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NONLINEAR ICE ROD -STRUCTURE VIBRATIONS

A. Abramian, S. Vakulenko

Institute of Problems of Mechanical Engineering, RAS
Bolshoy pr. 61, V.O., Saint Petersburg, Russia
andabr55@gmail.com

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Abstract. *In this paper, we introduce a new nonlinear model for the moving ice rod-structure interaction. Nonlinear vibrations of that system is a complicated process, which involves an ice failure, as a result of the interaction between a moving ice and a structure. Here we propose a model, which develops the known models in this field, in particular, the Matlock-Sodhi model. Similar to the Matlock model, a structure is considered as a single oscillator. In this paper, we study the ice rod-structure interaction in more details. The deformations of the ice rod are described taking into account a permanent contact between the oscillator and the rod. For calculation of the oscillator-ice rod interaction, an extrusion effect is taken into account. The aforementioned effects make the problem more complicated: partial differential equations (PDE's) for the ice rod and ordinary differential equations, (ODE's) for the structure are involved. The main difficulty of the problem is that these PDE and ODEs are coupled via boundary conditions for ice rod interaction force and displacement. Nonetheless, it is possible to resolve this problem using a new asymptotic approach. This approach allows us to find the ODE for the oscillator, where the ice displacement is excluded. This equation describes nonlinear oscillations of the oscillator. The terms in that equation admit transparent physical interpretations and relate to: 1) the effect of water extrusion under an ice rod pressure, which leads to a particular type of a friction force and nonlinear effects; 2) the contact interaction between ice rod and the oscillator, which leads to additional nonlinearities and to the oscillator frequency shift. The main result of the asymptotic investigation and numerical simulations is the origin of a negative friction for some ice rod velocities. Moreover, the resonance like peak shape of the amplitude on velocity dependence is a result of an instability of the vibrations regime. The investigation of an instability onset for these vibrations lead us to the conclusion that instability can be the reason of a lock-in regime of vibrations.*

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1 INTRODUCTION

In this paper, we investigate ice induced vibrations (IIV) of structures such as offshore drilling platforms and lighthouses. This problem has important applications for engineering in Arctic region. As a result of an ice floe interaction with such structures large amplitude vibrations where observed, which break their functioning and even destroying them. The IIV are generated by a complicated process involving ice failure, nonlinear dynamics of structures and an interaction between ice and structures.In this paper, we propose a model extending the prevous ones, in particular, suggested in [1, 2] and [3]. Following [1, 2, 3] we consider structures as oscillators, however, we suppose that these oscillators can involve a number of interaction modes. A novelty with respect to previous investigations is that we study the ice sheets and ice-structure interactions in more detail. We describe deformations of the ice sheets taking into account a contact between structures and ice. For oscillator-ice interactions, we take into account extrusion effects. This consideration leads to a diffucult problem, which involves partial differential equations (PDE's) for the ice sheets and ordinary differential equations (ODE's) for structures. The main difficulty is that these PDE and ODE are coupled via boundary conditions for the ice sheet deformations on a contact line between the ice sheet and the structure. This contact line is unknown. Such contact problems are difficult, nonetheless, we are capable to resolve our problem using a new asymptotic approach. This approach exploits mechanical properties of the ice sheet model, namely, we assume that the ice sheet internal friction is small whereas the sound velocity of the ice is large. This asymptotic approach based on such assumptions allows us to find an ODE for the structure, where ice deformations are excluded. This equation describes (for a single mode approximation) a linear oscillators perturbed different nonlinear terms. These terms admit transparent physical interpretations and describe the following effects:

i the effect of water mass extrusion under the ice sheet action that leads to a friction, nonlinear effects and a time periodic forcing;

Effect **ii** essentially depends on the ice velocity V. For V this effect reinforces the resonance. Moreover, we obtained that increase of V can descrease the friction, which can become even negative, and it also decreases the effective structure mass (effect of negative "added mass"). These results are consistent with experimental data.

2 STATEMENT OF THE PROBLEM

Our model is defined by a system of two equations. Following [1, 2, 3], we consider the structure as a rigid body having a contact with the ice rod which is imaginary cut of the floe. The first equation describes an oscillator dynamics and has the form

$$q_{tt} + \Omega^2 q + \alpha q_t = \mu, \tag{1}$$

where $q = q(t)$ is a unknown function of time t, which defines the structure vibrations, Ω^2 is an oscillator natural frequency. The term αq_t with $\alpha > 0$ defines an internal structure friction.

The term μ expressed the force which is acting on the structure from the ice rod and has the form

$$\mu = \epsilon(u_x + \frac{u_x^2}{2})|_{x=q(t)}, \tag{2}$$

where $u = u(x, t)$ is a displacement of the ice rod, ϵ is a positive parameter, the terms u_x and u_x^2 define linear and nonlinear deformation effects, respectively.

The second equation describes the displacement $u(x,t)$ of the ice rod, which occupies the domain $I_q = \{x : q < x < L\}$. This equation is as follows:

$$u_{xx} + a_0 u_{xx} u_x - c_0^{-2} u_{tt} - \beta u_t - k_0 u = -\beta s_t - k_0 s, \tag{3}$$

where $u(x,t)$ is a unknown ice rod displacement, and β, a_0 and k_0 are positive parameters, which determine the internal friction and the ice rod compression by the neighboring ice rods , respectively. Here c_0 stands for the ice sound velocity. Below we set $c_0 = 1$ to simplify formulas, and in future dealing with dimensionless magnitudes.

The function $s(t)$ is function which describe the ice rod movement and we suppose that $s(t)$ is defined by

$$s(t) = s_0 - vt + \rho(t), \quad v > 0, \tag{4}$$

where s_0 is a constant, which defines an initial distance from the floe to the structure, and $\rho(t)$ is described the process of producing ice crashing peaces:

$$\rho(t) = \sum_{n=1}^{\infty} d_n H(t - t_n). \tag{5}$$

We set the following boundary conditions

$$u(q,t) = q(t), \quad u(L-q,t) = 0 \tag{6}$$

(we suppose that the process of the crashing is fast, and a contact between the ice rod and the oscillator always exist) and the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0. \tag{7}$$

3 EQUATION FOR OSCILLATOR VIBRATIONS

3.1 Asymptotics for ice displacement u

Here we express the displacement $u(x,t)$ via q and obtain an equation involving q only. We use the following asymptotic assumptions:

$$0 < \beta, k_0 << 1, \quad a_0 = O(1), \tag{8}$$

$$0 < \epsilon << \Omega << 1. \tag{9}$$

We suppose that the length L of the ice rod is large: $L >> 1$. The following assumption is important: $L^{-1/2}$ is much less than all the rest small parameters, i.e., we suppose $L^{-1/2} << \beta, k, \Omega$. Note that the contact problems are very complicated. Here, however, we can find an asymptotic solution. The main idea is as follows. We represent u as a sum $u = u^{(0)} + u^{(1)} + ...$, where the main term $u^{(0)}$ is the first approximation of the solution

$$\mathcal{L}u^0 = -\beta s - ks, \tag{10}$$

where $\mathcal{L}$ is the linear operator $\mathcal{L} = D_x^2 - D_t^2 - \beta D_t - k$, under the boundary conditions

$$u^{(0)}(q,t) = q(t), \quad u^{(0)}(L-q,t) = 0. \tag{11}$$

We introduce the variable $z = x - q + t$ and find the following approximating solution:

$$u^{(0)} = (V(t) - V(z) + x)\exp(-a(x-q)), \quad a > 0 \tag{12}$$

where $a >> L^{-1}$ and $a << \max\{\beta, \kappa, \Omega, \epsilon\}$ and $V(t)$ is a solution of the equation

$$V_{tt} + \beta V_t + k_0 V = \beta s_t + k_0 s. \tag{13}$$

We see that the boundary condition at $x = L$ is satisfied, up to exponentially small terms and eq. (10) is satisfied up to small terms of the order Ω. Below we omit sometimes the factor $\exp(-a(x-q)$ and the terms of the order a. For $u^{(1)}$ we obtain

$$u^{(1)}_{xx} - u^{(1)}_{tt} = \mathcal{L}u^{(0)} = g_1 + g_2 + g_3, \tag{14}$$

where

$$g_1 = -\beta V^{'}(z)(1-q_t) - k_0 V(z) + k_0 x,$$
$$g_2 = 2u^{(0)}_{qt} q_t + u^{(0)}_{qq} {q_t}^2 + u^{(0)}_q q_{tt}$$

and

$$g_3 = -a_0 u^{(0)}_{xx} u^{(0)}_x,$$

where $V^{'}(z) = dV/dz,\ V^{''}(z) = d^2V/dz^2$. We substitute (12) into expressions for g_k that gives

$$g_2 = \Big(2V^{''}(z)q_t - V^{''}(z)q_t^2 + V^{'}(z)q_{tt}\Big)\exp(-a(x-q)), \tag{15}$$

$$g_3 = a_0(1 - V^{'}(z))V^{''}(z)\exp(-a(x-q)). \tag{16}$$

The boundary conditions for $u^{(1)}$ have the form

$$u^{(1)}(q,t) = 0, \quad u^{(1)}(L-q,t) = 0. \tag{17}$$

From (14) one obtains

$$u^{(1)} = k_0\frac{x^3 - q^3}{3} + \frac{(x-q)}{2}W, \tag{18}$$

where $W = W_1(z) + W_2(x,q,t)$ and

$$W_1(z) = (V^{'}(z) - \frac{{V^{'}}^2}{2}(z)) + 2V^{'}(z)q_t + V(z)q_{tt} - V^{'}(z)q_t^2, \tag{19}$$

$$W_2(x,q,t) = -\beta V(z) + \beta V(z)q_t - k_0\int_0^x V(s-q+t)ds. \tag{20}$$

Solution (18) satisfies (14) up to small corrections of the order Ω and smaller ones.

Equation for u^2 involves a number of terms however we take into account only terms, which give contributions in friction. Then

$$u^{(2)}_{xx} - u_{tt} = -a_0(u^{(0)}_{xx}u^{(1)}_x + u^{(1)}_{xx}u^{(0)}_x) = a_0 q_t R(x,t), \tag{21}$$

where

$$R(x,t) = -2V^{''}(z) + 3V^{'}V^{''}(z) + O(x-q), \tag{22}$$

where $O(x-q)$ denotes terms proportional to $x-q$. Those terms do not give contributions into $u_x(q)$ and thus they are not essential in a resulting equation for the oscillator displacement q. For $u^{(2)}$ one has

$$u^{(2)} = a_0 q_t (x-q)\big(-2V^{'}(z) + \frac{3{V^{'}}^2(z)}{2} + O(x-q)\big)\exp(-a(x-q)). \tag{23}$$

3.2 Deformation at the edge of ice rod

The relations obtained in the previous subsection imply that the deformation consists of the three main contributions:

$$u_x(q) = F_0(t) + q_t F_1(t) + q_{tt} F_2(t) + F_3(t) q_t^2 + kq^2 = S(t, q, q_t), \tag{24}$$

where

$$F_0(t) = 1 - V^{'}(t) + (\frac{V^{'}(t)}{2} - \frac{V^{'}(t)}{4}) - \beta V^{'}(t) - k_0 \int_0^q V(s - q + t) ds + O(a), \tag{25}$$

$$F_1(t) = 2V^{'}(t) - a_0(- 2V^{'}(t) + \frac{3V^{'2}(t)}{2}) + \beta V(t), \tag{26}$$

$$F_2(t) = V(t), \quad F_3(t) = -V^{'}(t), \tag{27}$$

These terms play quite different roles as we will see in coming sections. The main instability effect is induced by $q_t F_1(t)$. The term $q_{tt} F_2(t)$ defines an adjoint mass of the platform.

3.3 Nonlinear equation for q

Using relations (24), (25), (26), and (27), one obtains the following equation for q:

$$q_{tt} + \Omega^2 q + \alpha q_t = \epsilon \xi(q, q_t), \tag{28}$$

where ξ is defined by

$$\xi = S(t, q, q_t) + \frac{S(t, q, q_t)^2}{2} + higher\ order\ corrections. \tag{29}$$

After some transformations, we have

$$\xi = f_0 + f_1 q_t + f_2 q_{tt} + f_3 q^2 + f_4 q_t^2 + f_5 q_t q_{tt} + f_6 q_{tt}^2 + f_7 q_t^3 + f_8 q^2 q_{tt} + f_9 q_t^2 q_{tt} + f_{10} q^2 q_t + f_{11} q^4, \tag{30}$$

where

$$f_0 = F_0 + F_0^2, \quad f_1 = F_1 + 2F_0 F_1, \tag{31}$$

$$f_2 = V, \quad f_7 = 2F_1 F_3, \quad f_8 = 2kF_2, \tag{32}$$

$$f_9 = 2F_3 F_2, \quad f_{10} = 2kF_1. \tag{33}$$

In this equation the ice rod displacement is excluded. This main equation for structure vibrations describes a weakly perturbed linear oscillator with the non-perturbed frequency Ω^2. The perturbation involves many different contributions. First, here a term $F(T)$ which can be interpreted as an external force induced by the ice rod, which does not depend on q. For $F(t)$ one has a rough asymptotics

$$\xi_{ext} = F_0(t) \approx 1 - V^{'}(t) + O(\beta) + O(k_0).$$

Moreover, we have nonlinear terms depending on q_{tt} only and which can describe a shift of frequency (that can be interpreted as an adjoint mass). At last there are contributions associated with a linear and a nonlinear friction, namely

$$\xi_{Lf} = q_t f_1, \tag{34}$$

$$\xi_{Nf} = f_{10} q^2 q_t \tag{35}$$

The nonlinear friction is much weaker than the linear one since it involves a small quantity q^2.

Below we discuss possible mechanisms of the structure instability.

3.4 Computation of V

To simplify computations, let us assume that $s(t)$ is a periodic function with the period $T = d/v$ for the case $d_n = d$. To compute V, involved in all terms above, we solve eq. (13) for V by the Fourier expansion. Computing the Fourier coefficients, we have

$$s(t) = \bar{s} + \sum_{n \in Z} \hat{s}_n \exp(i2\pi nt/T), \quad T = d/v, \tag{36}$$

where T is the period and

$$\hat{s}_n = \frac{iT}{2\pi n}. \tag{37}$$

Therefore,

$$V(t) = \bar{s} + \sum_{n \in Z} \hat{V}_n \exp(i2\pi nt/T) + \tilde{V}(t), \tag{38}$$

where

$$\tilde{V}(t) = (V(0)\cos(\omega_0 t) + V'(0)\sin(\omega_0 t))\exp(-\frac{\beta t}{2}),$$
$$\hat{V}_n = (\frac{ik_0 T}{2\pi n} - \beta)\big(-(2\pi n/T)^2 + k_0 + i2\pi\beta n/T\big)^{-1}, \tag{39}$$

and $\bar{s}$ is a mean shift of the platform center. Moreover,

$$V^{'}(t) = \sum_{n \in Z}(i2\pi n/T)\hat{V}_n \exp(i2\pi nt/T), \tag{40}$$

$$\int_0^x V(s-q+t)ds = \sum_{n \in Z}(2i\pi n/T)^{-1}\hat{V}_n \exp(i2\pi nt/T)\big(\exp(i2\pi n(x-q)/T) - \exp(i2\pi n(x-q)/T)\big). \tag{41}$$

Note that the term $\tilde{V}(t)$ is exponentially decreasing in t and therefore we can remove that term for large times $t >> \beta^{-1}$.

4 ASYMPTOTIC ANALYSIS OF NONLINEAR EQUATION FOR q

Under some assumptions we can consider (28) as an equation that describes a weakly nonlinear oscillator with a weak damping. By this equation, we can describe the mechamisms of instability.

The asymptotic approach to study such equations is well known, see [4]. Let $\tau = \epsilon t$ is a slow time. We assume that $\alpha = \epsilon\bar{\alpha}$, where $\bar{\alpha} < 1$. We seek solutions in the form

$$q = A(\tau)\sin(\Omega t + \phi(\tau)) + \epsilon q_1(t, \tau) + ..., \tag{42}$$

where A and ϕ are unknown slowly varying in time the amplitude and the phase,respectively. We have

$$q_{tt} = -\Omega^2 q + 2\epsilon\Omega(A_\tau \cos(\Omega t + \phi(\tau)) - A\phi_\tau \sin(\Omega t + \phi(\tau)) + O(\epsilon^2).$$

For any smooth function $H(q, q_t)$

$$H(q, q_t) = H_0 + O(\epsilon), \quad H_0 = g(A\omega\cos(\Omega t + \phi(\tau)), A\sin(\Omega t + \phi(\tau))).$$

Using these relations, for q_1 one has

$$q_{1tt} + \Omega^2 q_1 = S(t, \tau, \epsilon), \tag{43}$$

where

$$S(t,\tau) = 2\Omega(-A_\tau \cos(\Omega t + \phi(\tau)) + A\phi_\tau \sin(\Omega t + \phi(\tau)) + R(A,\phi,t), \tag{44}$$

and

$$R(A,\phi,t) = xi(A\sin(\Omega t + \phi(\tau)), A\Omega\cos(\Omega t + \phi), t) - \bar{\alpha} A\Omega \cos(\Omega t + \phi). \tag{45}$$

For large times $t = O(\epsilon^{-1})$ equation (43) has a bounded solution if and only if

$$\lim_{T\to+\infty} T^{-1} \int_0^T S(t,\tau)\cos(\Omega t + \phi)dt = 0, \tag{46}$$

and

$$\lim_{T\to+\infty} T^{-1} \int_0^T S(t,\tau)\sin(\Omega t + \phi)dt = 0. \tag{47}$$

Finally, by (44) and (45) these relations lead to the following system of equations for the amplitude A and the phase ϕ:

$$\Omega A_\tau = \lim_{T\to+\infty} T^{-1} \int_0^T R(A,\phi,t)\cos(\Omega t + \phi)dt, \tag{48}$$

and

$$\Omega A\phi_\tau = -\lim_{T\to+\infty} T^{-1} \int_0^T R(A,\phi,t)\sin(\Omega t + \phi)dt = 0. \tag{49}$$

We investigate this system in the next section. To simplify formulas we will use notation

$$\langle f \rangle = -\lim_{T\to+\infty} T^{-1} \int_0^T f(t)dt \tag{50}$$

for averaged quantities.

5 AMPLITUDE EVOLUTION

Let us focus our attention on the equation for amplitude A. Assuming that $A << 1$, and therefore $|q| << 1$, we obtain then that (48) reduces to

$$\Omega \frac{dA}{d\tau} = D_1 A + D_2 A^2 + D_3 A^3 + D_4 A^4 + \bar{a}, \tag{51}$$

where $D_l, \bar{a}$ are coefficients depending on the system parameters. We compute those coefficients using relations for ξ obtained above. In particular, one has

$$D_1 = -\bar{\alpha} + \langle f_1(t)\cos^2(\Omega t)\rangle, \tag{52}$$

$$D_3 = \langle f_7(t)\Omega^3\cos^4(\Omega t) - f_8(t)\Omega^2\sin^3(\Omega t)\cos(\Omega t) + \Omega^4 f_9(t)\cos^3(\Omega)\sin(\Omega t) + f_{10} sin^2(\Omega t)\cos^2(\Omega t)\rangle, \tag{53}$$

and

$$\bar{a} = \langle f_0(t)\cos^2(\Omega t)\rangle. \tag{54}$$

Assume that resonances are absent, i.e., for all positive integers m we have

$$\Omega \neq m\omega_{ice}, \quad \omega_{ice} = 2\pi/T. \tag{55}$$

Then our expressions for D_i can be simplified. We obtain that $D_2, D_4, \bar{a} \approx 0$ and

$$D_1 = -\bar{\alpha} + \frac{1}{2}\langle f_1(t)\rangle, \tag{56}$$

Figure 1: The dependence of oscillator amplitude on the ice speed v.

$$D_3 = \frac{3\Omega^3}{8}\langle f_7(t)\rangle + \frac{\Omega}{8}\langle f_{10}\rangle. \tag{57}$$

As a result, we have

$$\Omega\frac{dA}{d\tau} = D_1 A - D_3 A^3 + \ smaller \quad terms. \tag{58}$$

Note that D_1, D_3 depend on the ice floe speed v. Moreover, equation (58) is fundamental in the phase transition theory, it is the famous Landau model.

Equation (58) can describe bifurcations and thus an instability onset. To see it, we assume that $D_3 > 0$ (this assumption will be checked in the next section). Let $D_1 < 0$. Then we have a single equilibrium close to 0. This solution corresponds to an oscillation regime of a small amplitude. For $D_1 > 0$ we have the oscillations of not small amplitudes.

The numerical computations show that such transition exists due to the fact that $D_1(v)$ changes its sign as the ice speed v increases.

6 INSTABILITY

Let us compute coefficients D_1 and D_3. One has

$$D_3 = \frac{3\Omega^3}{4}\langle F_1 F_3\rangle + \frac{\Omega k_0}{8}\langle F_1\rangle. \tag{59}$$

Using the expressions for F_1, F_3 we obtain

$$D_3 = \frac{3\Omega^3}{4}((\langle 2V^{'}(1-a_0) - \frac{3a_0}{2}V^{'2}\rangle) + \frac{3a_0\Omega k_0}{8}\langle V^{'2}\rangle. \tag{60}$$

Note that $\langle V^{'}\rangle = 0$ and

$$\langle V^{'2}\rangle = \sum_{n=1}^{\infty}(2\pi n/T)^2|\hat{V}_n|^2. \tag{61}$$

Therefore, finally one has

$$D_3 = \frac{a_0\Omega\langle V^{'2}\rangle}{4}(k_0 + 6(1-a_0)\Omega^2), \tag{62}$$

where

$$\langle V^{'2}\rangle = \sum_{n=1}^{\infty}\frac{(k_0^2 + \beta^2 z_n^2}{((z_n^2 - k_0)^2 + \beta^2 z_n^2}, \quad z_n = 2\pi n/T. \tag{63}$$

For D_1 one finds the following relation:

$$D_1 = \frac{(3a_0 - 8)\Omega\langle V^{'2}\rangle}{2} - \alpha\epsilon^{-1}. \tag{64}$$

One can show that $D_1 < 0$ for small v but it may be positive for some v under appropriate parameter choice. For $v >> 1$ the coefficient D_1 is again negative.

7 CONCLUSIONS

The main result of the investigation is that the ice rod-structure interaction may produce a negative input into the friction. These “negative “ friction lead to instability of the oscillator vibrations regime with a sharp increase of the amplitude, which then suddenly drops when the velocity of the ice rod reach some critical value. That mechanism, in our opinion, described the lock-in regime of the oscillator vibrations, when oscillator vibrate on a frequency which is close to its natural frequency, and which exists in some ice rod velocity interval.

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