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SMOOTH AND NON-SMOOTH SOLUTIONS FOR GEOMETRICALLY EXACT BEAMS SUBJECTED TO IMPACT

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Abstract. *Shock and Impact problems have been the focus of research for decades in many engineering areas such as structural engineering, automobile industry, aerospace, multibody dynamics. In general, unlike classical dynamics, impact requires specials techniques to correctly model the underlying physics. The solutions for the problems of impact have been developed in different ways by many authors with each of them using different hypothesis. However, formulations of impact of beams have received limited attention. The solution methods for impact can be divided into two main groups: smooth and non-smooth approaches. In this paper we will develop both smooth and nonsmooth formulations for impact between projectiles and beams. The adopted beam kinematics is geometrically exact. In smooth formulations, the velocities and forces are continuous functions. The impact forces is determined by the corresponding deformations of the projectile, the deformation of beam is calculated with the applied impact force. This approach is appropriate when the projectile is deformable and it reflects the reality of many impact scenarios.*
When it comes to impact between rigid projectile and deformable beams, the problem becomes more complex and should be treated using non-smooth formulations where the velocities and the impact forces may have jumps, thus, they are discontinuous. Set-valued force laws are used to account for unilateral conditions on both displacement and velocities. The appropriate framework is differntial measure combined with convex analysis The coefficient of restitution plays also an important role. The integration is also split into smooth and non-smooth part. Though, having stable time integration schemes for geometrically exact beams is not a straightforward task. For both formulations, the use of a newly developed energy-momentum time integration scheme conserve perfectly the total energy and the momentum of the system during and after the impact. A range of numerical applications will be delivered to show excellent performances of the new formulations.

1 Summary of kinematics of beams and strain measure

In this section we briefly introduce the beam's strain measures. We consider a general Cartesian co-ordinate basis $\boldsymbol{e}_i, i = 1, 2, 3$ and a curve $\boldsymbol{X}_0(s)$, with s being the arc-length. We understand $\boldsymbol{X}_0(s)$ as describing the centre line of the rod cross section. However, we restrict the deformations to planar ones in the $\boldsymbol{e}_1 - \boldsymbol{e}_2$ plane and introduce the tangent vector $\boldsymbol{G}_0 = \frac{d\boldsymbol{X}_0(s)}{ds}$. Perpendicular to this vector, we define $\boldsymbol{N}$ to be the normal vector with z as the corresponding coordinate in the direction of $\boldsymbol{N}$.

In addition to the Cartesian system, we define a suitable convected curvilinear coordinate system given by the triple s, z, x_3. We define the vector $\boldsymbol{X}(s, z) = \boldsymbol{X}_0(s) + z\boldsymbol{N}(s)$ as the position vector of points in the direction of $\boldsymbol{N}$ at the reference configuration. In a general cross section the boundary is defined by z being a function of x_3. Note that the third direction is not explicitly included in this equation, though it is implicitly understood that it is in the direction of $\boldsymbol{e}_3$ and that the deformation is independent of that direction. Consequently, a local basis in the reference configuration is defined by the triple $(\boldsymbol{G}, \boldsymbol{N}, \boldsymbol{e}_3)$, with $\boldsymbol{G} = \frac{\partial \boldsymbol{X}}{\partial s}, \ \boldsymbol{N} = \frac{\partial \boldsymbol{X}}{\partial z} = \frac{\partial \boldsymbol{X}}{\partial z}|_{z=0}$ and $\boldsymbol{G}_0 = \boldsymbol{G}\ |_{z=0}$. We have also the following relations $\boldsymbol{G} \cdot \boldsymbol{N} = \boldsymbol{G}_0 \cdot \boldsymbol{N} = 0, \ \ |\boldsymbol{N}| = 1, \ \ \boldsymbol{N} = \boldsymbol{e}_3 \times \frac{\boldsymbol{G}}{|\boldsymbol{G}|} = \boldsymbol{e}_3 \times \boldsymbol{G}_0$, where $| \bullet |$ denotes the absolute value of a vector, $\times$ and $(\cdot)$ denotes the cross and scalar product of vectors, respectively. The corresponding contra-variant basis vectors are then given by $(\boldsymbol{G}^*, \boldsymbol{N}, \boldsymbol{e}_3)$, with $\boldsymbol{G}^* = \frac{\boldsymbol{G}}{|\boldsymbol{G}|^2}$.

The deformed configuration is given as $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X})$ which defines the actual configuration. For our geometrically exact beam theory we make use of the Bernoulli model where the cross sections are assumed to be rigid and remain perpendicular to the center line. Thus, the corresponding tangent vectors at the deformed configuration are defined as $(\boldsymbol{g}, \boldsymbol{n}, \boldsymbol{e}_3)$ with $\boldsymbol{g}_0 = \boldsymbol{g}|_{z=0}$, and the following relations hold: $\boldsymbol{g} = \frac{\partial \boldsymbol{x}}{\partial s}, \ \boldsymbol{n} = \boldsymbol{e}_3 \times \frac{\boldsymbol{g}}{|\boldsymbol{g}|} = \boldsymbol{e}_3 \times \frac{\boldsymbol{g}_0}{|\boldsymbol{g}_0|}$. The position vector in the direction of $\boldsymbol{n}$ can be characterized by the following relation

$$\boldsymbol{x} = \boldsymbol{X}(s) - z\boldsymbol{N}(s) + \boldsymbol{u}(s) + z\boldsymbol{n}(s) = \boldsymbol{X}_0(s) + \boldsymbol{u}(s) + z\boldsymbol{n}(s), \tag{1}$$

where $\boldsymbol{u}(s)$ is the displacement vector of the centre line. From this we obtain immediately $\boldsymbol{g} = \boldsymbol{X}_{0,s} + \boldsymbol{u}_{,s} + z\boldsymbol{n}_{,s}$. The deformation gradient is written down in the curvilinear basis system as $\boldsymbol{F} = \boldsymbol{g} \otimes \boldsymbol{G}^* + \boldsymbol{n} \otimes \boldsymbol{N} + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$. The Green strain tensor is defined as $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{1})$, where $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$. It has only one non-trivial component $E_{11} = \boldsymbol{u}_{,s} \cdot \boldsymbol{X}_{0,s} + \frac{1}{2}\boldsymbol{u}_{,s} \cdot \boldsymbol{u}_{,s} + z\left(\boldsymbol{n}_{,s} \cdot (\boldsymbol{X}_{0,s} + \boldsymbol{u}_{,s}) - \boldsymbol{N}_{,s} \cdot \boldsymbol{X}_{0,s}\right)$, where the term in z^2 has been neglected. E_{11} is split into two components as $E_{11} = \varepsilon_{11} + z\kappa$, the definition of which are given by $\varepsilon_{11} = \boldsymbol{u}_{,s} \cdot \boldsymbol{X}_{0,s} + \frac{1}{2}\boldsymbol{u}_{,s} \cdot \boldsymbol{u}_{,s}, \ \kappa = \boldsymbol{n}_{,s} \cdot (\boldsymbol{X}_{0,s} + \boldsymbol{u}_{,s}) - \boldsymbol{N}_{,s} \cdot \boldsymbol{X}_{0,s}$. The first is the axial strain and the second is the classical change of curvature.

2 Smooth formulation and equations of motion

2.1 Dynamic equations and force integration method

In this section, the projectile is assumed to be elastically deformable and modelled by a spring with a mass, Fig. 1. The mass can only move in the horizontal direction which is parallel to vector $\boldsymbol{e_2}$ with an initial velocity V. Before contact, the projectile moves freely without friction and gravity, therefore the equation of motion of the projectile is:

$$M\ddot{U}_2 = 0, \tag{2}$$

where M is the weight of the projectile, $U_2 = \boldsymbol{U} \cdot \boldsymbol{e}_2$. $\boldsymbol{U}$ denotes the displacement of the mass measured from the moment that the spring hits the beam. Let's call the contact force F, so during contact, the equation of motion of the projectile is written as follows:

$$M\ddot{U}_2 + R_1 (U_2 - u_2) = 0, \;\; F = R_1 (U_2 - u_2). \tag{3}$$

where u_2 is the second component of the displacement vector of the contact point on beam, R_1 is the stiffness coefficient of spring. Starting from Hamilton's principle for our conservative mechanical system, the dynamics equation for our beam is written down as follows

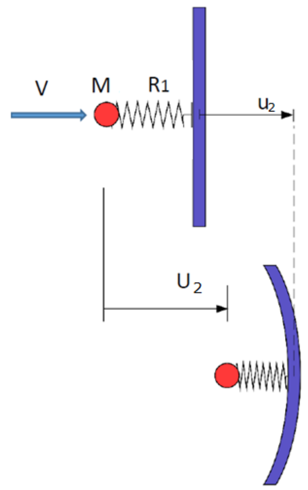

Figure 1: Impact model

$$\int_L \rho A \ddot{\boldsymbol{u}} \cdot \delta \boldsymbol{u}\, ds\, dt + \int_L \rho I \ddot{\boldsymbol{n}} \cdot \delta \boldsymbol{n}\, ds + \int_L (EA\varepsilon_{11}\delta\varepsilon_{11} + EI\kappa\delta\kappa)\, ds - F\delta u_2 = 0. \tag{4}$$

where E is Young's Modulus of the material, V is the volume of the system, $\boldsymbol{u}(s)$ is the displacement vector at coordinate s, $F = R_1 (U_2 - u_2) > 0$ during the contact and $F = 0$ otherwise, I is the moment of inertia of the section and L is the length of the beam, ρ is the material density.

2.2 Energy-momentum method and force integration

After the spatial discretisation which is standard as presented in[1], the numerical approach is completed by using a newly developed energy-momentum method for geometrically exact Euler-bernoulli beams which has been proven to have long-term stability and perfect conservation of energy and momentum [1]. The summary of the energy-momentum method is given as follows:

$$\varepsilon_{n+\frac{1}{2}} = \varepsilon_n + \frac{1}{2}\Delta T \dot{\varepsilon}_{n+\frac{1}{2}}, \;\; \kappa_{n+\frac{1}{2}} = \kappa_n + \frac{1}{2}\Delta T \dot{\kappa}_{n+\frac{1}{2}}, \;\; \dot{\boldsymbol{n}}_{n+\frac{1}{2}} = \dot{\boldsymbol{n}}_n + \frac{1}{2}\Delta T \ddot{\boldsymbol{n}}_{n+\frac{1}{2}}, \tag{5}$$

The mathematical proof of conservations are presented in [1]. The projectile displacement is updated using the classical midpoint rule. The value of force during the contact is integrated flowing $F^{n+\frac{1}{2}} = R1\left(U_2^{n+\frac{1}{2}} - u_2^{n+\frac{1}{2}}\right)$ with condition $F > 0$ to be verified at each time step, otherwise it's contact free and F is set to be zero.

2.3 Examples

In this example, we study the impact of an elastic projectile with an in-plane Euler-Bernoulli beam Fig. 2. Parameters of the problem are given below:

Projectile's mass: $M = 500\ Kg$
Projectile's initial velocity: $V = 20\ m/s$
Projectile's initial position:
$d = -0.2\ m$ - spring length
Spring stiffness: $R_1 = 7.5E5\ N/m$
Beam length $L = 3\ m$
Cross section area $A = 1E3\ cm^2$
Cross section inertia $I = 8330\ cm^4$
Young's Modulus $E = 0.2E10\ Pa$
Density $\rho = 3000\ Kg/m^3$
Number of elements = 6
Time increment $\Delta T = 1E - 4\ s$

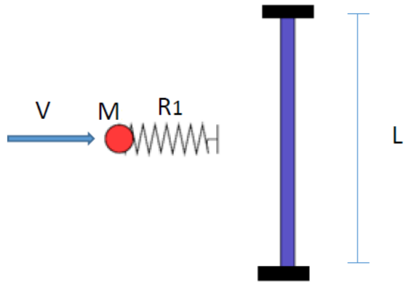

Figure 2: Impact model

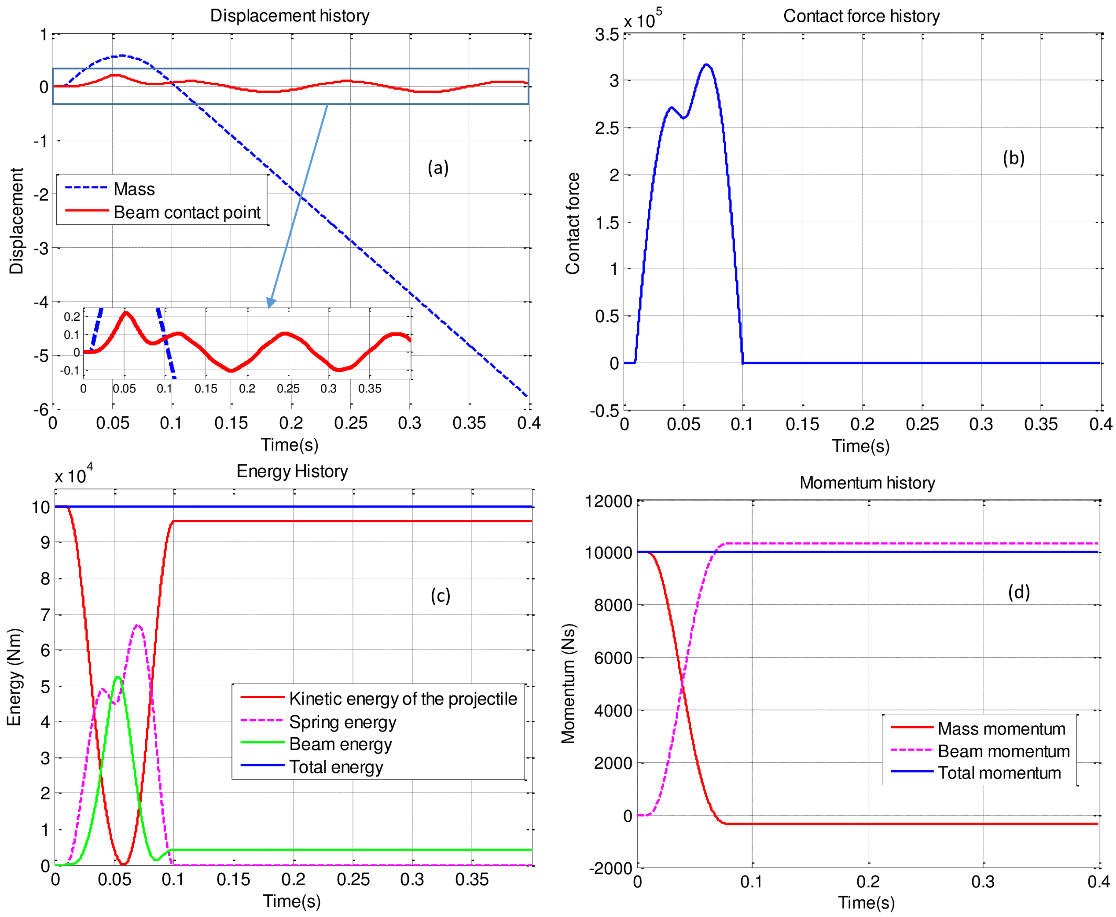

Figure 3: (a)displacement history, (b)contact force, (c)energy history, (d)momentum history

Fig. 3a illustrates the displacement of the mass compared to the displacement of the beam contact point. The segment where the displacement of the mass is larger than the beam corresponds to the contact period which is also represented by a positive contact force period in Fig.3b. Fig.3c shows that the total energy of the system is perfectly conserved. To verify the momentum conservation, the same test has been carried out considering the same beam but without support. The total momentum is also conserved perfectly, Fig. 3d. We note that when increasing the spring stiffness, not only one but several separated contact scenarios can happen.

3 Nonsmooth formulation and impact equations

3.1 Equations of motion

Hamilton's principle in nonsmooth mechanics has been studied by many authors [2, 5]. For our conservative system, the classical form still holds:

$$\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0, \tag{6}$$

where t refers to time and t_1 and t_2 are boundaries of the time interval, $\mathcal{L}$ is defined as the Lagrangian given by

$$\mathcal{L} = T - \Psi_{int} - \Psi_{ext}, \tag{7}$$

where T is the kinetic energy of the system and Ψ_{int}, Ψ_{ext} are respectively the internal and the external potential energies. Our body is non-conducting linear elastic solid and thermodynamic effects are not included in the system. The quantities in (7) are defined as

$$\delta T = \int_V \rho \dot{\boldsymbol{x}} \cdot \delta \dot{\boldsymbol{x}}\, dV, \;\; \delta \Psi_{int} = \int_V (E E_{11} \delta E_{11})\, dV, \;\; \delta \Psi_{ext} = \boldsymbol{\lambda}_N \delta \boldsymbol{u}, \tag{8}$$

where E is Young's modulus of the material, V is the volume of the system, L is the length of the beam, $\boldsymbol{\lambda}_N$ is the contact force during the impact time and equal to zero elsewhere. Let t_k be the moment of impact. t_k^- and t_k^+ are the starting and the closing time of the impact event, as a result, $\Delta t = t_k^+ - t_k^- = \frac{1}{n}$, with $n \to +\infty$. The impact event is assumed to happen in an infinitesimal time. The displacement is continuous but the velocity is discontinuous[2, 3, 4, 5], which read $\boldsymbol{x}\left(t_k^-\right) = \boldsymbol{x}\left(t_k^+\right)$, $\dot{\boldsymbol{x}}\left(t_k^-\right) \neq \dot{\boldsymbol{x}}\left(t_k^+\right)$. For $t < t_k^-$ and $t > t_k^+$ the equation of motion has its regular form, from (6), (7), together with an integration over the cross section, and due to the fact that the variations vanish at the boundaries, standard arguments of the calculus of variation deliver

$$\int_L \rho A \ddot{\boldsymbol{u}} \cdot \delta \boldsymbol{u}\, ds + \int_L \rho I \ddot{\boldsymbol{n}} \cdot \delta \boldsymbol{n}\, ds + \int_L \left(EA\varepsilon_{11}\delta\varepsilon_{11} + EI\kappa\delta\kappa\right) ds = 0, \tag{9}$$

where A and I is the area and the moment of inertia of the cross section, respectively. The equation of motion for projectile holds

$$M_1 \ddot{\boldsymbol{y}} = 0, \tag{10}$$

where M_1 and $\boldsymbol{y}$ are the mass and the position vector of the projectile, respectively. For $t_k^- < t < t_k^+$, from (6), (7), we have

$$\int_{t_k^-}^{t_k^+} \int_V \dot{\boldsymbol{x}} \delta \dot{\boldsymbol{x}} dV dt - \int_{t_k^-}^{t_k^+} \int_V E E_{11} \delta E_{11} dV dt + \int_{t_k^-}^{t_k^+} \boldsymbol{\lambda}_N \delta \boldsymbol{u} dt = 0. \tag{11}$$

Let's define the acceleration $\{\ddot{\boldsymbol{x}}\}$ as follows $\{\ddot{\boldsymbol{x}}\} \xrightarrow[n \to +\infty]{} \frac{\dot{\boldsymbol{x}}(t_k^+) - \dot{\boldsymbol{x}}(t_k^-)}{t_k^+ - t_k^-}$. From (11), the fact that the variations vanish at the boundary and the calculation of the integral over Δt and then the integral over the cross sectio, we obtain

$$\int_L A\left(\dot{\boldsymbol{u}}\left(t_k^+\right) - \dot{\boldsymbol{u}}\left(t_k^-\right)\right)\delta \boldsymbol{u} ds + \int_L I\left(\dot{\boldsymbol{n}}\left(t_k^+\right) - \dot{\boldsymbol{n}}\left(t_k^-\right)\right)\delta \boldsymbol{n} ds$$
$$+ \int_L \left(EA\varepsilon_{11}\delta\varepsilon_{11} + EI\kappa\delta\kappa\right)\Delta t\, ds - \boldsymbol{\Lambda}_N \delta \boldsymbol{u} = 0, \tag{12}$$

where $\boldsymbol{\Lambda}_N = \boldsymbol{\lambda}_N \Delta t$ is the impulsive force during impact. The equation of motion for projectile during impact holds

$$M_1 \left(\dot{\boldsymbol{y}}^+ - \dot{\boldsymbol{y}}^-\right) = -\boldsymbol{\lambda}_N. \tag{13}$$

(9), (10), (12) and (13) are fundamental equations of motion for our system before, after and within the impact event.

3.2 Unilateral constraints

Let's define $\boldsymbol{D}_N$ as the relative displacement between the projectile and the contact point on the beam. On displacement level, we have the following constraint between contact forces and $\boldsymbol{D}_N$ holds $\boldsymbol{D}_N \geq 0$, $\boldsymbol{\lambda}_N \geq 0$ and $\boldsymbol{D}_N \boldsymbol{\lambda}_N = 0$. Due to the equivalence between Signorini's force laws and normal cone inclusion of force laws[2, 3, 4], one can write the previous relations in a very compact form: $\boldsymbol{D}_N \in \mathcal{N}_{\mathcal{B}}\left(-\boldsymbol{\lambda}_N\right)$, where $\mathcal{B}$ is the convex set containing all available values of $-\boldsymbol{\lambda}_N$. Equivalently, at velocity level, the constraint is written as $\boldsymbol{\zeta}_N \in \mathcal{N}_{\mathbb{R}_0^-}\left(-\boldsymbol{\Lambda}_N\right)$, where $\boldsymbol{\zeta}_N$ is the relative velocity between the projectile and the beam contact point. Please refer to [2, 3, 4] for more details on the solution of normal cone inclusion as proximal point.

3.3 Newton's impact laws

Newton's impact laws states: $\boldsymbol{\zeta}_N^+ = -\alpha \boldsymbol{\zeta}_N^-$, where $\boldsymbol{\zeta}_N^-$ and $\boldsymbol{\zeta}_N^+$ are the relative velocities just before and after impact, $\alpha = [0, 1]$ is the coefficient of restitution(CoR). Therefore, the constraint on velocity level can be generalized to have the form:

$$\boldsymbol{\zeta}_N^+ + \alpha \boldsymbol{\zeta}_N^- \in \mathcal{N}_{\mathbb{R}_0^-}\left(-\boldsymbol{\Lambda}_N\right). \tag{14}$$

(14) can be solved using proximal point equation [2, 3, 4] as follows

$$-\boldsymbol{\Lambda}_N = prox_{\mathcal{B}}^{\mathbb{R}_0^-}\left(-\boldsymbol{\Lambda}_N + R^{-1}\left(\boldsymbol{\zeta}_N^+ + \alpha \boldsymbol{\zeta}_N^-\right)\right), \tag{15}$$

where R is a chosen parameter to only speed up the convergence and it doesn't affect the converged numerical results.

3.4 Space and time discretisation and local iteration

Before and after impact ($t < t_k^-,\ t > t_k^+,\ \boldsymbol{D}_N > 0$), the motion of the beam is regular and is described by (9). Thus, the time integration scheme is similar to the smooth case, Cf.5. During impact ($t_k^- < t < t_k^+,\ \boldsymbol{D}_N = 0$), the value of the impulsive force is determined via a internal iteration by solving the proximal point equation [2, 3, 4]:

$$\boldsymbol{\Lambda}_N^{j+1} = max\left(\boldsymbol{\Lambda}_N^j - R^{-1}\left(\boldsymbol{\zeta}_{Nn+1}^+ + \alpha \boldsymbol{\zeta}_{Nn}^-\right)\right),\ |\boldsymbol{\Lambda}_N^{j+1} - \boldsymbol{\Lambda}_N^j| \leq error. \tag{16}$$

In our case R is chosen to be equal to $\frac{M_1 m}{M_1 + m}$, where m is the weigh of one beam element.

3.5 Examples

To assess the effectiveness of the formulation in dealing with impact problems, we first examine an example of impact between a rigid projectile and a clamped Euler-Bernoulli beam at both ends, Fig 4. The mass; the initial velocity of the projectile and the beam parameters are given below:

Projectile's mass: $M_1 = 100\ Kg$
Projectile's initial velocity: $V_1 = -40\ m/s$
Projectile's initial position: $d = 2\ cm$
Beam length $L = 3\ m$
Cross section area $A = 1E3\ cm^2$
Cross section inertia $I = 8330\ cm^4$

Young's Modulus $E = 0.2E12\ Pa$
Density $\rho = 3000\ Kg/m^3$
Number of elements = 6
Time increment $\Delta T = 1E-5\ s$
Coefficient of restitution = 1

In Fig. 5a, both the motion of the projectile as well as the motion of the contact point on the beam are depicted. Jumps in velocities are illustrated in Fig. 5c. Impulsive force are plotted in Fig. 5b which have value only at the moments of impact. At the same time, Fig. 5d shows clearly the conservation of total energy. To verify the conservation of momentum, the same example has been studied but the beam doesn't have any support. The momentum is perfectly conserved, total momentum is independent of the coefficient of restitution and always equal to the initial momentum of the projectile, Fig. 5e.

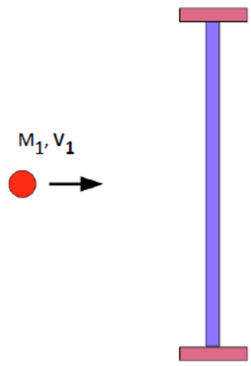

Figure 4: Nonsmooth impact model

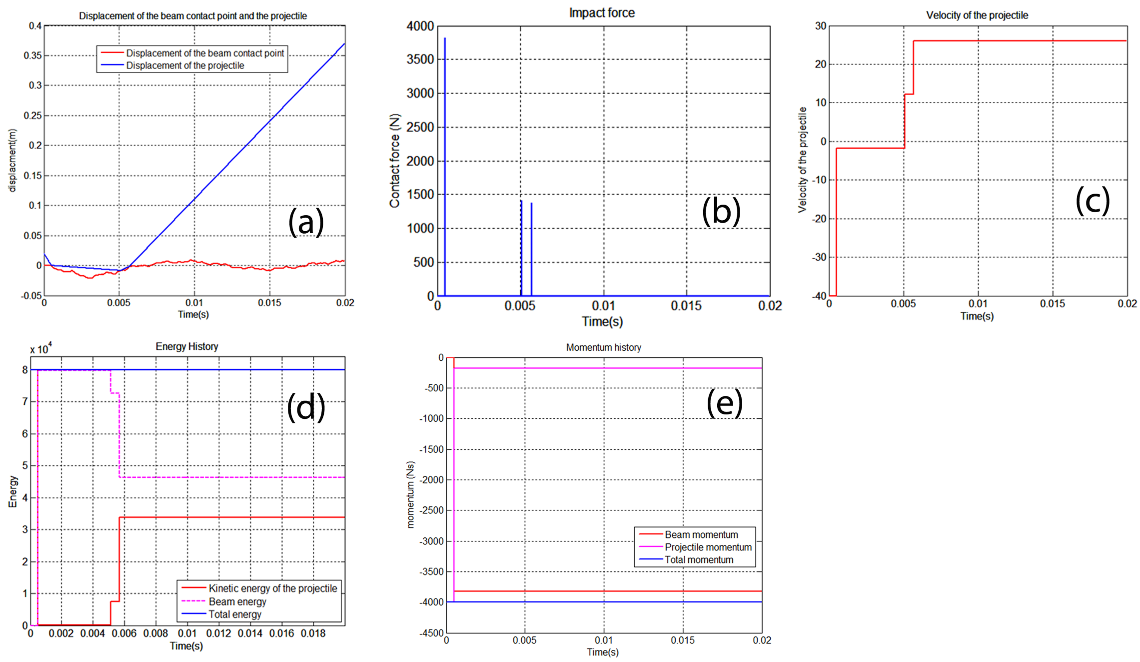

Figure 5: (a)displacement history, (b)contact force, (c)velocity of projectile, (d)energy history, (e)momentum history

The same test has been carried out, but in this case the value of coefficient of restitution is set to be zero. Similar results are obtained Fig.6 but we can see that preserved energy is proportional to the coefficient of restitution (Fig. 6d, 6f, 5d) and total momentum is conserved independently of the coefficient of restitution.

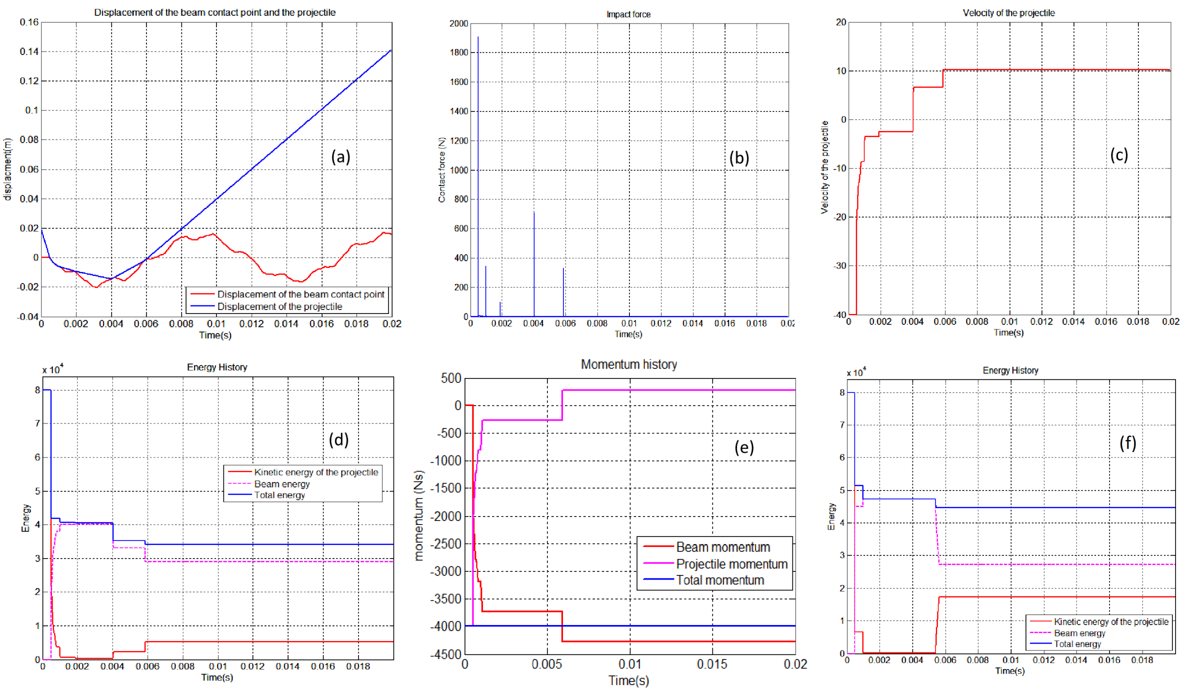

Figure 6: (a)displacement history, (b)contact force, (c)velocity of projectile, (d)energy history, (e)momentum history, (f)energy history (CoR=0.5)

4 Conclusions

In this work, two algorithms for impact on beam have been presented. Both smooth and nonsmooth approaches have shown excellent performances in term of accuracy, stability and conservation properties even for strongly non-linear situations. In the smooth formulation, the impact force is calculated directly from projectile deformations, however the whole together has to respect collision laws, dynamics equations for the beams and the projectile. The nonsmooth ones are specially used when projectiles are rigids. The method is able to tackle perfectly the discontinuities of velocities and impact forces via set-value force laws of normal cone and also Newton's Impact law where coefficient of restitution is incorporated.

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