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# APPLICATION OF THE WAVE FINITE ELEMENT APPROACH TO THE STRUCTURAL FREQUENCY RESPONSE OF STIFFENED STRUCTURES

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The present work shows many aspects concerning the use of the wave methodology for the response computation of periodic structures, through the use of substructures and single cells. Applying Floquet principle, continuity of displacements and equilibrium of forces at the interface, an eigenvalue problem, whose solutions are the waves propagation constants and wavemodes, is defined. With the use of single cells, thus a double periodicity, the dispersion curves of the waveguide under investigation are obtained and a validation of the results is performed with analytic ones, both for isotropic and composite material. Two different approaches are presented, instead, for computing the forced response of stiffened structures, through substructures of the whole periodic structure. The first one, dealing with the condensed-to-boundaries dynamic stiffness matrix, proved to drastically reduce the problem size in terms of degrees of freedom, with respect to more mature techniques such as the classic FEM. Moreover it proved to be the most controllable one. The other approach presented deals with waves propagation and reflection in the structure. However it suffers more numerical conditioning and requires a proper choice of the reflection matrices to the boundaries, which has been one of the most delicate passages of the whole work, as the effects of the direct excitation. However this last approach can deal with the response and loads applied on any inner point. The results show a good agreement with numerical classic-FEM except for damping needed to be trimmed for perfect agreement . The drastic reduction of DoF is evident, even more when the number of repetitive substructures is high and the substructures itself is modelled in order to get the lowest number of DoF at the boundaries.

## 1 Introduction

One of the most used and appealing methods for solving problem concerning the dynamics of continuous structures is the finite element method (FEM). Typically used for modal/dynamic-response applications, this method enhances the operator to obtain information about the vibrational level from the model of the whole structure in every frequency range. However, in many engineering applications, high frequency vibrations become significant, in particular where sound transmission and loading have to be considered, such as in the cases, for example, of the transmission loss


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evaluation of a panel or structure response due to aeroacoustic excitation produced by a launcher on its own structure.

The continous research on new and better performing techniques, in therms of computational cost, is addressed to the computational cost itself getting soon unaffodable, as soon as the frequency of analysis increases, even with modern CPU power.

Moreover, the development of new models is complicated because of the wide frequency range of interest. In the Low Frequency range the response exhibits isolated modal resonances and it has local characteristics, like spatial response peculiarities or constraint effects on the global response. In this case the deterministic techniques is still the preferred one. At High Frequencies the response is diffuse and does not present specific resonances. The averaged response in any point is all we need to describe it. On the other hand the Mid Frequencies are a transition zone, for which well-established prediction techniques are not yet available.

The Wave Finite Element Method, which candidates itself to be a feasible approach to overcome such issues, uses the Periodic Structure Theory [1] to re-built a spatially distrubuted response with the analysis of a single element or cell.

Periodic structures, very common in engeneering fields, are supposed to be constituted by a set of identical elementary cell, repeated along one or two directions. The Wave and Finite Element Method allows to investigate only one cell of the structure, modelled through FE.

Once the cell characteristic are obtained, the entire method is based on a post-processing which leads, to one or more eigenvalue problems.

The research in this peculiar field is very intense nowadays. The potentiality of WFE is thrilling and it may give to modern industry a toll which could enable to overcome actual bareers of computational cost.

# 2 Theoretical Background

A wave can be described as a disturbance that travels through a medium, transporting energy (or informations) from its source location to others, without matter flux. A medium is intended as a material that carries the wave, and it should be considered as a collection of particles capable to interact with each other [5].

## 2.1 Bloch-Floquet Principle

A plane free wave that propagates along a structure is assumed to take the form of a Bloch wave. Bloch's theorem represents a generalisation, in solid–state physics, of the Floquet's theorem for 1–dimensional problem. A Bloch wave is a wavefunction valid for a particle in a periodic media, whose wavebasis is of the type of eq.1. Bloch showed how electron wave functions in a crystals have a basis consisting entirely of Bloch wave energy eigenstates. This reflect in periodic structures to exhibit pass–bands and stop–bands, in that each disturbance can propagate freely only in specific frequency ranges, otherwise they decay with distance [4]. Generally the number of propagation surfaces equals the number of degrees of freedom at each node of the structure considered. However some so called "branches" are an artifact of the FE discretization, but this will be investigated later.

$$\{\psi_r\} = e^{-i\beta r}\{u_r\} \tag{1}$$

Where r is the distance vector, $\beta$ is the propagation constant, $u_r$ is a spatially periodic function.

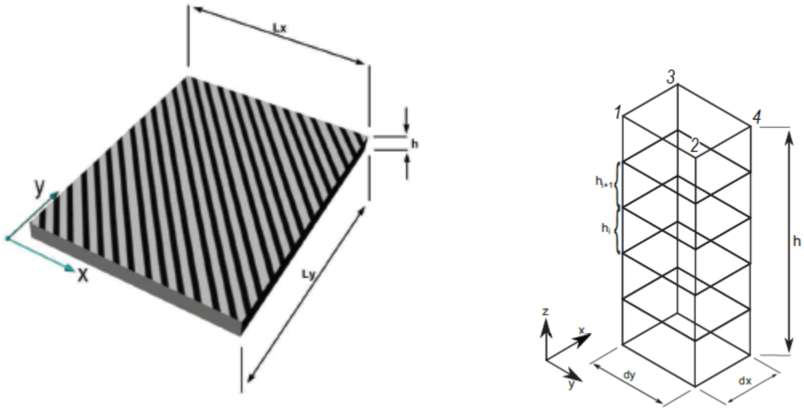


**Figure 1** Rectangular Panel and its modelled periodic cell

The field at a certain point is related to that at distance S by a complex function, frequency dependent. In an infinite periodic media the fields at $w$ and $w + S$ must differ only by a constant attenuation and phase shift [11].

Lets consider a rectangular panel [Fig.1] with $Lx$, $Ly$ its dimensions and $h$ its thickness. A periodic segment of the panel with dimensions $dx$ and $dy$ is modelled using FE. No displacement field is supposed, as in SFEM, and the cell matrix can be obtained through extraction from common FE commercial software. The entries for each Degree of Freedom (DoF), of every node laying on the same edge of the segment, say edges 1, 2, 3 and 4, are placed in the mass and stiffness matrices so that the vector of displacements can be written as [9]: $\{q\} = \{q1 \quad q2 \quad q3 \quad q4\}$.

The time-harmonic equation of motion of the segment assuming uniform and structural damping for all the DoF can be written as

$$[K(1 - \eta i) - \omega^2 M]\{q\} = \{F\} \tag{2}$$

Where $\eta$ is the structural damping coefficient, $\omega$ is the angular frequency and F the vector of the nodal forces [9]. It is important to remind that damping can be introduced also with a specific matrix, not merely as structural one. Different damping model can be used, but then, the dynamic stiffness matrix, can always be written as:

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

So using the Floquet theory, or Bloch's theorem, for a rectangular segment and assuming a time-harmonic response, the displacements and forces of each edge can be written, as before, as a function of a single edge/corner displacement and forces [9]. Taking edge 1 (see Fig. 1) as the edge of reference we have:

$$\{q_2\} = \lambda_X\{q_1\} \qquad \{q_3\} = \lambda_Y\{q_1\} \qquad \{q_4\} = \lambda_X\lambda_Y\{q_1\} \tag{3}$$

$$\{F_2\} = \lambda_X\{F_1\} \qquad \{F_3\} = \lambda_Y\{F_1\} \qquad \{F_4\} = \lambda_X\lambda_Y\{F_1\} \tag{4}$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \mathbf{I} \\ \lambda_X \mathbf{I} \\ \lambda_Y \mathbf{I} \\ \lambda_X \lambda_Y \mathbf{I} \end{Bmatrix} \{q_1\}$$

Since equilibrium requires that the sum of the nodal forces at each node is zero, we have

$$\left\{\mathbf{I} \quad \lambda_X^{-1}\mathbf{I} \quad \lambda_Y^{-1}\mathbf{I} \quad \lambda_X^{-1}\lambda_Y^{-1}\mathbf{I}\right\} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \mathbf{0}$$

With $\lambda_j = e^{-i\mu_j \Delta_j}$, where $\mu$ is the propagation constant and $\Delta_j$ the distance in "j" direction between reference an target cell.

### 2.2 Periodicity Conditions: Single Cell Scale

The approach which will be presented is often used for the analysis of dispersion properties of cells, as will be in next sections. As for classic FEM, the dynamic stiffness matrix of eq.2 is a NxN matrix, were N is the number of the total DOF of the cell. Substituting periodicity conditions in eq.2, we obtain a reduced matrix of size "n", equal to the reference side, in this case corner "1".

$$\left\{\mathbf{I} \quad \lambda_X^{-1}\mathbf{I} \quad \lambda_Y^{-1}\mathbf{I} \quad \lambda_X^{-1}\lambda_Y^{-1}\mathbf{I}\right\} \mathbf{D} \begin{Bmatrix} \mathbf{I} \\ \lambda_X \mathbf{I} \\ \lambda_Y \mathbf{I} \\ \lambda_X \lambda_Y \mathbf{I} \end{Bmatrix} \{q_1\} = \mathbf{0}$$

$$[\mathbf{K}_r - \omega^2 \mathbf{M}_r]\{q\} = \mathbf{D}_r\{q\} = \mathbf{0} \tag{5}$$

### 2.3 The Eigenvalue Problem

Using eq.5 and exploiting all the therms, it is possible to write the problem in this form:

$$\begin{aligned} [(\mathbf{D}_{11} + \mathbf{D}_{22} + \mathbf{D}_{33} + \mathbf{D}_{44})\lambda_X\lambda_Y + \\ (\mathbf{D}_{12} + \mathbf{D}_{34})\lambda_X^2\lambda_Y + \\ (\mathbf{D}_{13} + \mathbf{D}_{24})\lambda_X\lambda_Y^2 + \mathbf{D}_{32}\lambda_X^2 + \mathbf{D}_{23}\lambda_Y^2 + \\ (\mathbf{D}_{21} + \mathbf{D}_{43})\lambda_Y + (\mathbf{D}_{31} + \mathbf{D}_{42})\lambda_X + \\ \mathbf{D}_{14}\lambda_X^2\lambda_Y^2 + \mathbf{D}_{41}] \quad \{q\} = \mathbf{0} \end{aligned} \tag{6}$$

Considering the transpose of equation 6 divided by $\lambda_Y\lambda_X$, it can be proved that the solutions come in pairs involving $(\lambda_Y, \lambda_X)$ and $(\lambda_Y^{-1}, \lambda_X^{-1})$ for a given real frequency $\omega$. These of course represent the same disturbance propagating in the four directions $\pm\vartheta$, $\pi \pm \vartheta$ [4].

Equations 6 gives eigenproblems relating $\lambda_Y$, $\lambda_X$ and $\omega$, whose solutions give FE estimates of the wave modes and dispersion relations for the continuous structure. Three different algebraic eigenvalue problems are possible since the problem is three parametric. If $(\mu_x)$ and $(\mu_x)$ are assigned and real, a linear eigenvalue problem results in $\omega$ for propagating waves. If the frequency

$\omega$ and wavenumber are given, the other one is extracted. This is the case of an incident wave on a boundary, whose trace wavenumber is known. In this case the problem becomes a quadratic eigenproblem with twice the solutions.

When the frequency and $\vartheta$ are prescribed the problem might become a trascendental eigenvalue problem [4].

### 2.4 Periodicity Conditions: Substructure Scale

The description of waves travelling in a periodic structure is achieved by considering the eigenproblem which derives from imposing the Floquet's conditions to a cell or a substructure, whose stiffness and mass matrices can be gathered from classic FE models.

Starting from the DSM problem of a partial assemble of cells, reordering the DoFs so that internal, left and right nodes's degrees of freedom are properly separated in the following way, we can move to the transfer matrix.

$$\begin{bmatrix} \mathbf{D}_{ll} & \mathbf{D}_{lr} & \mathbf{D}_{li} \\ \mathbf{D}_{rl} & \mathbf{D}_{rr} & \mathbf{D}_{ri} \\ \mathbf{D}_{il} & \mathbf{D}_{ir} & \mathbf{D}_{ii} \end{bmatrix} \begin{Bmatrix} q_L \\ q_R \\ q_I \end{Bmatrix} = \begin{Bmatrix} F_L \\ F_R \\ 0 \end{Bmatrix}$$

Eliminating internal degrees of freedom (I) we get our base-work equation, as previously described.

$$\begin{bmatrix} \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_L \\ \mathbf{q}_R \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_L \\ \mathbf{F}_R \end{Bmatrix}$$

With:

$$\mathbf{D}_{BB} = \mathbf{D}_{BB} - \mathbf{D}_{Bi}\mathbf{D}_{ii}^{-1}\mathbf{D}_{iB} \tag{7}$$

Where $B$ stands for cell boundary, left or rigth.

Imposing continuity of displacements and equilibrium of forces at the interface between adjacent cell S and (S+1), and putting all in a matrix form we can get the Transfer Matrix, **T**. It relates the nodal displacements and forces (evaluated on the left side) between two adjacent substructures.

$$\begin{bmatrix} \mathbf{T} \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & \mathbf{D}_{LR}^{-1} \\ -\mathbf{D}_{RL} + \mathbf{D}_{RR}\mathbf{D}_{LR}^{-1}\mathbf{D}_{LL} & -\mathbf{D}_{RR}\mathbf{D}_{LR}^{-1} \end{bmatrix}$$

It has been shown [12] that the eigenvalues of the transfer matrix occur in reciprocal pairs as , $\lambda_j^+ = 1/\ \lambda_j^-$ corresponding to pairs of positive (+) and negative (-) going waves, respectively. Associated with these eigenvalues are the positive and negative going right eigenvectors $\phi_j^-$ and $\phi_j^+$ respectively, which will also be referred to as wavemodes [12]. Every wavemode can be partitioned into a sub-vector of DoFs and internal forces/moments.

Positive waves are characterized by $|\lambda_j^+| < 1$, which means that if the wave propagates its amplitude must decrease in travelling. If $|\lambda_j^+| = 1$ then the time average power transmission in the positive direction is evaluated to select the positive and negative going waves[12]. With that, one can group the wavemodes as

$$\Phi_q^+ = \begin{bmatrix} [\phi_{1,q}^+] & [\phi_{n,q}^+] \end{bmatrix} \qquad \Psi_q^+ = \begin{bmatrix} [\psi_{1,q}^+]^T & [\psi_{n,q}^+]^T \end{bmatrix} \tag{8}$$

A transformation between the physical domain, where the system's behaviour is described in terms of $q$ and $f$ , and the wave domain, where the behaviour is described in terms of waves of amplitudes $Q^+$ ans $Q^-$ travelling in the positive and negative directions is derived through these matrices [12].

Lets now consider a structure made by $N$ substructures assembled with a 1D periodicity and subjected to arbitrary boundary conditions on its left and right ends. Expanding the vectors of displacements/rotations and forces/moments, on the left or right boundary of a substructure $s$, using wave bases [10], we get:

$$\begin{aligned} \mathbf{q}_L^s &= \Phi_q^+ \mu^{s-1} \mathbf{Q}^+ + \Phi_q^- \mu^{N+1-s} \mathbf{Q}^- \\ -\mathbf{f}_L^s &= \Phi_f^+ \mu^{s-1} \mathbf{Q}^+ + \Phi_f^- \mu^{N+1-s} \mathbf{Q}^- \end{aligned} \tag{9}$$

Where $\mathbf{Q}$ are the wave amplitudes at the left and right ends of the whole periodic structure, and $\mu$ is a diagonal matrix of the positive propagating eigenvalues.

## 2.5 Condensed Dynamic Stiffness Matrix of a Periodic Structure

Denoting $D_s$ as the dynamic stiffness matrix of the periodic structure which is condensed on its left and the right ends, the dynamic equilibrium equation of the periodic structure can be written as

$$\mathbf{D}_s \begin{Bmatrix} \mathbf{q}_L^1 \\ \mathbf{q}_R^N \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_L^1 \\ \mathbf{F}_R^N \end{Bmatrix}$$

Considering Eq.9, the wave-base expansion can be re-formulated in matrix form.

$$\begin{Bmatrix} \mathbf{q}_L^1 \\ \mathbf{q}_R^N \end{Bmatrix} = \begin{bmatrix} \Phi_q^+ & \Phi_q^- \mu^N \\ \Phi_q^+ \mu^N & \Phi_q^- \end{bmatrix} \begin{Bmatrix} \mathbf{Q}^+ \\ \mathbf{Q}^- \end{Bmatrix} \qquad \begin{Bmatrix} \mathbf{F}_L^1 \\ \mathbf{F}_R^N \end{Bmatrix} = \begin{bmatrix} -\Phi_f^+ & -\Phi_f^- \mu^N \\ \Phi_f^+ \mu^N & \Phi_f^- \end{bmatrix} \begin{Bmatrix} \mathbf{Q}^+ \\ \mathbf{Q}^- \end{Bmatrix}$$

To derive the condensed dynamic stiffness matrix $D_s$ of the periodic structure. As it turns out, the vectors of wave amplitudes $\mathbf{Q}$ can be expressed in this form:

$$\begin{Bmatrix} \mathbf{Q}^+ \\ \mathbf{Q}^- \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & (\Phi_q^+)^{-1} \Phi_q^- \mu^N \\ (\Phi_q^-)^{-1} \Phi_q^+ \mu^N & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} (\Phi_q^+)^{-1} & \mathbf{0} \\ \mathbf{0} & (\Phi_q^-)^{-1} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_L^1 \\ \mathbf{q}_R^N \end{Bmatrix}$$

The version perorted is the one of Silva, [10], which proposes to make left multiply to avoid ill-conditioning of matrices, as explained by Mencik [16].

Finally the condensed dynamic stiffness matrix of the whole structure can be derived combining previous equations.

$$\mathbf{D}_s = \begin{bmatrix} -\Phi_f^+ & -\Phi_f^- \mu^N \\ \Phi_f^+ \mu^N & \Phi_f^- \end{bmatrix} \begin{bmatrix} \mathbf{I} & (\Phi_q^+)^{-1} \Phi_q^- \mu^N \\ (\Phi_q^-)^{-1} \Phi_q^+ \mu^N & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} (\Phi_q^+)^{-1} & \mathbf{0} \\ \mathbf{0} & (\Phi_q^-)^{-1} \end{bmatrix}$$

## 2.6 The Model Reduction

The proper selection of propagating waves can be done considering the imaginary part of the propagating contant of each wavemode. The criteria used are to be chosen case by case. In the present work the criteria used can be summarized in the following way, considering $j$ as a tolerance parameter:

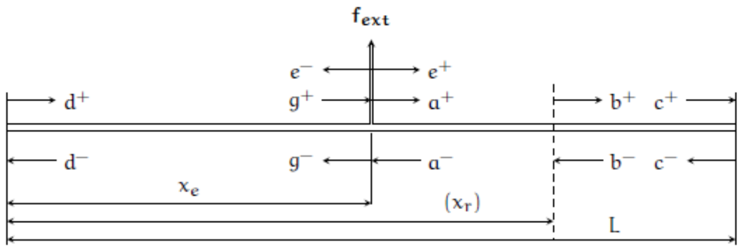


**Figure 2** Waves in a finite structure excited by point load [5]

$$imag|k_i L_i| < j \tag{10}$$

Physically this means considering the contribution, to structural response, of the only propagating or close-to propagating waves. At each frequency step, when and if new wavemodes cut-on, these are included in the propagating matrix.

In this case the problem has to take a different form since most of the matrices we were inverting before suddenly became rectangular. This is the case of the $\Phi_{q,f}$ which are now $DoF\ x\ m$, where $m$ is the number of selected wavemodes.

The boundary conditions can then be reformulated:

$$\begin{bmatrix} \mathbf{I} & (\Phi s_f^+)^P \Phi s_f^- \mu_s^N \\ (\Phi s_q^-)^P \Phi s_q^+ \mu_s^N & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mu_s^{-N} \end{bmatrix} \begin{Bmatrix} \mathbf{Q}_1^+ \\ \mathbf{Q}_1^- \end{Bmatrix} = \begin{Bmatrix} -(\Phi s_f^+)^P \mathbf{F}_0 \\ (\Phi s_q^-)^P \mathbf{q}_0 \end{Bmatrix}$$

Where the sign $P$ above the matrices indicates the pseudo-inverse. This is the case of a Neumann-to-Dirichlet problem.

# 3 Forced Vibrations for Inner Points Excitation

The following section will deal with a variant of the previous approach which can be more flexible in dealing with the inner-point excitation. We can write the continuity of displacements and the equilibrium of the force at the excitation point using the wave-base expansion, from which we can rewrite the equilibrium equations in matrix form.

$$\begin{bmatrix} \Phi s_q^+ & -\Phi s_q^- \\ \Phi s_f^+ & -\Phi s_f^- \end{bmatrix} \begin{Bmatrix} e^+ \\ e^- \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ f_{ext} \end{Bmatrix}$$

The inversion of the above left-hand side matrix can lead to numerical errors since it could be ill-conditioned, especially for complicated structures. A solution to avoid this kind of problem is exploiting the orthogonality of the left and right eigenvectors, premultiplying by the matrix of left eigenvectors properly rearranged to get to the following form [5] [14].

$$\begin{Bmatrix} e^+ \\ e^- \end{Bmatrix} = \begin{Bmatrix} \Psi s_f^+ f_{ext} \\ -\Psi s_f^- f_{ext} \end{Bmatrix}$$

This, of course, requires a left-eigenvalue problem to be solved too.

The wave amplitudes $e^{-,+}$ are the amplitudes of the directly excited waves. These move in both the sides of the now uni-dimensionalised problem. Fig 2 can be used as illustrative reference.

### 3.1 Waves at Boundaries

Waves incident upon discontinuities and boundaries are partially reflected, transmitted and absorbed. Instead, in the case of elastic boundary conditions, an incident wave is only reflected, without any transmission [5].

Considering a generic wave of amplitude $\mathbf{h}^{+}$ travelling in the medium, we can model the reflection and the subsequent opposite-going wave amplitude, $\mathbf{h}^{-}$, with the use of reflection matrices at boundaries.

Considering also $\mathbf{R}$ as the matrix of reflection coefficients, which depends on the type of constraint, the wave problem at the boundaries can be expressed as $\mathbf{h}^{-} = \mathbf{R}\mathbf{h}^{-}$. Each boundary condition can always be expressed in the form [5]: $\mathbf{Af} + \mathbf{Bq} = \mathbf{0}$.

Substituting the wave base expansion for forces and displacements:

$$
\begin{aligned}
\mathbf{R}_{right} &= -(\mathbf{A}\Phi s_f^{-} + \mathbf{B}\Phi s_q^{-})^{-1}(\mathbf{A}\Phi s_f^{+} + \mathbf{B}\Phi s_q^{+}) \\
\mathbf{R}_{left} &= -(\mathbf{A}\Phi s_f^{+} + \mathbf{B}\Phi s_q^{+})^{-1}(\mathbf{A}\Phi s_f^{-} + \mathbf{B}\Phi s_q^{-})
\end{aligned}
\tag{11}
$$

Where the matrices $\mathbf{A}$ and $\mathbf{B}$ are dependent on the type of constrain, as said. In the case of force-free boundaries, for example, $\mathbf{A} = I$ and $\mathbf{B} = 0$.

### 3.2 Waves Propagation

Moving in the medium, the amplitude of all the waves changes, depending on the distance and the wave characteristics itself.

Their variations can be derived by applying the definition of propagation constant. For instance, if the waveguide has $n$ wave components, the waves amplitudes at two points a distance x apart are given by [5]: $\mathbf{h}^{+} = \mathbf{Tr}\ \mathbf{s}^{+}$, where $\mathbf{Tr}$ is the wave propagation matrix. It can be expressed as:

$$
\mathbf{Tr}(x) = diag\left(e^{-ik_1 x}, e^{-ik_2 x}, \ldots, e^{-ik_n x}\right) \tag{12}
$$

All the elements of the wave propagation matrix have a magnitude less or equal to the unity, by definition.

### 3.3 Waves Superposition for Response Computation

Once the amplitudes of directly excited are known, we can calculate the waves amplitudes at a given response point by considering the excitation, reflection and propagation relations.

Using again Fig 2 as reference we can evaluate the amplitude of waves in the reference point. Using the reflection relations at the boundaries and considering the propagating matrices, then, the wave amplitudes at excitation point are ( [5] is suggested for more detailed passages):

$$
a_{+} = [\mathbf{I} - \mathbf{Tr}(x_{exc})\mathbf{R}_{left}\mathbf{Tr}(L)\mathbf{R}_{right}\mathbf{Tr}(L - x_{exc})]^{-1}[e_{+} + \mathbf{Tr}(x_{exc})\mathbf{R}_{left}\mathbf{Tr}(x_{exc})e_{-}] \tag{13}
$$

$$
a_{-} = [\mathbf{I} - \mathbf{Tr}(L - x_{exc})\mathbf{R}_{right}\mathbf{Tr}(L)\mathbf{R}_{left}\mathbf{Tr}(x_{exc})]^{-1}[e_{-} + \mathbf{Tr}(L - x_{exc})\mathbf{R}_{right}\mathbf{Tr}(L - x_{exc})e_{+}] - e_{-} \tag{14}
$$

The adopted approach allows numerical stability. In fact, the above solutions are well-conditioned because the matrices being inverted are diagonally dominant and the element of the wave propagation matrices are less than or equal to the unity [5].

The response in the reception point can be then calculated applying the propagation relations. For example, if the response point is over the excitation one:

$$
\begin{aligned}
Q_+ &= \mathbf{Tr}(x_{resp} - x_{exc})a_+ \\
Q_- &= \mathbf{Tr}(L - x_{resp})\mathbf{R}_{right}\mathbf{Tr}(L - x_{resp})Q_+
\end{aligned}
\tag{15}
$$

# 4 Numerical Models

## 4.1 Dispersion curves for 2D waveguides

The eigenvalue problem, as said, is a three-parametric one. Main interest has been placed on quadratic eigenvalue problem, especially for the cases of curved waveguides. In this kind of eigenproblem, in fact, one wavenumber, i.e. in $x$ direction, is given. In the case of cylinders we also know the wavenumber around the circumference. Hence equation 6 becomes a quadratic eigenvalue problem in $k_y$, for which there are 2n solutions:

$$
[\mathbf{A}_2\lambda_Y^2 + \mathbf{A}_1\lambda_Y + \mathbf{A}_0] \quad \{\mathbf{q}_1\} = \mathbf{0} \tag{16}
$$

The WFE method can be applied also to curved and doubly curved waveguides. A small rectangular shell or solid element is taken and, as shown in Fig. 3, it is supposed to subtend a small angle $L_\alpha$. In the FE method, curved structures are approximated by piece–wise–flat surfaces.

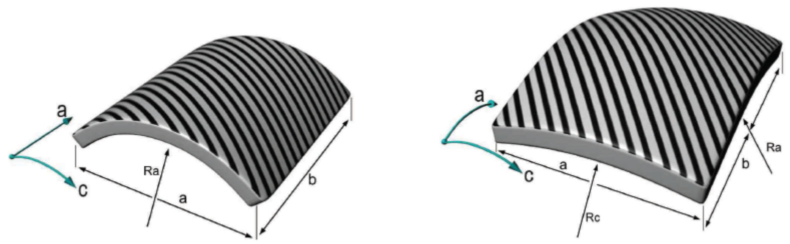


**Figure 3** Singly and Doubly curved panel modelled within the current approach [15]

As in the case of flat waveguides, the DOFs of the cell are arranged in the following way: $\{q\}$ = $\{q1 \quad q2 \quad q3 \quad q4\}$. Before applying the periodicity condition, the local coordinates must be rotated to model the curvature of the panel [4][9]. For example, the DOFs of node 2 and 4 should be transformed to global coordinates by a rotation through an angle $L_\alpha$ as shown in Fig.3. Again, to model the other curvature, a rotation of the DOFs of node 3 and 4, with angle $L_\beta$ is performed.

A transformation matrix $\mathbf{r}$ is defined and assembled in a block diagonal matrix , $\mathbf{R}$, produced by a repeated pattern of the former. It is intended to be done for each curvature.

Following the scheme in Fig. 3 we can define these matrices for direction $a$ and $c$.

$$\mathbf{R}_{a,c} = \begin{bmatrix} \mathbf{r}_{a,c} & 0 & \dots & 0 \\ 0 & \mathbf{r}_{a,c} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & & \mathbf{r}_{a,c} \end{bmatrix}$$

Finally, the Mass and Stiffness matrices are obtained as:

$$\begin{aligned} \mathbf{M} &= \mathbf{R}_c^T \mathbf{R}_a^T \mathbf{M}_{flat} \mathbf{R}_c \mathbf{R}_a \\ \mathbf{K} &= \mathbf{R}_c^T \mathbf{R}_a^T \mathbf{K}_{flat} \mathbf{R}_c \mathbf{R}_a \end{aligned} \tag{17}$$

**The ring frequency** For curved panels the ring frequency is defined as the frequency at which one longitudinal wavelength equals the cylinder perimeter, or bettere, the frequency at which the wavelength becomes so small that it cannot be aware of the curvature itself.

$$\omega_R = \frac{1}{R}\sqrt{\frac{E}{\rho(1-\nu^2)}} \tag{18}$$

This frequency plays an important role in indicating the frequency range for which the curvature effects are important. Further aspects can be explored in [4].

**Singly and Doubly Curved Panels** Since the phase change of a wave as it propagates around a circumference (closed structure) must be a multiple of $2\pi$, the circumferential wavenumber can only take the discrete values $k_\alpha$ = n (n = 0, 1, 2, . . .) which defines the order n of the wave mode [4][9]. The modes n are independent and can be analysed separately.

$$\lambda_{circ} = e^{-inL_{circ.dir.}/R} \tag{19}$$

Using the same material described in previous sections and $SOLID_{45}$ properties, the disperision curves are calculated for thin curved shell-like structures. The radii of curvature are in figure description. The two radii towards directions $a$ and $c$ are equal to $1.5m$. The dispersion curves obtained in both cases of singly and doubly curved panel are then compared.

It is possible to see how the behaviour of the curved panel is completely different from the one of the flat plate only before the ring frequency is reached. The it behaves asymptotically as the flat one.

The cut-on frequency of the first wave changes from about 500 Hz to 600 Hz. This change shows the effect of the geometry parameters on the longitudinal propagation characteristics [15]. It is also observed that the flexural wavenumber for the doubly curved panel below the transition frequency is greater that the one of the singly curved panel, as in the case of Chronopoulos [15].

**Laminated Panel** The dispersion curves of a laminated panel are calculated through a WFE approach. In this case a first agreement with analic wavenumbers has been obtained. Then, increasing frequency, new wavenumbers appear, which can not be extracted with CLPT (Classical Laminate Plate Theory). A comparison is here presented for a layup lamination as described below. The analytical model used is the one described in [6]

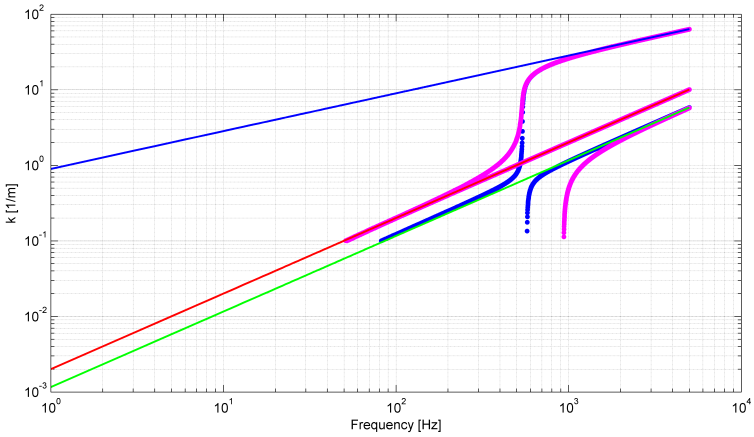


**Figure 4** Dispersion curves of the singly and doubly curved panel; R = 1.5 m; n = 0; WFEM results sc = (.b), dc = (.m), Analytic (-) Fexural (b), Shear (r) and Longitudinal (g) waves for flat panel

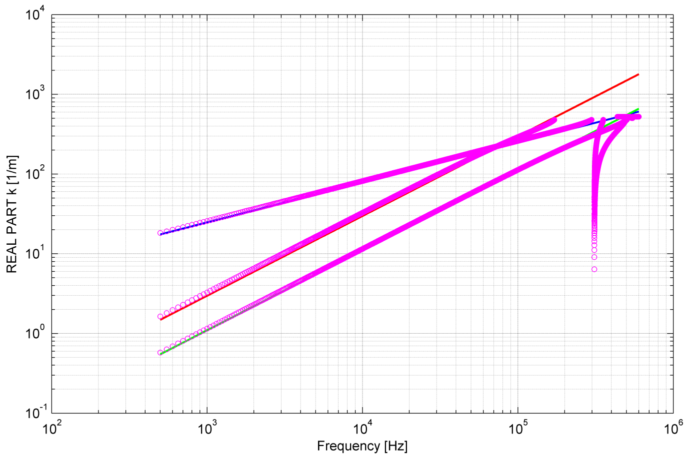


**Figure 5** WFEM results with $SOLID_{45}$ (om) compared with analitic dispersion curves. Analytic (-) Fexural (b), Shear (r) and Longitudinal (g) waves

| $E_x$ | $E_y$ | $G_{xy}$ | $\nu_{xy}$ | Layup |
|---|---|---|---|---|
| 125 GPa | 12.5 GPa | 6.89 GPa | 0.38 | $[0, 90, +45, -45, 0]_{sym}$ |

It is possible to see how in Fig.5, which is calculated using a solid cell, new wavemodes arise at a certain frequency, around 300 KHz, which can not be computed using classic CLPT.

This is caused by the wavelength, at that frequency, becoming close to the thickness of the plate. The plate indeed becomes thick and new waveforms arise when the frequency limit is reached and overtaken.

## 4.2 Forced Response: Stiffened Cylinder Bay

In the case of the stiffened cylinder bay the approach followed is similar. In this case much attention had to be focused on node numbering in order to avoid numerical issues.

The number of chosen consecutive substructures is 35, with forces applied on two consecutive nodes; one on the skin and the other in correspondence of a stiffener. The same base geometry as in the previous cylinder, was used for this case.

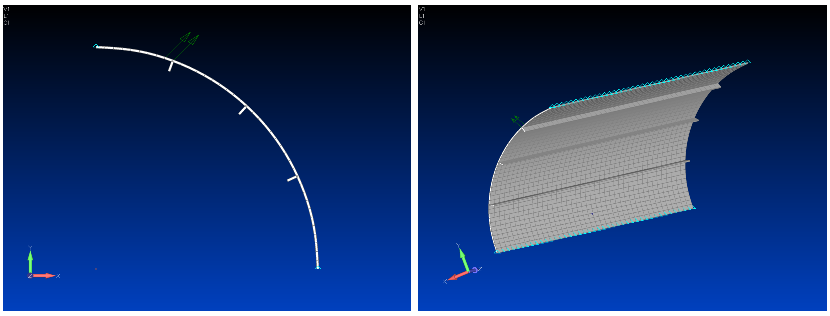

**Figure 6** Finite Element model of the whole stiffened cylinder bay

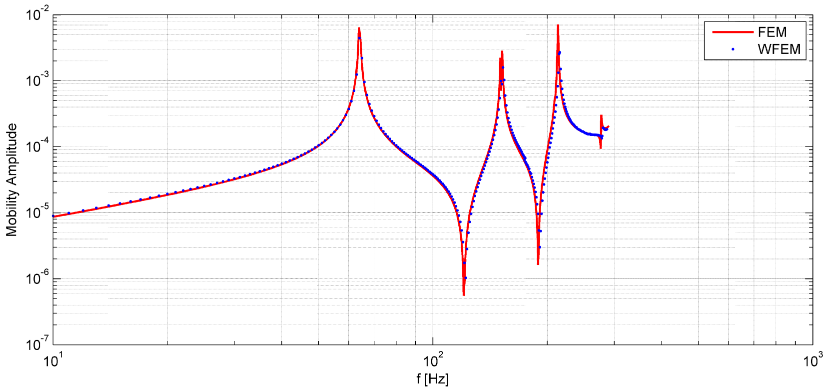



**Figure 7** Mobility amplitude of a node for the stiffened cylinder bay - VX

A strong reduction of the problem size, in therms of DoF is achieved, passing from 8988 to 516 degrees of freedom.

The results are in good agreement with FEM ones, whose model can be seen in Fig.6

## 4.3 Forced Response: Fuselage Bay Model

In the following case a doubly stiffened cylinder bay, as in the case of a simple fuselage bay, will be investigated. Frames and stiffeners will be used to recreate the fuselage. The skins will be modelled using the composite material used for dispersion curves in previous chapter, while the frames and the stiffeners are in aluminium.

A $7.2m$ long Fuselage-like structure, with a $4m$ diameter, is analysed. Stiffeners are modelled using beam properties and frames and skins with shell ones. Of course the latter ones are of different material and thickness: laminated graphite-epoxy for the skins and aluminium for the frames.

The problem size reduces to $480DoF$ instead of 24226, with a reduction of 98%.

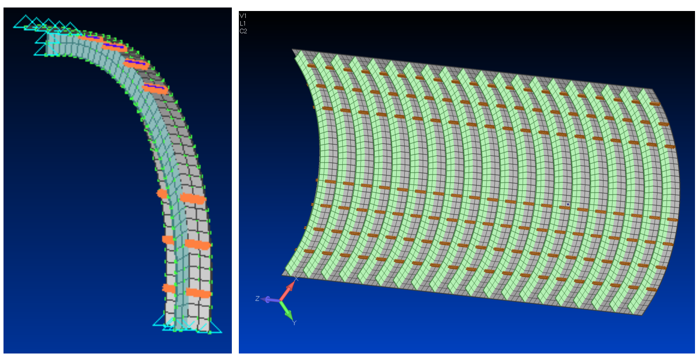

**Figure 8** Finite Element model of the Fuselage Substructure and whole structure

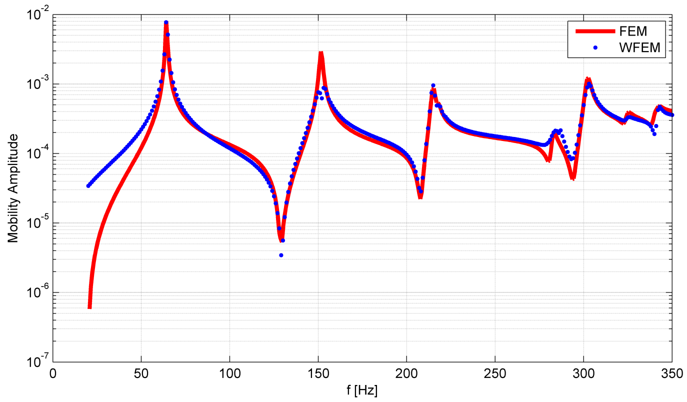


**Figure 9** Mobility amplitude of a node, in the case of inner point load - VY

The theoretic approach used is the one presented at the end of previous section, with simply-supported boundary conditions at the end.

The Wave model provides an overall accurate response. Again the reduced computational cost allowed to drastically reduce the problem size in therms of degrees of freedom to deal with. Working on the same machine and the same software there is no doubt the WFEM would allow a sensible time saving.

## 5 Conclusions and Future Works

The present work has shown many aspect concerning the use of the wave methodology for the response computation of period structures, through the use of substructures, thus forcing a 1D periodicity.

Two different approaches have been presented for computing the forced response of 2D waveguides, whose assemble represents a repetitive substructure of the whole periodic structure. The first one, dealing with the condensed dynamic stiffness matrix, proved to be the most controllable one, which is a very appreciated quality. The method, anyway, has some limits, especially in excitation localization.

The other approach presented suffers more numerical issues and requires a proper selection of the waves in computing the response. However it has not the same limits as the first one and can deal with loads applied on inner points.

The results show a good agreement with numerical classic-FEM and analytic one, for the case of dispersion curves, except for damping, whose modelling and relation with dissipative classic models, requires deeper research. The drastic reduction of DoF is evident, even more when the number of repetitive substructures is high and the cell itself is modelled in order to get the lowest number of DoF at the boundaries.

Numerical issues too, can be managed through a problem reformulation for well conditioning. Most of times, numerical problems arising after the eigenvalue problem are caused by a wrong waves selection, which must be chosen with a criteria slightly different in each case.

Despite a more complicated approach and tricky numerical issues, the advantages provided by the present approach, especially in mid-frequency, are far superior and justify all the research moving in this direction.

Dealing with 2D periodicity, thus using a single cells instead of a substructure, is an actual challenge which could enable a total breakthrough in dynamic analysis of structures, especially for vibro-acoustic response.

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