCHHOFF PLATES

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Abstract. *The paper deals with the implementation of multiple boundary conditions in the radial basis function-based pseudospectral method. This numerical technique belongs to the wide range of meshless discretization methods, combining the flexibility of the meshless discretization with efficiency and simplicity of the pseudospectral techniques. The method can be directly applied to lower order equation, where the number of boundary conditions at a boundary node corresponds to the number of degrees of freedom at this node. For higher order equations the problem arises during collocation procedure due to multiple boundary conditions. In the present paper two approaches to overcome this inconvenience are shown. One is based on the Hermite interpolation idea while the other is a slight modification of the direct approach. The convergence and accuracy of both approaches are examined in the problem of free vibration analysis of plates. Square plates as well as irregular shaped plates with various types of boundary conditions are taken under consideration. Uniform as well as scattered node distributions are applied. The results confirm that both approaches can be useful in the application to plate vibration analysis.*

1 INTRODUCTION

In recent years a significant effort has been made to develop meshless numerical techniques [1, 2]. The main feature of these methods is the use of scattered nodes to discretize the domain of the problem under consideration. Due to this feature, they can easily discretize irregular shapes and problems defined in more than two dimensions. This type of discretization is also very effective in association with adaptation techniques.

Several formulations of meshless methods have been developed till now. Some of them are derived from the weak formulation, while the others are used to discretize directly differential equation. The method investigated in the present paper, called the radial basis function-based pseudospectral method (RBF-PS) falls into the second category. It uses radial basis functions (RBFs) to approximate derivatives in a governing equation in the view of the pseudospectral mode. Using this approximation and applying the collocation procedure, the governing equation is transformed into the set of algebraic equations. The method has been used to solve problems from various disciplines of science. A collection of papers by Fasshauer [3] and Ferreira and Fasshauer [4, 5] devoted to the method and its application to engineering problems deserves a particular attention. It is also known a slight modification of this method called the radial basis function-based differential quadrature (RBF-DQ) [6]. Some applications of the latter can be found in papers by Shu [6, 7] and others [8].

Since the method uses collocation procedure, one discrete equation can be associated with one degree of freedom at a node. Therefore, a problem arises for differential equations possessing multiple boundary conditions. In this case the number of boundary conditions usually exceeds the number of degrees of freedom at boundary node. This problem has been undertaken in [9], where the Hermite interpolation with RBFs has been proposed to approximate the sought solution of a problem. It has introduced additional degrees of freedom at boundary nodes and therefore has allowed the multiple boundary conditions to be conveniently implemented. The numerical tests carried out till now [9, 10] indicate that the method is effective in eigenvalue problems, although the problem of ill-conditioned interpolation matrix following from the RBF interpolation is particularly important in this approach due to larger sized matrices.

In the present paper another approach that allows the multiple boundary conditions to be easily implemented is proposed. In this approach none of additional degrees of freedom are introduced, therefore, the size of the interpolation matrix is appropriately decreased. The proposed approach is compared with the above-mentioned one in the problem of free vibration analysis of thin plates. The layout of the paper is as follows: In section 2 the RBF-PS method as well as its extension based on the Hermite interpolation are briefly described and then the proposed approach is explained. In section 3 the method is applied to free vibration analysis of square plates as well as irregular shaped plates. Convergence and accuracy of these two approaches are compared as a result of numerical tests. On this base some conclusions are drawn in section 4.

2 RADIAL BASIS FUNCTION-BASED PSEUDOSPECTRAL METHOD

In the RBF-PS method the sought solution of a differential equation is approximated by a global interpolation function of the form

$$u(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \varphi\left(\left\|\mathbf{x} - \boldsymbol{\xi}_j\right\|\right) \tag{1}$$

In Eq. (1) α_j are the interpolation coefficients, $\varphi_j(\mathbf{x}) = \varphi\left(\left\|\mathbf{x}-\xi_j\right\|\right)$ are the RBFs, where ξ_j denote the special points called the centers. These special points coincide with the nodes imposed $\mathbf{x}_i$, $i = 1, \ldots, N$.

Using interpolation conditions

$$\sum_{j=1}^{N} \alpha_j \varphi\left(\left\|\mathbf{x}_i - \mathbf{x}_j\right\|\right) = u_i, \quad i = 1,\ldots,N \tag{2}$$

the interpolation coefficients can be expressed in terms of sought function values u_i at the nodes, what can be put in the matrix notation as

$$\boldsymbol{\alpha} = \boldsymbol{\Phi}^{-1}\mathbf{u} \tag{3}$$

where $\boldsymbol{\alpha} = \begin{bmatrix}\alpha_1 & \cdots & \alpha_N\end{bmatrix}^T$, $\mathbf{u} = \begin{bmatrix}u_1 & \cdots & u_N\end{bmatrix}^T$ and $\boldsymbol{\Phi}_{ij} = \varphi(\left\|\mathbf{x}_i - \mathbf{x}_j\right\|),\ i, j = 1,\ldots,N$.

To obtain the approximation for derivatives contained in the governing equation, appropriate differential operator L has to be imposed on the interpolant (1) and obtained expression has to be evaluated at each interior node $\mathbf{x}_i^I, i = 1,\ldots,N^I$, what yields

$$\mathbf{u}_L = \boldsymbol{\Phi}_L \boldsymbol{\alpha} \tag{4}$$

where $\left(\mathbf{u}_L\right)_i = \left[Lu(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_i^I}$, $i = 1,\ldots,N^I$, $\left(\boldsymbol{\Phi}_L\right)_{ij} = \left[L\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\substack{\xi=\mathbf{x}_j \\ \mathbf{x}=\mathbf{x}_i^I}}$, $i = 1,\ldots,N^I$, $j = 1,\ldots,N$.

In Eq. (4) L acts on the radial function treated as a function of $\mathbf{x}$ variable. Introducing Eq. (3) into Eq. (4) one can express derivative $\mathbf{u}_L$ in terms of the sought function values from all over domain as

$$\mathbf{u}_L = \boldsymbol{\Phi}_L \boldsymbol{\Phi}^{-1}\mathbf{u} \tag{5}$$

where $\boldsymbol{\Phi}_L \boldsymbol{\Phi}^{-1}$ is so-called the differentiation matrix.

Taking advantage of this approximation and using collocation procedure the governing equation can be discretized at each interior node. To impose boundary conditions, boundary equations should be collocated at boundary nodes $\mathbf{x}_i^B, i = 1,\ldots,N^B$ and possible derivatives contained in these equations are approximated similarly as in steps (4)-(5).

Since there is one degree of freedom at each node (the sought function value u_i) only one boundary condition can be easily imposed at each boundary node according to collocation technique. This fact makes it difficult to solved higher order equations that are characterized by multiple boundary conditions.

2.1 Hermite interpolation in RBF-PS

To extend the possibilities of the application of the RBF-PS method to higher order equations, one can modify the interpolant (1) introducing additional degrees of freedom at boundary nodes. This idea, known in the interpolation theory as Hermite interpolation, has been applied for the RBF-PS approach in [9]. For coherence of the present paper the main features of this approach are briefly presented below. Now, the global interpolation function has the form

$$u(\mathbf{x}) = \sum_{j=1}^{N^I} \alpha_j \varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\Big|_{\xi=\mathbf{x}_j^I} + \sum_{j=1}^{N^B} \beta_j \left[B_1^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B} + \sum_{j=1}^{N^B} \gamma_j \left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B} \tag{6}$$

In Eq. (6), it is assumed that at each boundary node two degrees of freedom, represented by differential operators B_1^{ξ} and B_2^{ξ}, are introduced. These differential operators correspond to those contained in boundary conditions and act on the radial function treated as a function of ξ variable. Remaining symbols used in Eq. (6) denote: N^I and N^B – numbers of interior nodes $\mathbf{x}^I$ and boundary nodes $\mathbf{x}^B$, respectively and α, β, γ – interpolation coefficients.

Enforcing interpolation conditions in the following form

$$\sum_{j=1}^{N^I}\alpha_j\varphi\left(\left\|\mathbf{x}_i^I-\xi\right\|\right)\Big|_{\xi=\mathbf{x}_j^I}+\sum_{j=1}^{N^B}\beta_j\left[B_1^{\xi}\varphi\left(\left\|\mathbf{x}_i^I-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}+\sum_{j=1}^{N^B}\gamma_j\left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}_i^I-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B} \\ =u(\mathbf{x}_i^I),\quad i=1,\dots,N^I \tag{7}$$

$$\sum_{j=1}^{N^I}\alpha_j\left[B_1^{\mathbf{x}}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\substack{\xi=\mathbf{x}_j^I\\ \mathbf{x}=\mathbf{x}_i^B}}+\sum_{j=1}^{N^B}\beta_j\left[B_1^{\mathbf{x}}\left[B_1^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}\right]_{\mathbf{x}=\mathbf{x}_i^B} \\ +\sum_{j=1}^{N^B}\gamma_j\left[B_1^{\mathbf{x}}\left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}\right]_{\mathbf{x}=\mathbf{x}_i^B}=\left[B_1^{\mathbf{x}}u(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_i^B},\quad i=1,\dots,N^B \tag{8}$$

$$\sum_{j=1}^{N^I}\alpha_j\left[B_2^{\mathbf{x}}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\substack{\xi=\mathbf{x}_j^I\\ \mathbf{x}=\mathbf{x}_i^B}}+\sum_{j=1}^{N^B}\beta_j\left[B_2^{\mathbf{x}}\left[\mathrm{B}_1^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}\right]_{\mathbf{x}=\mathbf{x}_i^B} \\ +\sum_{j=1}^{N^B}\gamma_j\left[B_2^{\mathbf{x}}\left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}\right]_{\mathbf{x}=\mathbf{x}_i^B}=\left[B_2^{\mathbf{x}}u(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_i^B},\quad i=1,\dots,N^B \tag{9}$$

it is possible to express the interpolation coefficients in terms of the sought function values at interior nodes and differential operators evaluated at boundaries. In Eq. (8)-(9) $B_1^{\mathbf{x}}$ and $B_2^{\mathbf{x}}$ denote the same differential operators as B_1^{ξ} and B_2^{ξ}, but acting on the radial function viewed as a function of **x** variable. It makes the coefficient matrix of the system (7)-(9) be a symmetric one.

Then, the differential operator from the governing equation is imposed on interpolant (6) and evaluated at each interior node, what yields

$$\left[L^{\mathbf{x}}u(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_i^I}=\sum_{j=1}^{N^I}\alpha_j\left[L^{\mathbf{x}}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\substack{\xi=\mathbf{x}_j^I\\ \mathbf{x}=\mathbf{x}_i^I}}+\sum_{j=1}^{N^B}\beta_j\left[L^{\mathbf{x}}\left[B_1^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}\right]_{\mathbf{x}=\mathbf{x}_i^I} \\ +\sum_{j=1}^{N^B}\gamma_j\left[L^{\mathbf{x}}\left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}\right]_{\mathbf{x}=\mathbf{x}_i^I},\quad i=1,\dots,N^I \tag{10}$$

Introducing the interpolation coefficients determined from Eqs. (7)-(9) into Eq. (10), one can express the derivatives contained in the governing equation $\left[L^{\mathbf{x}}u(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_i^I}$ in terms of the sought function values at interior nodes as well as in terms of the derivatives contained in boundary conditions, evaluated at boundary nodes $\left[B_1^{\mathbf{x}}u(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_i^B}$, $\left[B_2^{\mathbf{x}}u(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_i^B}$. It can be put in matrix notation as

$$\mathbf{u}_{L^{\mathbf{x}}}=\bar{\mathbf{\Phi}}_L\cdot\bar{\mathbf{\Phi}}^{-1}\cdot\bar{\mathbf{u}} \tag{11}$$

In Eq. (11) $\bar{\mathbf{\Phi}}_L \cdot \bar{\mathbf{\Phi}}^{-1}$ can be found as a generalization of the differentiation matrix, where

$$\bar{\mathbf{\Phi}}_L = \begin{bmatrix} \mathbf{\Phi}_{L^{\mathbf{x}}} & \mathbf{\Phi}_{L^{\mathbf{x}}B_1^{\xi}} & \mathbf{\Phi}_{L^{\mathbf{x}}B_2^{\xi}} \end{bmatrix},\ \bar{\mathbf{\Phi}} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Phi}_{B_1^{\xi}} & \mathbf{\Phi}_{B_2^{\xi}} \\ & \mathbf{\Phi}_{B_1^{\mathbf{x}}B_1^{\xi}} & \mathbf{\Phi}_{B_1^{\mathbf{x}}B_2^{\xi}} \\ sym & & \mathbf{\Phi}_{B_2^{\mathbf{x}}B_2^{\xi}} \end{bmatrix} \text{ and } \bar{\mathbf{u}} = \begin{bmatrix} \mathbf{u} \\ \mathbf{u}_{B_1^{\mathbf{x}}} \\ \mathbf{u}_{B_2^{\mathbf{x}}} \end{bmatrix}$$

In the above expressions the block matrices have following forms

$$\left(\mathbf{\Phi}_{L^{\mathbf{x}}}\right)_{ij} = \left[L^{\mathbf{x}}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\substack{\xi=\mathbf{x}_j^I \\ \mathbf{x}=\mathbf{x}_j^I}}, \quad i,j=1,\dots,N^I$$

$$\left(\mathbf{\Phi}_{L^{\mathbf{x}}B_1^{\xi}}\right)_{ij} = \left[L^{\mathbf{x}}\left[B_1^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}\right]_{\mathbf{x}=\mathbf{x}_i^I}, \quad i=1,\dots,N^I,\ j=1,\dots,N^B$$

$$\left(\mathbf{\Phi}_{L^{\mathbf{x}}B_2^{\xi}}\right)_{ij} = \left[L^{\mathbf{x}}\left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}\right]_{\mathbf{x}=\mathbf{x}_i^I}, \quad i=1,\dots,N^I,\ j=1,\dots,N^B$$

$$\mathbf{\Phi}_{ij} = \varphi\left(\left\|\mathbf{x}_i^I-\xi\right\|\right)\Big|_{\xi=\mathbf{x}_j^I}, \quad i,j=1,\dots,N^I$$

$$\left(\mathbf{\Phi}_{B_1^{\xi}}\right)_{ij} = \left[B_1^{\xi}\varphi\left(\left\|\mathbf{x}_i^I-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}, \quad i=1,\dots,N^I,\ j=1,\dots,N^B$$

$$\left(\mathbf{\Phi}_{B_2^{\xi}}\right)_{ij} = \left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}_i^I-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}, \quad i=1,\dots,N^I,\ j=1,\dots,N^B$$

$$\left(\mathbf{\Phi}_{B_1^{\mathbf{x}}B_1^{\xi}}\right)_{ij} = \left[B_1^{\mathbf{x}}\left[B_1^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}\right]_{\mathbf{x}=\mathbf{x}_i^B}, \quad i,j=1,\dots,N^B$$

$$\left(\mathbf{\Phi}_{B_1^{\mathbf{x}}B_2^{\xi}}\right)_{ij} = \left[B_1^{\mathbf{x}}\left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}\right]_{\mathbf{x}=\mathbf{x}_i^B}, \quad i,j=1,\dots,N^B$$

$$\left(\mathbf{\Phi}_{B_2^{\mathbf{x}}B_2^{\xi}}\right)_{ij} = \left[B_2^{\mathbf{x}}\left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^B}\right]_{\mathbf{x}=\mathbf{x}_i^B}, \quad i,j=1,\dots,N^B$$

$$\mathbf{u} = u(\mathbf{x}_i^I),\ \left(\mathbf{u}_{B_1^{\mathbf{x}}}\right)_i = \left[B_1^{\mathbf{x}}u(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_i^B},\ \left(\mathbf{u}_{B_2^{\mathbf{x}}}\right)_i = \left[B_2^{\mathbf{x}}u(\mathbf{x})\right]_{\mathbf{x}=\mathbf{x}_i^B}, i=1,\dots N^B$$

In the similar manner any number of boundary conditions can be involved during discretization of a differential operator from a governing equation. The boundary conditions are not discretized separately, what is another advantage of the approach.

2.2 Simplified implementation of boundary conditions in RBF-PS

The approach presented in section 2.1 introduces additional degrees of freedom at the boundary. The size of the interpolation matrix $\bar{\mathbf{\Phi}}$ from Eq. (11) is then increased comparing to matrix $\mathbf{\Phi}$ from direct approach (Eq. (3)). It is well-known that the size of the interpolation matrix is one of factors that influence conditioning of the problem in the case of the RBF interpolation. To maintain the same number of degrees of freedom as in the conventional RBF-PS method and its simplicity, in the present section another approach is proposed.

In this approach, the boundary nodes are split into appropriate number of groups. This number corresponds to the number of boundary conditions. The nodes are chosen by turns to appropriate group. Each boundary condition is implemented at nodes from another group, what is shown in Fig. 1 in the case of two boundary conditions.

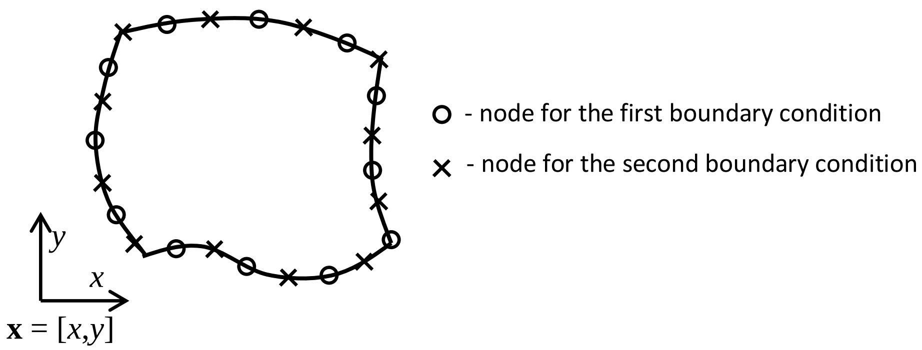

Figure 1: Implementation of boundary conditions.

To derive the differentiation matrix one can modify interpolant (6) in the following way

$$u(\mathbf{x})=\sum_{j=1}^{N^I}\alpha_j\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\Big|_{\xi=\mathbf{x}_j^I}+\sum_{j=1}^{N^{B_1}}\beta_j\left[B_1^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^{B_1}}+\sum_{j=1}^{N^{B_2}}\gamma_j\left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^{B_2}} \qquad (12)$$

where N^{B_1} and N^{B_2} denote the numbers of boundary nodes for the groups of nodes $\mathbf{x}^{B_1}$ and $\mathbf{x}^{B_2}$ associated with the first and the second boundary condition, respectively. Note, that $N^{B_1}+N^{B_2}=N^B$ is the number of all nodes imposed on boundaries of the domain.
Then, one should enforce interpolation conditions for interpolant (12), similarly as in Eqs. (7)-(9), but using appropriate group of boundary nodes to evaluate boundary conditions. Finally, differential operator from governing equation has to be imposed on the interpolation function as in Eq. (10) and the differentiation matrix, as in Eq. (11), can be derived. Now, this matrix has similar structure as in section 2.1 but the block matrices connected with boundary conditions are evaluated using different nodes. All of the block matrices, used in the present approach, have the form

$$\left(\mathbf{\Phi}_{L^{\mathbf{x}}}\right)_{ij}=\left[L^{\mathbf{x}}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\substack{\xi=\mathbf{x}_j^I\\ \mathbf{x}=\mathbf{x}_j^I}},\quad i,j=1,\ldots,N^I$$

$$\left(\mathbf{\Phi}_{L^{\mathbf{x}}B_1^{\xi}}\right)_{ij}=\left[L^{\mathbf{x}}\left[B_1^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^{B_1}}\right]_{\mathbf{x}=\mathbf{x}_i^I},\quad i=1,\ldots,N^I,\ j=1,\ldots,N^{B_1}$$

$$\left(\mathbf{\Phi}_{L^{\mathbf{x}}B_2^{\xi}}\right)_{ij}=\left[L^{\mathbf{x}}\left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^{B_2}}\right]_{\mathbf{x}=\mathbf{x}_i^I},\quad i=1,\ldots,N^I,\ j=1,\ldots,N^{B_2}$$

$$\mathbf{\Phi}_{ij}=\varphi\left(\left\|\mathbf{x}_i^I-\xi\right\|\right)\Big|_{\xi=\mathbf{x}_j^I},\quad i,j=1,\ldots,N^I$$

$$\left(\mathbf{\Phi}_{B_1^{\xi}}\right)_{ij}=\left[B_1^{\xi}\varphi\left(\left\|\mathbf{x}_i^I-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^{B_1}},\quad i=1,\ldots,N^I,\ j=1,\ldots,N^{B_1}$$

$$\left(\mathbf{\Phi}_{B_2^{\xi}}\right)_{ij}=\left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}_i^I-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^{B_2}},\quad i=1,\ldots,N^I,\ j=1,\ldots,N^{B_2}$$

$$\left(\mathbf{\Phi}_{B_1^{\mathbf{x}}B_1^{\xi}}\right)_{ij}=\left[B_1^{\mathbf{x}}\left[B_1^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^{B_1}}\right]_{\mathbf{x}=\mathbf{x}_i^{B_1}},\quad i,j=1,\ldots,N^{B_1}$$

$$\left(\mathbf{\Phi}_{B_1^{\mathbf{x}}B_2^{\xi}}\right)_{ij}=\left[B_1^{\mathbf{x}}\left[B_2^{\xi}\varphi\left(\left\|\mathbf{x}-\xi\right\|\right)\right]_{\xi=\mathbf{x}_j^{B_2}}\right]_{\mathbf{x}=\mathbf{x}_i^{B_1}},\quad i=1,\ldots,N^{B_1},\ j=1,\ldots,N^{B_2}$$

$$\left(\mathbf{\Phi}_{B_2^{\mathbf{x}}B_2^{\xi}}\right)_{ij} = \left[B_2^{\mathbf{x}} \left[B_2^{\xi} \varphi\left(\left\|\mathbf{x}-\xi\right\|\right) \right]_{\xi=\mathbf{x}_j^{B_2}} \right]_{\mathbf{x}=\mathbf{x}_i^{B_2}}, \quad i, j = 1,\ldots, N^{B_2}$$

$$\mathbf{u} = u(\mathbf{x}_i^I),\ \left(\mathbf{u}_{B_1^{\mathbf{x}}}\right)_i = \left[B_1^{\mathbf{x}} u(\mathbf{x}) \right]_{\mathbf{x}=\mathbf{x}_i^{B_1}}, i = 1,\ldots N^{B_1},\ \left(\mathbf{u}_{B_2^{\mathbf{x}}}\right)_i = \left[B_2^{\mathbf{x}} u(\mathbf{x}) \right]_{\mathbf{x}=\mathbf{x}_i^{B_2}}, i = 1,\ldots N^{B_2}$$

By splitting boundary nodes into appropriate number of groups it is possible to implement any number of boundary conditions using presented approach.

3 APPLICATION OF RBF-PS TO VIBRATION ANALYSIS OF PLATES

To assess the usefulness of presented approaches for a dynamic problem described by higher order equation, they have been applied to free vibration analysis of quadrilateral, thin, isotropic plates of various shapes. The governing equation for this problem is as follows

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \Omega^2 w \tag{13}$$

where w denotes the mode of vibration and Ω is the free vibration parameter related to free vibration frequency ω by the formula: $\Omega = \omega a^2 \sqrt{\rho h/D}$ (ρ – density of the plate material, D – plate stiffness, h – plate thickness, a – characteristic plate dimension).

Taking into account the quadrilateral plate, where two boundary conditions are applied at kth edge of this plate, the interpolant for displacement function w, in the view of the Hermite RBF-PS method (section 2.1), should be written as

$$w(\mathbf{x}) = \sum_{j=1}^{N^I} \alpha_j \varphi\left(\left\|\mathbf{x}-\xi\right\|\right)_{|\xi=\mathbf{x}_j^I} + \sum_{k=1}^{4}\left[\sum_{j=1}^{N_k^B} \beta_{kj} \left[B_{k1}^{\xi} \varphi\left(\left\|\mathbf{x}-\xi\right\|\right) \right]_{\xi=\mathbf{x}_j^{Bk}} + \sum_{j=1}^{N_k^B} \gamma_{kj} \left[B_{k2}^{\xi} \varphi\left(\left\|\mathbf{x}-\xi\right\|\right) \right]_{\xi=\mathbf{x}_j^{Bk}} \right] \tag{14}$$

By slight modification of Eq. (14) one can obtain the interpolant used in the approach described in section 2.2

$$w(\mathbf{x}) = \sum_{j=1}^{N^I} \alpha_j \varphi\left(\left\|\mathbf{x}-\xi\right\|\right)_{|\xi=\mathbf{x}_j^I} + \sum_{k=1}^{4}\left[\sum_{j=1}^{N_k^{B_1}} \beta_{kj} \left[B_{k1}^{\xi} \varphi\left(\left\|\mathbf{x}-\xi\right\|\right) \right]_{\xi=\mathbf{x}_j^{Bk1}} + \sum_{j=1}^{N_k^{B_2}} \gamma_{kj} \left[B_{k2}^{\xi} \varphi\left(\left\|\mathbf{x}-\xi\right\|\right) \right]_{\xi=\mathbf{x}_j^{Bk2}} \right] \tag{15}$$

In the present paper, the plates with combination of simply supported, clamped and free boundary conditions are considered. The general description of these boundary conditions for the kth edge in terms of differential operators contained in Eq. (14) and (15) can be written as

$$B_{k1}^{\mathbf{x}} w = 0,\ B_{k2}^{\mathbf{x}} w = 0 \tag{16}$$

where

$$\begin{aligned} &B_{k1}^{\mathbf{x}} = 1 \\ &B_{k2}^{\mathbf{x}} = \left(\cos^2(\theta) + \nu \sin^2(\theta)\right)\frac{\partial^2}{\partial x^2} + \left(\sin^2(\theta) + \nu \cos^2(\theta)\right)\frac{\partial^2}{\partial y^2} + 2(1-\nu)\cos(\theta)\sin(\theta)\frac{\partial^2}{\partial x \partial y} \end{aligned} \tag{17}$$

for the simply supported edge (S)

$$B_{k1}^{\mathbf{x}} = 1$$
$$B_{k2}^{\mathbf{x}} = \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} \tag{18}$$

for the clamped edge (C) and

$$B_{k1}^{\mathbf{x}} = \left(\cos^2(\theta) + \nu\sin^2(\theta)\right)\frac{\partial^2}{\partial x^2} + \left(\sin^2(\theta) + \nu\cos^2(\theta)\right)\frac{\partial^2}{\partial y^2} + 2(1-\nu)\cos(\theta)\sin(\theta)\frac{\partial^2}{\partial x\,\partial y}$$
$$B_{k2}^{\mathbf{x}} = \sin(\theta)(1-\cos^2(\theta)(\nu-1))\frac{\partial^3}{\partial y^3} + \cos(\theta)(-2\cos^2(\theta)(\nu-1) + \sin^2(\theta)(\nu-1) + \nu)\frac{\partial^3}{\partial y^2 \partial x} \tag{19}$$
$$+\sin(\theta)(\cos^2(\nu-1) - 2\sin^2(\theta)(\nu-1) + \nu)\frac{\partial^3}{\partial x^2 \partial y} + \cos(\theta)\left(1-\sin^2(\theta)(\nu-1)\right)\frac{\partial^3}{\partial x^3}$$

for the free edge (F).

In Eqs. (17)-(19) θ is the angle between the normal to the plate boundary and the x-axis, ν is the Poisson's ratio.

With interpolants (14) and (15), appropriate differentiation matrices for biharmonic operator contained in Eq. (13) have been determined. Then, Eq. (13) has been transformed into the algebraic eigenvalue problem of the form

$$\mathbf{A}\mathbf{w} = \Omega^2\mathbf{w} \tag{20}$$

where **A** is the differentiation matrix modified by deleting the columns associated with degrees of freedom at boundary nodes and **w** denotes the vector containing function values at interior nodes.

3.1 Numerical results

As a benchmark problem, the square plate has been considered. Uniform grid node distribution as well as irregular one has been assumed. The obtained results, in the form of convergence tests for non-dimensional free vibration frequency Ω and number of nodes used N, are presented in Figs. 2-5.

The results from Figs. 2-5 indicate that the approaches preserve a proper convergence trend. Increasing the number of nodes makes the results more accurate, regardless of the assumed node distribution (uniform or irregular). In the view of these results one can conclude that the method based on the Hermite interpolation (section 2.1) gives faster convergence than the RBF-PS approach from section 2.2, in the case of irregular node distribution. In the case of the uniform node distribution the results are comparable. Note that the use of the Hermite interpolation increases the number of degrees of freedom, while the number of nodes N remains the same, what can explain faster convergence.

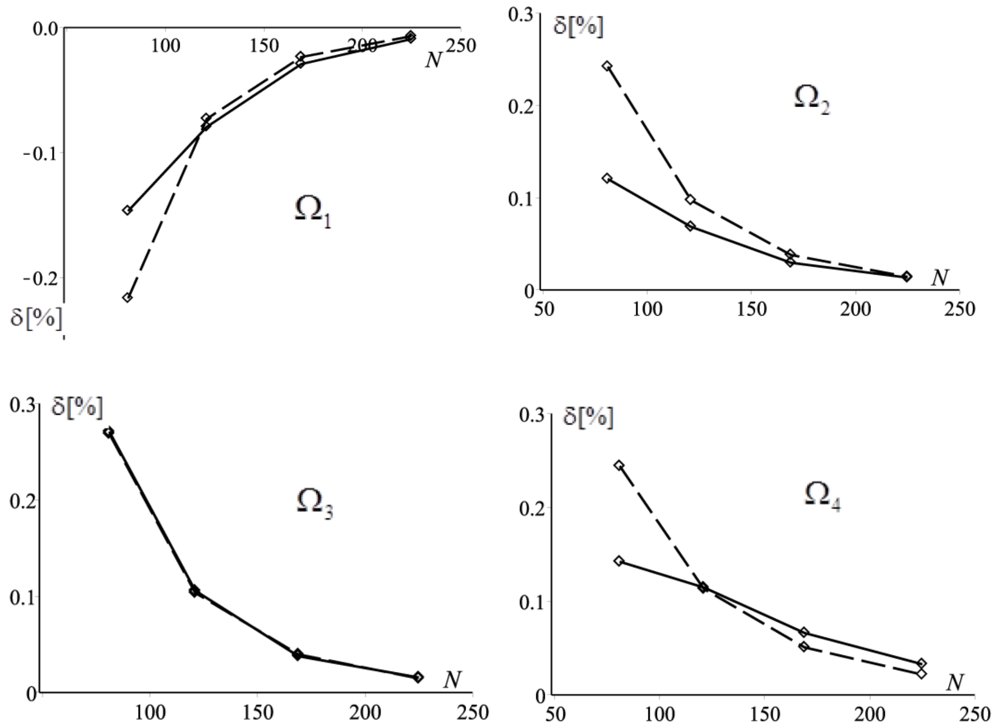

Figure 2: Results for the square SSSS plate obtained with uniform grid. Dash line – the method from section 2.1, solid line - the method from section 2.2.

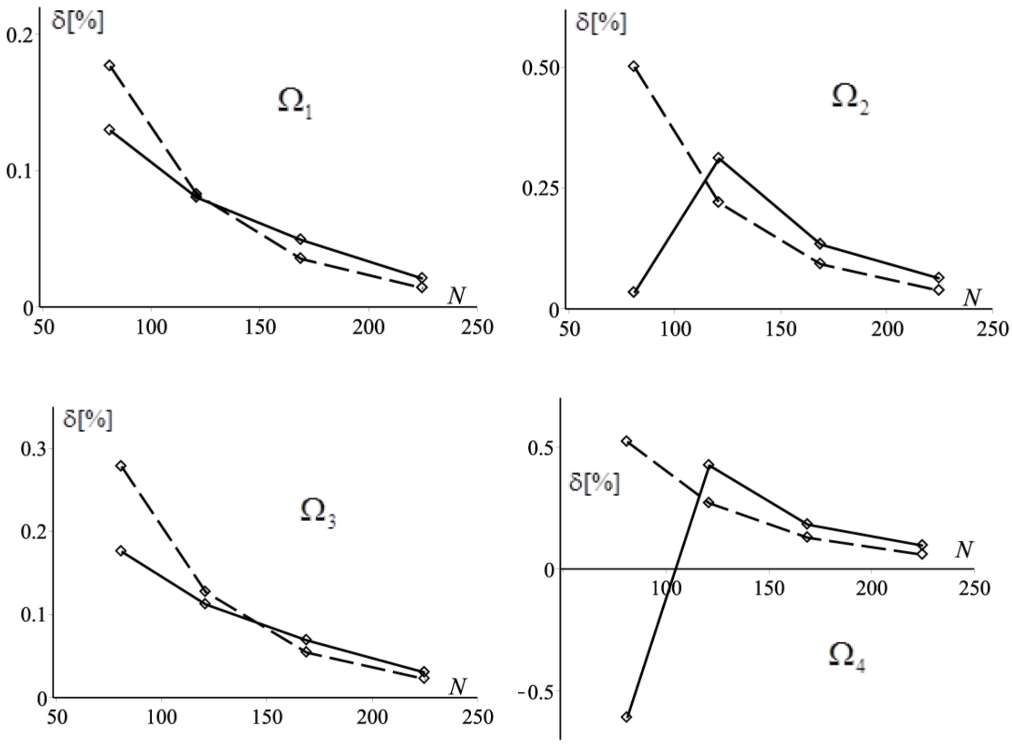

Figure 3: Results for the square SCSC plate obtained with uniform grid. Dash line – the method from section 2.1, solid line - the method from section 2.2.

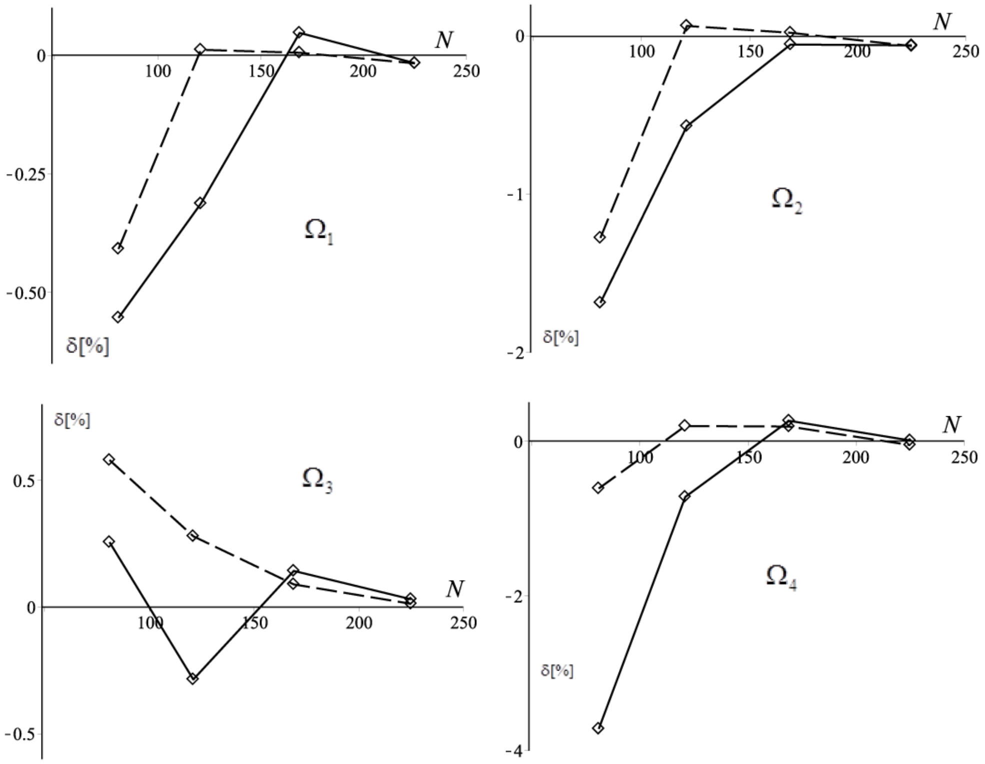

Figure 4: Results for the square CCCC plate obtained with irregular grid. Dash line – the method from section 2.1, solid line - the method from section 2.2.

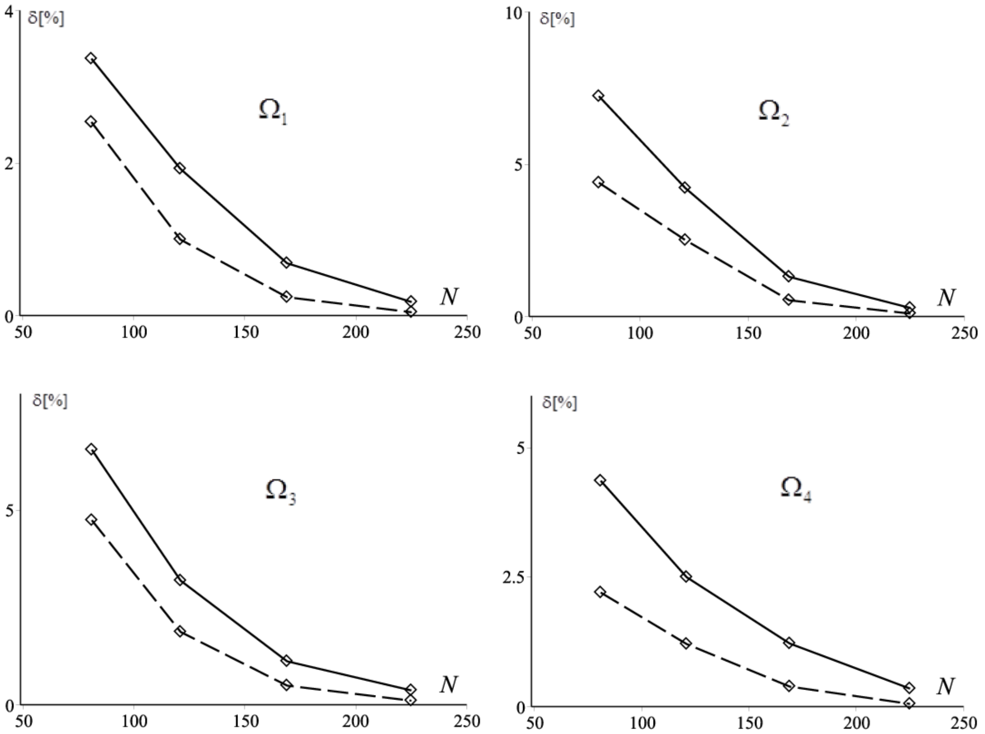

Figure 5: Results for the square SFSF plate obtained with irregular grid. Dash line – the method from section 2.1, solid line - the method from section 2.2.

Similar computation has been carried out for the irregular shaped plate presented in Fig. 6. The obtained results are shown in Tab. 1 and Tab. 2.

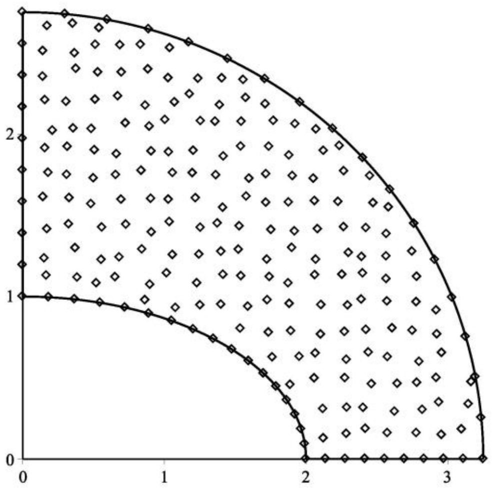

Figure 6: Quarter section of the elliptical plate with an example of the node distribution.

It should be noted that in this computation, besides the different numbers of interior nodes, two types of boundary grid patterns have been assumed. The latter differ from each other by the density of nodes imposed on boundaries.

It has been found that the method based on the Hermite interpolation provides accurate results regardless of the density of boundary nodes, while the RBF-PS approach from section 2.2 gives accurate results with the pattern possessing dense enough boundary node distribution (Tab. 2).

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSSS					
N =233, N^I = 181	4.800	7.520	11.424	16.032	16.614
N =374, N^I = 322	4.832	7.586	11.500	16.020	16.701
Reference results	4.894	7.598	11.436	16.086	16.635
CCCC					
N =233, N^I = 181	9.286	12.746	16.709	21.206	23.890
N =374, N^I = 322	9.617	12.833	17.194	22.604	24.099
Reference results	9.595	12.717	16.743	22.275	24.068
SCSC					
N =233, N^I = 181	9.138	12.068	15.158	19.675	23.490
N =374, N^I = 322	9.134	11.958	15.428	19.767	23.303
Reference results	9.142	12.003	15.321	19.920	23.493

Table 1: Results for the quarter section of the elliptical plate obtained by the Hermite RBF-PS approach.

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
SSSS					
N =245, N^I = 181	4.815	7.559	11.359	15.987	16.618
N =386, N^I = 322	4.815	7.519	11.471	16.014	16.679
Reference results	4.894	7.598	11.436	16.086	16.635
CCCC					
N =245, N^I = 181	9.654	12.210	16.729	20.426	24.023
N =386, N^I = 322	9.601	12.718	16.715	22.098	23.928
Reference results	9.595	12.717	16.743	22.275	24.068
SCSC					
N =245, N^I = 181	9.238	11.608	15.294	18.372	23.549
N =386, N^I = 322	9.134	12.062	15.468	19.831	23.432
Reference results	9.142	12.003	15.321	19.920	23.493

Table 2: Results for the quarter section of the elliptical plate obtained by the RBF-PS approach from section 2.2.

In all computations, the multiquadric RBFs have been used with the so-called shape parameter assumed as 0.7. The reference results contained in Tab. 1 and 2 have been obtained with the use of differential quadrature method with domain transformation. The details of this method can be found in [11].

4 CONCLUSION

In the paper two approaches for the implementation of multiple boundary conditions in the RBF-PS method have been presented and examined on the example of the free vibration analysis of plates. The main advantage of this method is its meshless character that allows easily to apply it for problems with irregular domains. The presented approaches can be successfully used for various types of boundary conditions encountered in plate analysis. The method based on the Hermite interpolation seems to be more versatile and works correctly regardless of the boundary node distribution, while the RBF-PS approach described in section 2.2 needs to have dense enough boundary grid to provide acceptable results.

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