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OPTIMAL REDUCTION OF NUMERICAL DISPERSION FOR WAVE PROPAGATION PROBLEMS. APPLICATION TO ISOGEOMETRIC ELEMENTS.

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Abstract. *Based on the optimal coefficients of the stencil equation, a numerical technique for the reduction of the numerical dispersion error has been suggested. New isogeometric elements with the reduced numerical dispersion error for wave propagation problems have been developed with the suggested approach. By the minimization of the order of the dispersion error of the stencil equation, the order of the dispersion error is improved from order 2p (the conventional isogeometric elements) to order 4p (the isogeometric elements with reduced dispersion) where p is the order of the polynomial approximations. Because all coefficients of the stencil equation are obtained from the minimization procedure, the obtained accuracy is maximum possible. The corresponding elemental mass and stiffness matrices of the isogeometric elements with reduced dispersion are calculated with help of the optimal coefficients of the stencil equation. The analysis of the dispersion error of the isogeometric elements with the lumped mass matrix has also shown that independent of the procedures for the calculation of the lumped mass matrix, the second order of the dispersion error cannot be improved with the conventional stiffness matrix. However, the dispersion error with the lumped mass matrix can be improved from the second order to order 2p by the modification of the stiffness matrix. The numerical examples confirm the computational efficiency of the isogeometric elements with reduced dispersion that significantly reduce the computation time at a given accuracy. The numerical results obtained by the new and conventional isogeometric elements may include spurious oscillations due to the dispersion error. These oscillations can be quantified and filtered by the two-stage time-integration technique developed recently. The approach developed can be directly applied to other space-discretization techniques with similar stencil equations.*

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1 INTRODUCTION

Wave propagation in an isotropic homogeneous medium is described by the following scalar wave equation in domain Ω:

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0\,, \tag{1}$$

with the boundary conditions $\boldsymbol{n}\cdot\bigtriangledown u = g_1$ on Γ^t and $u = g_2$ on Γ^u, and the initial conditions $u(\boldsymbol{x}, t = 0) = g_3$, $v(\boldsymbol{x}, t = 0) = g_4$ in Ω. Here, u is the field variable, $v = \dot{u}$ is the velocity, c is the wave velocity, t is the time, Γ^t and Γ^u denote the natural and essential boundaries, g_i $(i = 1, 2, 3, 4)$ are the given functions, $\boldsymbol{n}$ is the outward unit normal on Γ^t. The application of the continuous Galerkin approach and the space discretization (e.g., the finite elements, spectral elements, isogeometric elements; see [9, 10, 18] and others) to Eq. (1) leads to a system of ordinary differential equations in time

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + c^2\boldsymbol{K}\boldsymbol{U} = \boldsymbol{0}\,, \tag{2}$$

with

$$\boldsymbol{M} = \sum_e \boldsymbol{M}^e\,, \qquad \boldsymbol{K} = \sum_e \boldsymbol{K}^e\,, \tag{3}$$

where $\boldsymbol{U}(t)$ is the vector of the field variable, the global mass $\boldsymbol{M}$ and stiffness $\boldsymbol{K}$ matrices have a banded structure and are obtained by the summation of the corresponding local (element Ω^e) matrices $\boldsymbol{M}^e$ and $\boldsymbol{K}^e$:

$$\boldsymbol{M}^e = \int_{\Omega^e} \boldsymbol{N}^T\boldsymbol{N}d\Omega^e\,, \tag{4}$$

$$\boldsymbol{K}^e = \int_{\Omega^e}^{1} [\frac{\partial\boldsymbol{N}}{\partial\boldsymbol{x}}]^T[\frac{\partial\boldsymbol{N}}{\partial\boldsymbol{x}}]d\Omega^e\,. \tag{5}$$

Here, $\boldsymbol{N}$ and $\frac{\partial\boldsymbol{N}}{\partial\boldsymbol{x}}$ are the shape matrix and its derivative with respect to the physical coordinate $\boldsymbol{x}$; see [9, 10, 6]. Due to the space discretization, the exact solution to Eq. (2) contains the numerical dispersion error. Usually the analysis of the numerical dispersion error and its improvement for many space-discretization techniques such as the finite elements, spectral elements, isogeometric elements and others starts with the analysis and modifications of the elemental mass and stiffness matrices; see [10, 1, 2, 7, 8, 15, 16, 17, 18, 22, 11, 14, 20, 21]. For example, one simple and effective finite-element technique for acoustic and elastic wave propagation problems is based on the calculation of the mass matrix $\boldsymbol{M}$ in Eq. (2) as a weighted average of the consistent and lumped mass matrices; see [15, 16, 17, 18] and others. For the 1-D case and the linear finite elements, this approach reduces the error in the wave velocity for harmonic waves from the second order to the fourth order of accuracy. However, for harmonic wave propagation in the 2-D and 3-D cases, these results are not valid (nevertheless, in the multi-dimensional case, the averaged mass matrix yields more accurate results compared with the standard mass matrix).

An interesting technique with implicit and explicit time-integration methods is suggested in [22] for acoustic waves. It is based on the modified integration rule for the calculation of the mass and stiffness matrices for the linear finite elements. In contrast to the averaged mass matrix, the use of the modified integration rule increases the accuracy for the phase velocity from the second order to the fourth order in the general multi-dimensional case of acoustic waves. A similar improvement in the order of the numerical dispersion error for the linear elements in the 1-D case yields the selective mass scaling technique developed in [19]. Using the mimetic finite difference approach, similar results for acoustic waves and electric waves in the 2-D case have been obtained in [8, 4] for the linear finite elements with reduced dispersion and reduced numerical anisotropy.

The dispersion reduction technique for the high-order finite elements have been suggested in [2] for acoustic waves. This technique is based on the calculation of the mass matrix $\boldsymbol{M}$ in Eq. (2) as a weighted average of the consistent and lumped mass matrices for the high-order finite elements. It was also shown in [2] that the same results can be obtained with the modified integration rule for the mass matrix. With this technique, the dispersion error is improved from the order $2p$ to the order $2p+2$ (p is the order of polynomial approximations).

New isogeometric elements with a higher order of continuity across elements are suggested in [10, 6] for dynamics problems. It has been shown in these papers that the isogeometric elements yield more accurate numerical results for wave propagation problems compared with the high-order finite elements. The modification of the non-diagonal mass matrix in [20, 21] for the isogeometric elements allows the increase in the order of the dispersion error from order $2p$ to order $2p+2$ in the 1-D case and for one specific direction of harmonic waves in the 2-D case (these techniques do not improve the order of the dispersion error in the general 2-D case). It is also necessary to mention that in contrast to the high order finite and spectral elements, the stencil equation for the isogeometric elements on uniform meshes is the same for all internal degrees of freedom (for the high order finite and spectral elements there are several stencil equations with different structures depending on the location of nodes). This simplifies the analysis of the numerical dispersion error for the isogeometric elements. However, along with the advantages, there are some issues with the isogeometric elements. For example, the lumped mass matrix for the isogeometric elements decreases the order of the dispersion error to the second order of accuracy; e.g., see [3]. Therefore, a special iterative procedure for linear dynamics problems was developed in [3] in order to obtain the same order of the dispersion error for the lumped and consistent mass matrices.

Here, we will present a new approach with new quadratic isogeometric elements in the 1-D case for which the order of the numerical dispersion error is improved from order $2p$ to order $4p$. This order is optimal and cannot be further improved. More detailed derivations of the new technique for quadratic and cubic isogeometric elements in the 1-D and 2-D cases as well as the corresponding numerical examples can be found in our papers [12, 13].

2 A NEW APPROACH FOR THE QUADRATIC ISOGEOMETRIC ELEMENTS WITH REDUCED DISPERSION IN THE 1-D CASE

Below we will present the the derivation of new quadratic isogeometric elements in the 1-D case. More detailed derivations of the new technique for quadratic and cubic isogeometric elements in the 1-D and 2-D cases can be found in our papers [12, 13].

Inserting time-harmonic solutions

$$u(x,t) \,=\, exp(i\,\omega t)u(x) \tag{6}$$

into the wave equation (1) in the 1-D case ($\nabla^2 u = \frac{\partial^2 u}{\partial x^2}$) leads to its reduction to the Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + k^2 u = 0\,, \tag{7}$$

where ω is the angular velocity, $k = \omega/c$ is the wave number, $i = \sqrt{-1}$. Eq. (7) allows the following exact solutions:

$$u(x) \,=\, exp(\pm ikx)\,. \tag{8}$$

After the space discretization, Eq. (7) reduces to

$$(\boldsymbol{K} \,-\, k^2\boldsymbol{M}\,)\,\boldsymbol{U} \,=\, \boldsymbol{0}\,, \tag{9}$$

where the matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ are calculated with the help of Eqs. (3)-(5).

For the derivation of new quadratic isogeometric elements with reduced dispersion let us start with the stencil equation with unknown coefficients for the degree of freedom u_A of the system of equations Eq. (9) that can be written as follows:

$$k^2h^2(m_1(u_{A-2}+u_{A+2})+m_2(u_{A-1}+u_{A+1})+m_3u_A)-(k_1(u_{A-2}+u_{A+2})+k_2(u_{A-1}+u_{A+1})+k_3u_A) \,=\, 0\,, \tag{10}$$

where unknown coefficients m_j and k_j ($j \,=\, 1, 2, 3$) are calculated in terms of the elemental mass and stiffness matrices, Eq. (3), respectively (at this point, these elemental matrices are not defined). For example, for the conventional quadratic isogeometric elements, the coefficients of the stencil equation (10) are (e.g., see [10, 6, 20, 21]):

$$m_1 \,=\, 1\,,\; m_2 \,=\, 26\,,\; m_3 \,=\, 66\,,\; k_1 \,=\, -20\,,\; k_2 \,=\, -40\,,\; k_3 \,=\, 120\,. \tag{11}$$

Similar to the conventional isogeometric elements we assumed the symmetry for coefficients m_j and k_j in the stencil equation (10) for the degrees of freedom u_j ($j = A-1, A+1, A-2, A+2$) symmetrically located with respect to the degree of freedom u_A.

The idea of the new approach is to find the optimal coefficients of the stencil equation Eq. (10) that reduce the order of the dispersion error. Eq. (10) allows the following solutions (similar to Eq. (8)):

$$u_A \,=\, exp(\pm ik_hhA)\,, \tag{12}$$

where k_h is the numerical wave number. Inserting Eq. (12) into Eq. (10) we can find the following relation between the exact and numerical wave numbers k and k_h:

$$\frac{k}{k_h} \,=\, \frac{1}{(k_hh)}\frac{\sqrt{2k_1\cos(2(k_hh)) + 2k_2\cos(k_hh) + k_3}}{\sqrt{2m_1\cos(2(k_hh)) + 2m_2\cos(k_hh) + m_3}}\,. \tag{13}$$

Expanding the right-hand side of Eq. (13) into a Taylor series at small $h \ll 1$ and analyzing the first terms of a Taylor series we can find that the following constrain $k_3 = -2k_1 - 2k_2$ should be met in order to get at least the first order of the dispersion error. In this case Eq. (13) can be rewritten as:

$$\frac{k}{k_h} \,=\, \frac{1}{(k_hh)}\frac{\sqrt{2}\sqrt{k_1\cos(2(k_hh)) + k_2\cos(k_hh) - k_1 - k_2}}{\sqrt{2m_1\cos(2(k_hh)) + 2m_2\cos(k_hh) + m_3}}\,. \tag{14}$$

Expanding again the right-hand side of Eq. (14) into a Taylor series at small $h \ll 1$ and equating the first term to one and the following three terms to zero, we can find the coefficients m_j and k_j ($j = 1, 2, 3$) from a system of four algebraic equations. They are

$$m_1 \,=\, \frac{23a_1}{2358}\,,\; m_2 \,=\, \frac{344a_1}{1179}\,,\; m_3 \,=\, a_1\,,\; k_1 \,=\, -\frac{155a_1}{786}\,,\; k_2 \,=\, -\frac{320a_1}{393}\,,\; k_3 \,=\, \frac{265a_1}{131}\,, \tag{15}$$

where a_1 is the arbitrary coefficient. Expanding the right-hand side of Eq. (14) into a Taylor series at small $h \ll 1$ with the coefficients given by Eq. (11) we get

$$\frac{k}{k_h} = 1 + \frac{(k_h h)^4}{1440} + \frac{(k_h h)^6}{6720} + \frac{23(k_h h)^8}{4147200} + O\left((k_h h)^{10}\right), \tag{16}$$

i.e., the conventional quadratic isogeometric elements yield the fourth order of the dispersion error for (k/k_h-1). Expanding the right-hand side of Eq. (14) into a Taylor series at small $h \ll 1$ with the coefficients given by Eq. (15) we get

$$\frac{k}{k_h} = 1 - \frac{79(k_h h)^8}{9525600} - \frac{2633(k_h h)^{10}}{2640496320} + O\left((k_h h)^{12}\right), \tag{17}$$

i.e., the stencil equation (10) with the coefficients given by Eq. (15) yields the 8th order of the dispersion error for (k/k_h-1).

Let us find the elemental mass and stiffness matrices that yield the stencil equation (10) with the coefficients m_j and k_j ($j = 1, 2, 3$) given by Eq. (15). Similar to the conventional mass and stiffness matrices, let us assume the following form of the new mass and stiffness matrices:

$$\boldsymbol{M}^e = h \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{12} \\ m_{13} & m_{12} & m_{11} \end{pmatrix}, \qquad \boldsymbol{K}^e = \frac{1}{h} \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{12} \\ k_{13} & k_{12} & k_{11} \end{pmatrix}, \tag{18}$$

where we use the symmetry of the coefficients of the mass and stiffness matrices for the degrees of freedom contributing to these matrices. In this case the matrices $\boldsymbol{M}^e$ and $\boldsymbol{K}^e$ depend on 8 unknown terms m_{11}, m_{12}, m_{13}, m_{22} and k_{11}, k_{12}, k_{13}, k_{22}. With the help of (3), the coefficients of the stencil equation (10) can be expressed in terms of the coefficients of the matrices $\boldsymbol{M}^e$ and $\boldsymbol{K}^e$ as follows:

$$m_1 = m_{13}\,,\ m_2 = 2m_{12}\,,\ m_3 = 2m_{11} + m_{22}\,,\ k_1 = k_{13}\,,\ k_2 = 2k_{12}\,,\ k_3 = 2k_{11} + k_{22}\,. \tag{19}$$

Solving simultaneously Eqs. (15) and (19) we can find the mass and stiffness matrices for the quadratic isogeometric elements with reduced dispersion. They have the following form:

$$\boldsymbol{M}^e = \frac{h}{120} \begin{pmatrix} a_2 & \frac{6880a_1}{393} & \frac{460a_1}{393} \\ \frac{6880a_1}{393} & 120a_1 - 2a_2 & \frac{6880a_1}{393} \\ \frac{460a_1}{393} & \frac{6880a_1}{393} & a_2 \end{pmatrix}, \qquad \boldsymbol{K}^e = \frac{1}{6h} \begin{pmatrix} a_3 & -\frac{320a_1}{131} & -\frac{155a_1}{131} \\ -\frac{320a_1}{131} & \frac{1590a_1}{131} - 2a_3 & -\frac{320a_1}{131} \\ -\frac{155a_1}{131} & -\frac{320a_1}{131} & a_3 \end{pmatrix}, \tag{20}$$

where a_j (j=1,2,3) are three arbitrary coefficients. The dispersion error is independent of the values of these coefficients a_j. Some coefficients a_j can be found from the following conditions. Let us assume that the velocity is the same for the entire domain. In this case we can calculate the kinetic energy for one element with the help of the mass matrix and analytically for the considered domain. Equating these two expressions we get

$$\boldsymbol{V}_0^T \boldsymbol{M}^e \boldsymbol{V}_0 = v_0^2 h\,, \tag{21}$$

where $\boldsymbol{V}_0 = (v_0\,, v_0\,, v_0)^T$ is the vector with three equal components. Calculating the left-hand side of Eq. (21) with the help of Eq. (20) we get that $a_1 = 131/210$. Next, let us assume that the displacement is the same for the entire domain. In this case strains, stresses and forces are zero. The forces for one element can be calculated with the help of the stiffness matrix:

$$\boldsymbol{K}^e \boldsymbol{U}_0 = \boldsymbol{0}\,, \tag{22}$$

where $\boldsymbol{U}_0 = (u_0\,, u_0\,, u_0)^T$ is the vector with three equal components. Calculating the left-hand side of Eq. (22) with the help of Eq. (20) we get that $a_3 = 95/42$. For $a_1 = 131/210$ and $a_3 = 95/42$, Eq. (20) can be rewritten as follows:

$$\boldsymbol{M}^e = \frac{h}{120}\begin{pmatrix} a_2 & \frac{688}{63} & \frac{46}{63} \\ \frac{688}{63} & \frac{524}{7} - 2a_2 & \frac{688}{63} \\ \frac{46}{63} & \frac{688}{63} & a_2 \end{pmatrix}, \qquad \boldsymbol{K}^e = \frac{1}{6h}\begin{pmatrix} \frac{95}{42} & -\frac{32}{21} & -\frac{31}{42} \\ -\frac{32}{21} & \frac{64}{21} & -\frac{32}{21} \\ -\frac{31}{42} & -\frac{32}{21} & \frac{95}{42} \end{pmatrix}. \tag{23}$$

Remark. All unknown coefficients of the stencil equation (10) have been found from the minimization of the order of the dispersion error; see Eq. (15). The presence of the arbitrary coefficient a_1 in Eq. (15) is explained by the form of the homogeneous discretized Helmholtz equation (9); i.e., the multiplication of the mass and stiffness matrices by any coefficient does not change this equation and the stencil equation (10). The unknown coefficient a_1 can be found from Eq. (21). This leads to the following constrain for the coefficients m_i of the stencil equation (10):

$$2m_1 + 2m_2 + m_3 \;=\; 1\,. \tag{24}$$

In this case it follows from Eqs. (15) and (24) that $a_1 = 131/210$. Because all coefficients of the stencil equation can be found from the analysis of the dispersion error, the 8th order of the dispersion error in Eq. (17) is maximum possible for the considered form of the stencil equation (that is related to the support of basis functions by a specific number of elements) for all quadratic isogeometric elements.

3 NUMERICAL EXAMPLE: A STANDING WAVE IN 1-D ELASTIC BAR

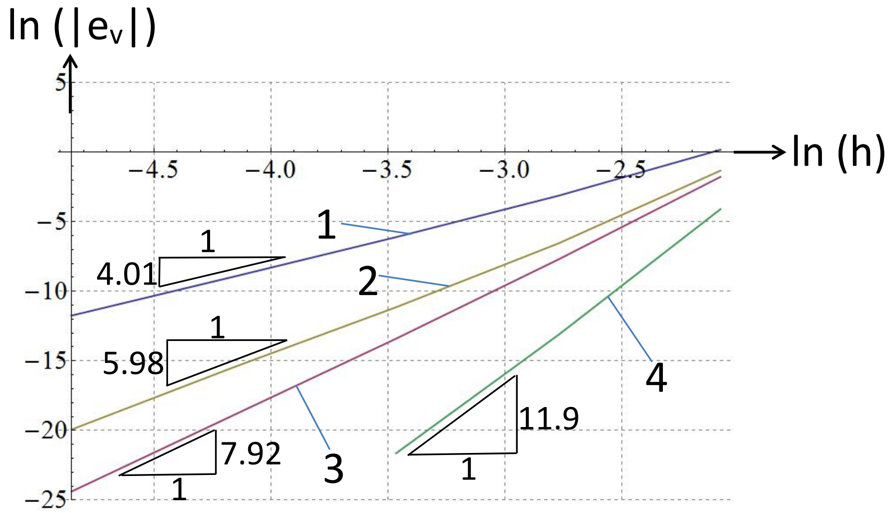

Figure 1: The error in velocity $e_v = |v_{exact}(x = L/2, t = T) - v_{num}(x = L/2, t = T)|$ as a function of the mesh size h in the logarithmic scale. Curves 1 and 2 correspond to the conventional quadratic and cubic isogeometric elements. Curves 3 and 4 correspond to the new quadratic and cubic isogeometric elements. The slopes of the curves at small h show the order of convergence of the corresponding techniques.

Similar to [5], the problem of a standing wave is used for the demonstration of the order of convergence of the new numerical technique. Let us consider an elastic bar of length $L = 1$. The wave velocity is chosen to be $c = 1$. A standing wave in the 1-D case can be described by the following exact solution to the wave equation:

$$u(x,t) = Sin\left(\frac{a\pi x}{L}\right) Cos\left(\frac{a\pi t}{L}\right) , \tag{25}$$

where $a = 5$ is used. The initial conditions at time $t = 0$ and the boundary conditions in terms of displacements at $x = 0$ and $x = L$ are selected according to the exact solution, Eq. (25); i.e., the initial displacements are $u(x, t = 0) = Sin\left(\frac{a\pi x}{L}\right)$, the initial velocities are zero and the two ends of the bar are fixed. The observation time is selected to be $T = L/a$. According to Eq. (25), the displacements at this observation time change the sign and the velocities at all points become zero again. The problem is solved by the conventional and new isogeometric elements on meshes with uniformly spaced control points. For the time integration the trapezoidal rule is used with very small time increments at which the error in time is very small and can be neglected. This means that the difference between numerical and analytical solutions is only related to the space-discretization error. Fig. 1 shows the convergence of the error in the velocity $e_v = |v_{exact}(x = L/2, t = T) - v_{num}(x = L/2, t = T)|$ in the center of the bar at mesh refinement where $v_{exact}(x = L/2, t = T)$ and $v_{num}(x = L/2, t = T)$ are the exact and numerical velocities in the center of the bar at the observation time T. We should mention that the maximum error in the velocity occurs in the center of the bar. h in Fig. 1 is the distance between uniformly spaced control points. At the same h, the meshes with quadratic and cubic isogeometric elements include the same number of degrees of freedom. The results in Fig. 1 are plotted in the logarithmic scale. Therefore, the slopes of the curves at small h shown in Fig. 1 correspond to the order of convergence (the order of accuracy) of the conventional and new isogeometric elements. The new isogeometric elements yield much more accurate results than those obtained by the conventional elements. Moreover, at the same number of degrees of freedom (at the same h), the new quadratic elements are more accurate compared to the conventional quadratic and cubic elements (see curves 1, 2 and 3 in Fig. 1). The results in Fig. 1 are in good agreement with the theoretical order of accuracy of the conventional and new isogeometric elements reported in the previous sections of the paper.

More numerical examples in the 1-D and 2-D cases solved by the new isogeometric elements can be found in our papers [12, 13].

3.1 CONCLUSIONS

By the analysis of the dispersion error of the stencil equation with arbitrary coefficients we have shown that these coefficients can be found by the minimization of the order of the dispersion error. In this case, the order of the dispersion error can be increased from the order $2p$ for the conventional high-order isogeometric elements to the order $4p$ for the new elements (p is the order of the polynomial approximations). Because all coefficients of the stencil equation can be found from the analysis of the dispersion error, the order $4p$ of the dispersion error is maximum possible for the considered form of the stencil equation (that is related to the support of basis functions by a specific number of elements) for the corresponding high-order isogeometric elements. We should mention that we have never seen in the literature such a significant improvement of the order of the dispersion error for the conventional finite elements, the spectral elements or the isogeometric elements. For example, for the linear and high-order finite and isogeometric elements (e.g., see [2, 22, 20, 11, 14]), the order of the dispersion error has been improved from the order $2p$ to the order $2p + 2$ and this leads to a significant decrease in the computation time at a given accuracy. The increase in accuracy for the new isogeometric elements with the order $4p$ of the dispersion error is much higher compared with the known approaches. This will lead to a much more significant reduction in the computation time at a given accuracy.

More detailed derivations of the new technique for quadratic and cubic isogeometric elements in the 1-D and 2-D cases as well as the corresponding numerical examples can be found in our papers [12, 13].

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