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AN ENERGY-MOMENTUM FORMULATION FOR NONLINEAR DYNAMICS OF PLANAR CO-ROTATING BEAMS

Sophy Chhang[1,2], **Mohammed Hjiaj**[1], **Jean-Marc Battini**[2] **and Carlo Sansour**[1]

[1]Structural Engineering Research Group /LGCGM , INSA de Rennes, Université Bretagne Loire, 20 Avenue des Buttes de Coësmes, 35708 Rennes Cédex 7, France
e-mail: Sophy.Chhang@insa-rennes.fr, Mohammed.Hjiaj@insa-rennes.fr, Carlo.Sansour@nottingham.ac.uk

[2]Department of Civil and Architectural Engineering, KTH, Royal Institute of Technology, SE-10044 Stockholm, Sweden
e-mail: Jean-Marc.Battini@byv.kth.se

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Abstract. *This article presents an energy-momentum integration scheme for the nonlinear dynamic analysis of planar Bernoulli/Timoshenko beams. The co-rotational approach is adopted to describe the kinematics of the beam and Hermitian functions are used to interpolate the local transverse displacements. In this paper, the same kinematic description is used to derive both the elastic and the inertia terms. The classical midpoint rule is used to integrate the dynamic equations. The central idea, to ensure energy and momenta conservation, is to apply the classical midpoint rule to both the kinematic and the strain quantities. This idea, developed by one of the authors in previous work, is applied here in the context of the co-rotational formulation to the first time. By doing so, we circumvent the nonlinear geometric equations relating the displacement to the strain which is the origin of many numerical difficulties. It can be rigorously shown that the proposed method conserves the total energy of the system and, in absence of external loads, the linear and angular momenta remain constant. The accuracy and stability of the proposed algorithm, especially in long term dynamics with a very large number of time steps, is assessed through two numerical examples.*

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1 INTRODUCTION

The co-rotational method is a very attractive approach to derive highly nonlinear beam elements [1, 4, 3, 2, 5, 6, 7, 8, 9, 10, 11, 12]. The fundamental idea is to decompose the motion of the element into rigid body and pure deformational parts through the use of a local system which continuously rotates and translates with the element. The deformational response is captured at the level of the local reference frame, whereas the geometric nonlinearity induced by the large rigid-body motion, is incorporated in the transformation matrices relating local and global quantities. The main interest is that the pure deformational parts can be assumed as small and can be represented by a linear or a low order nonlinear theory.

Regarding the inertia terms in the co-rotational context, Crisfield et al. [1, 2] used linear local interpolations although they took local cubic interpolations to derive the elastic terms. Then, the inertia terms are easily derived and the classical constant Timoshenko mass matrix is obtained. However, Le at al. [3] adopted the Interdependent Interpolation element (IIE) formulation [13], and hence cubic shape functions, to derive both the inertia and elastic terms. This leads to a formulation that requires a less number of elements but also to more complicated expressions for the inertia force vector and tangent dynamic matrix. The formulation was then extended to 3D beams without [8, 9] and with [10] warping.

Another important issue in the context of non-linear dynamics is the choice of the time stepping method. In commercial finite element programs, the Alpha method [14] is usually used. However, this approach introduces numerical dissipations and consequently, the energy in the system is not conserved [15, 16]. In the last decades, it has been recognized that energy conservation is a key for the stability of time-stepping algorithms in the dynamics of solids and structures. Simo and Tarnow [16] were the first authors to design energy momentum algorithms that inherit the conservation of momenta and energy for geometrically nonlinear problem involving quadratic Green- Lagrange strains. Much effort was devoted then to develop energy momentum methods for nonlinear rod dynamics [17, 19, 18, 20, 21]. Sansour et al. [22, 23] developped an energy-momentum method applicable to any nonlinear shell and to any nonlinear complexities involved in the strain-displacement relations. It has been extended to arbitrary continuum formulations [24] and to geometrically exact Bernoulli beam model [19]. Based on the generalization of the method in [24], Gams et al. [18] developed the for the geometrically exact planar Reissner beam. Besides, it has been applied to the problem of the dissipation of high frequency oscillations associated to energy-momentum methods [25].

In the co-rotational context, there have been some efforts to develop energy-momentum methods as well. Crisfield and Shi [11] proposed a mid-point energy-conserving time algorithm for two-dimensional truss elements. This concept was further developed by Galvanetto and Crisfield [5] for planar beam structures. Various end- and mid-point time integration schemes for the nonlinear dynamic analysis of 3D co-rotational beams are discussed in [1]. The authors concluded that the proposed mid-point scheme can be considered as an approximately energy conserving algorithm. Salomon et al. [12] showed the conservation of energy and momenta in the 2D and 3D analyses for the simulation of elastodynamic problems. They mentioned that, for some cases, the angular momentum is asymptotically preserved and a priori estimate is obtained. However, despite of all these works, the design of an effective time integration scheme for co-rotational elements that inherently fulfils the conservation properties of energy and momenta is still an open question.

In this paper, an energy momentum method in the context of co-rotational shear flexible 2D beam elements is proposed. The IIE cubic shape functions [13] are used together with a shallow arch beam thoery for the co-rotational framework. Based on the previous works of Sansour et al. [22, 23], the main idea is to apply the midpoint rule not only to nodal displacements, velocities and accelerations but also to the strain fields. It means that the strains are updated by using the strain velocities instead of by using directly the strain-displacement relation. The midpoint velocities are applied to the kinematic variables as well. The conservation of energy, linear and angular momentum are then validated by the numerical applications.

The paper is organized as follows: the beam kinematics and strain definitions are presented in Section 2. Section 3 is devoted to Hamilton's principle and conserving properties. The energy-momentum method is then developed in Section 4. In order to assess the numerical performances of the proposed method, two numerical applications are presented in Section 5. Finally, conclusions are presented in Section 6.

2 BEAM KINEMATICS AND STRAIN DEFINITIONS

2.1 Beam kinematics

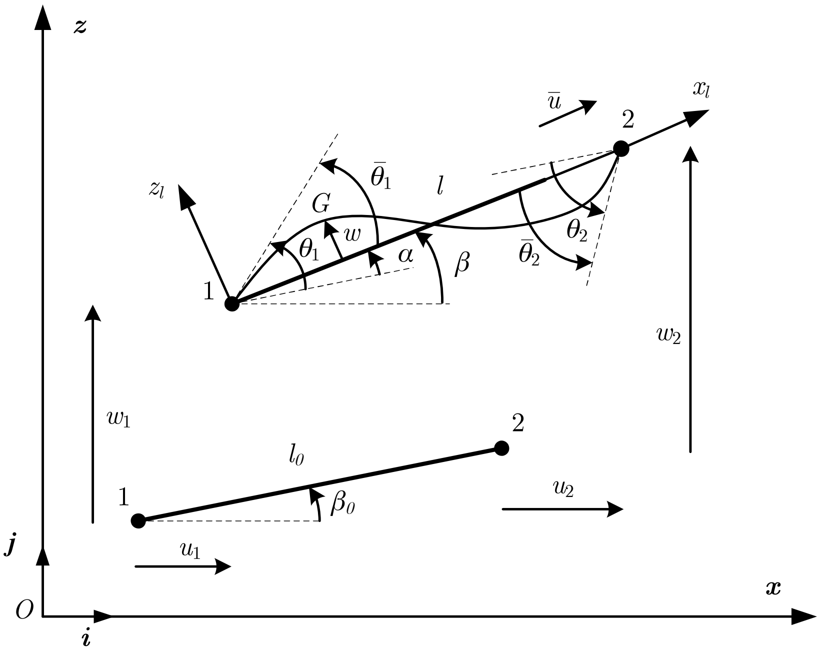

Figure 1: Beam kinematics.

The kinematics of the beam and all the notations used in this section are shown in Figure 1. The motion of the element is decomposed in two parts. In a first step, a rigid body motion is defined by the global translation (u_1,w_1) of the node 1 as well as the rigid rotation α. This rigid motion defines a local coordinate system (x_l, z_l) which continuously translates and rotates with the element. In a second step, the element deformation is defined in the local coordinate system. Assuming that the length of the element is properly selected, the deformational part of the motion is always small relative to the local co-ordinate systems. Consequently, the local deformations can be expressed in a simplified manner.

The vectors of global and local displacements are defined by

$$\boldsymbol{q} = \begin{bmatrix} u_1 & w_1 & \theta_1 & u_2 & w_2 & \theta_2 \end{bmatrix}^{\mathrm{T}} \tag{1}$$

and

$$\bar{\boldsymbol{q}} = \begin{bmatrix} \bar{u} & \bar{\theta}_1 & \bar{\theta}_2 \end{bmatrix}^{\mathrm{T}} \tag{2}$$

Explicitly, the components of $\bar{\boldsymbol{q}}$ are given by

$$\bar{u} = l - l_0 \tag{3}$$

$$\bar{\theta}_1 = \theta_1 - \alpha \tag{4}$$

$$\bar{\theta}_2 = \theta_2 - \alpha \tag{5}$$

where l_0 and l denote the initial and current lengths of the element, respectively:

$$l_0 = \sqrt{(x_2 - x_1)^2 + (z_2 - z_1)^2} \tag{6}$$

$$l = \sqrt{(x_2 + u_2 - x_1 - u_1)^2 + (z_2 + w_2 - z_1 - w_1)^2} \tag{7}$$

The current angle of the local system with respect to the global system is denoted as β and is given by

$$c = cos\beta = \frac{1}{l}\left(x_2 + u_2 - x_1 - u_1\right) \tag{8}$$

$$s = sin\beta = \frac{1}{l}\left(z_2 + w_2 - z_1 - w_1\right) \tag{9}$$

From Figure 1, the components u_G, w_G of the global positions of the centroid G and the global rotation θ_G of the cross section are obtained as

$$u_G = N_1\left(x_1 + u_1\right) + N_2\left(x_2 + u_2\right) - w\,\mathrm{sin}\beta \tag{10}$$

$$w_G = N_1\left(z_1 + w_1\right) + N_2\left(z_2 + w_2\right) + w\,\mathrm{cos}\beta \tag{11}$$

$$\theta_G = \theta + \alpha \tag{12}$$

where w is the local transversal displacement, θ is the local rotation of the cross-section and the linear interpolations N_1, N_2 are

$$N_1 = 1 - \frac{x}{l_0} \tag{13}$$

$$N_2 = \frac{x}{l_0} \tag{14}$$

2.2 Strain definitions

The shape functions of the IIE are used together with a shallow arch beam theory for the local formulation. The shallow arch longitudinal and shear strains are given by

$$\varepsilon_{11} = \varepsilon - \kappa\, z \tag{15}$$

$$\gamma = \frac{\partial w}{\partial x} - \theta \tag{16}$$

in which the axial strain ε and the curvature κ are defined by

$$\varepsilon = \frac{1}{l_0}\int_{l_0}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\mathrm{d}x \tag{17}$$

$$\kappa = \frac{\partial^2 w}{\partial x^2} \tag{18}$$

The shape functions of the IIE element [13] are based on the exact solution of the homogeneous form of the equilibrium equations for a Timoshenko beam. Consequently, the IIE element retains the accuracy inherent to the cubic interpolation including the bending and shear deformation. It should be noted that, in Eq.(18), the axial strain is averaged over the element in order to avoid membrane locking. The local axial displacement u, the local transversal displacement w and the local rotation θ are calculated by

$$u = N_2\,\bar{u} \tag{19}$$

$$w = N_3\,\bar{\theta}_1 + N_4\,\bar{\theta}_2 \tag{20}$$

$$\theta = N_5\,\bar{\theta}_1 + N_6\,\bar{\theta}_2 \tag{21}$$

where the shape functions of the IIE element are given by

$$N_3 = \mu\,x\left[6\,\Omega\left(1-\frac{x}{l_0}\right)+\left(1-\frac{x}{l_0}\right)^2\right] \tag{22}$$

$$N_4 = \mu\,x\left[6\,\Omega\left(\frac{x}{l_0}-1\right)-\frac{x}{l_0}+\frac{x^2}{l_0^2}\right] \tag{23}$$

$$N_5 = \mu\left(1+12\,\Omega-\frac{12\,\Omega\,x}{l_0}-\frac{4\,x}{l_0}+\frac{3\,x^2}{l_0^2}\right) \tag{24}$$

$$N_6 = \mu\left(\frac{12\,\Omega\,x}{l_0}-\frac{2\,x}{l_0}+\frac{3\,x^2}{l_0^2}\right) \tag{25}$$

$$\Omega = \frac{EI}{k_s\,GA\,l_0^2} \tag{26}$$

$$\mu = \frac{1}{1+12\,\Omega} \tag{27}$$

with E is Young's modulus; G is the shear modulus of the material; A is the area of the cross-section; I is the inertia moment of the cross-section and k_s is the shear correction coefficient. For a rectangular cross-section, k_s is equal to $5/6$.

For the dynamic terms, Ω is taken to 0 because this simplification does not affect the numerical results. In addition with $\Omega = 0$, the Hermitian shape functions of the classical Bernoulli element are recovered.

3 HAMILTON'S PRINCIPLE

Hamilton's principle states that the integral of the Lagrangian between two specified time instants t_1 and t_2 of a conservative mechanical system is stationary

$$\delta\int_{t_1}^{t_2}\mathcal{L}\,\mathrm{d}t = 0 \tag{28}$$

The Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \mathsf{K} - \mathsf{U}_{int} - \mathsf{U}_{ext} \tag{29}$$

with K is the kinetic energy. U_{int} and U_{ext} are respectively the internal and the external potential energies. The body is non-conducting linear elastic solid and thermodynamic effects are not included in the system. The kinetic energy is the sum of the translational and rotational kinetic energies:

$$\mathsf{K} = \frac{1}{2}\int_{l_0} \rho A\, \dot{u}_G^2\, \mathrm{d}x + \frac{1}{2}\int_{l_0} \rho A\, \dot{w}_G^2\, \mathrm{d}x + \frac{1}{2}\int_{l_0} \rho I\, \dot{\theta}_G^2\, \mathrm{d}x \tag{30}$$

The internal potential is defined as

$$\mathsf{U}_{int} = \frac{1}{2}\int_{l_0} EA\, \varepsilon^2 \mathrm{d}x + \frac{1}{2}\int_{l_0} EI\, \kappa^2 \mathrm{d}x + \frac{1}{2}\int_{l_0} k_s GA\, \gamma^2\, \mathrm{d}x \tag{31}$$

The external potential is defined as

$$\mathsf{U}_{ext} = -\sum_{i=1}^{6} P_i\, q_i \tag{32}$$

where ρ is the density of the material. P_i is the i component (concentrated forces and moments at the nodes) of external force vector $\boldsymbol{P}$.

By introducing Eqs.(30) to (32), the variation of Eq.(28) can be written as

$$\begin{aligned} &\int_{t_1}^{t_2} \left(\int_{l_0} \rho A\, \dot{u}_G\, \delta\dot{u}_G\, \mathrm{d}x + \int_{l_0} \rho A\, \dot{w}_G\, \delta\dot{w}_G\, \mathrm{d}x + \int_{l_0} \rho I\, \dot{\theta}_G\, \delta\dot{\theta}_G\, \mathrm{d}x \right) \mathrm{d}t \\ - &\int_{t_1}^{t_2} \left(\int_{l_0} EA\, \varepsilon\, \delta\varepsilon\, \mathrm{d}x + \int_{l_0} EI\, \kappa\, \delta\kappa\, \mathrm{d}x + \int_{l_0} k_s GA\, \gamma\, \delta\gamma - \sum_{i=1}^{6} P_i\, \delta q_i \right) \mathrm{d}t = 0 \end{aligned} \tag{33}$$

By using part integration for the first three terms of (33), the previous equation can be reformulated

$$\begin{aligned} &\int_{t_1}^{t_2} \left(\int_{l_0} \rho A\, \ddot{u}_G\, \delta u_G\, \mathrm{d}x + \int_{l_0} \rho A\, \ddot{w}_G\, \delta w_G\, \mathrm{d}x + \int_{l_0} \rho I\, \ddot{\theta}_G\, \delta\theta_G\, \mathrm{d}x \right) \mathrm{d}t \\ + &\int_{t_1}^{t_2} \left(\int_{l_0} EA\, \varepsilon\, \delta\varepsilon\, \mathrm{d}x + \int_{l_0} EI\, \kappa\, \delta\kappa\, \mathrm{d}x + \int_{l_0} k_s GA\, \gamma\, \delta\gamma\, \mathrm{d}x - \sum_{i=1}^{6} P_i\, \delta q_i \right) \mathrm{d}t = 0 \end{aligned} \tag{34}$$

The above equation is the starting point for further developments. The above Hamiltonian system exhibits the following properties. If external loads are conservative, the total energy of the beam element can be written as

$$\mathsf{K} + \mathsf{U}_{int} + \mathsf{U}_{ext} = \mathsf{constant} \tag{35}$$

The linear momentum is defined by

$$\mathbf{L} = \begin{bmatrix} \mathsf{L}_u \\ \mathsf{L}_w \end{bmatrix} = \int_{l_0} \rho A \begin{bmatrix} \dot{u}_G \\ \dot{w}_G \end{bmatrix} \mathrm{d}x \tag{36}$$

and the angular momentum by

$$\mathsf{J} = \int_{l_0} \rho A \begin{bmatrix} u_G \\ w_G \\ 0 \end{bmatrix} \times \begin{bmatrix} \dot{u}_G \\ \dot{w}_G \\ 0 \end{bmatrix} \mathrm{d}x + \int_{l_0} \rho I \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_G \end{bmatrix} \mathrm{d}x \tag{37}$$

In the above equations, $\dot{u}_G$, $\dot{w}_G$ and $\dot{\theta}_G$ are the global velocities of the centroid G of the cross-section.
The time derivative of the two momenta define the equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L} = \begin{bmatrix} P_1 + P_4 \\ P_2 + P_5 \end{bmatrix} \tag{38}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{J} &= (x_1+u_1)\,P_2 - (z_1+w_1)\,P_1 + (x_2+u_2)\,P_5 \\ &- (z_2+w_2)P_4 + P_3 + P_6 = M_{ext} \end{aligned} \tag{39}$$

from which it can be seen that, with vanishing external load, the linear momentum is a constant and, with vanishing external moments, the angular momentum is a constant. It should be noted that the expression "external load" refers to all possible loading conditions including reactions forces.

4 ENERGY-MOMENTUM METHOD

The classical midpoint time integration scheme is defined by the following equations:

$$\boldsymbol{q}_{n+\frac{1}{2}} = \frac{\boldsymbol{q}_{n+1}+\boldsymbol{q}_n}{2} = \boldsymbol{q}_n + \frac{1}{2}\Delta\boldsymbol{q} \tag{40}$$

$$\dot{\boldsymbol{q}}_{n+\frac{1}{2}} = \frac{\dot{\boldsymbol{q}}_{n+1}+\dot{\boldsymbol{q}}_n}{2} = \frac{\boldsymbol{q}_{n+1}-\boldsymbol{q}_n}{\Delta t} = \frac{\Delta\boldsymbol{q}}{\Delta t} \tag{41}$$

$$\ddot{\boldsymbol{q}}_{n+\frac{1}{2}} = \frac{\ddot{\boldsymbol{q}}_{n+1}+\ddot{\boldsymbol{q}}_n}{2} = \frac{\dot{\boldsymbol{q}}_{n+1}-\dot{\boldsymbol{q}}_n}{\Delta t} = \frac{2}{\Delta t^2}\Delta\boldsymbol{q} - \frac{2}{\Delta t}\dot{\boldsymbol{q}}_n \tag{42}$$

By extension of the classical midpoint rule, the average midpoint strains are developed in the context of energy-momentum method. This idea has been introduced in [22, 23]. The midpoint velocities are applied to both the kinematic fields and the strains because the nonlinear terms arise from both fields. This gives:

$$\int_{t_n}^{t_{n+1}} f(t)dt = f(t_{n+\frac{1}{2}})\Delta t = f_{n+\frac{1}{2}}\Delta t \tag{43}$$

$$f_{n+\frac{1}{2}} = f_n + \frac{\Delta t}{2}\dot{f}_{n+\frac{1}{2}} \tag{44}$$

where the function f can represent both the kinematic (u_G, w_G, θ_G) and deformational quantities $(\varepsilon, \kappa, \gamma)$.

The application of the midpoint rule (43) to the Hamilton's principle (34) gives

$$\begin{aligned} \Delta t \quad & \Bigg(\int_{l_0} \rho A\, \ddot{u}_{G,n+\frac{1}{2}}\, \delta u_{G,n+\frac{1}{2}}\, \mathrm{d}x + \int_{l_0} \rho A\, \ddot{w}_{G,n+\frac{1}{2}}\, \delta w_{G,n+\frac{1}{2}}\, \mathrm{d}x + \int_{l_0} \rho I\, \ddot{\theta}_{G,n+\frac{1}{2}}\, \delta\theta_{G,n+\frac{1}{2}}\, \mathrm{d}x \\ + \quad & \int_{l_0} EA\, \varepsilon_{n+\frac{1}{2}}\, \delta\varepsilon_{n+\frac{1}{2}}\, \mathrm{d}x + \int_{l_0} EI\, \kappa_{n+\frac{1}{2}}\, \delta\kappa_{n+\frac{1}{2}}\, \mathrm{d}x + \int_{l_0} k_s GA\, \gamma_{n+\frac{1}{2}}\, \delta\gamma_{n+\frac{1}{2}}\, \mathrm{d}x \\ - \quad & \sum_{i=1}^{6} P_{i,n+\frac{1}{2}}\, \delta q_{i,n+\frac{1}{2}} \Bigg) = 0 \end{aligned} \tag{45}$$

As the variation $\delta \boldsymbol{q}^{\mathrm{T}}$ is arbitrary, the dynamic equilibrium at time $n+\frac{1}{2}$ is obtained from the previous equation

$$\begin{aligned}
\boldsymbol{f}_{R,n+\frac{1}{2}} &= \int_{l_0} \rho A\, \ddot{u}_{G,n+\frac{1}{2}} \left(\frac{\partial u_{G,n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}}\right)^{\mathrm{T}} \mathrm{d}x + \int_{l_0} \rho A\, \ddot{w}_{G,n+\frac{1}{2}} \left(\frac{\partial w_{G,n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}}\right)^{\mathrm{T}} \mathrm{d}x \\
&+ \int_{l_0} \rho I\, \ddot{\theta}_{G,n+\frac{1}{2}} \left(\frac{\partial \theta_{G,n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}}\right)^{\mathrm{T}} + \int_{l_0} EA\, \varepsilon_{n+\frac{1}{2}} \left(\frac{\partial \varepsilon_{n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}}\right)^{\mathrm{T}} \mathrm{d}x \\
&+ \int_{l_0} EI\, \kappa_{n+\frac{1}{2}} \left(\frac{\partial \kappa_{n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}}\right)^{\mathrm{T}} \mathrm{d}x + \int_{l_0} k_s GA\, \gamma_{n+\frac{1}{2}} \left(\frac{\partial \gamma_{n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+\frac{1}{2}}}\right)^{\mathrm{T}} \mathrm{d}x \\
&- \boldsymbol{P}_{i,n+\frac{1}{2}} = 0
\end{aligned} \tag{46}$$

With the help of the second and third of Eqs. (40), the accelerations at midpoint are obtained as

$$\ddot{u}_{G,n+\frac{1}{2}} = \frac{2}{\Delta t}\, \dot{u}_{G,n+\frac{1}{2}} - \frac{2}{\Delta t}\, \dot{u}_{G,n} \tag{47}$$

$$\ddot{w}_{G,n+\frac{1}{2}} = \frac{2}{\Delta t}\, \dot{w}_{G,n+\frac{1}{2}} - \frac{2}{\Delta t}\, \dot{w}_{G,n} \tag{48}$$

$$\ddot{\theta}_{G,n+\frac{1}{2}} = \frac{2}{\Delta t}\, \dot{\theta}_{G,n+\frac{1}{2}} - \frac{2}{\Delta t}\, \dot{\theta}_{G,n} \tag{49}$$

and the strains at $n+\frac{1}{2}$ are obtained from Eq.(44) as

$$\varepsilon_{n+\frac{1}{2}} = \varepsilon_n + \frac{\Delta t}{2}\, \dot{\varepsilon}_{n+\frac{1}{2}} \tag{50}$$

$$\kappa_{n+\frac{1}{2}} = \kappa_n + \frac{\Delta t}{2}\, \dot{\kappa}_{n+\frac{1}{2}} \tag{51}$$

$$\gamma_{n+\frac{1}{2}} = \gamma_n + \frac{\Delta t}{2}\, \dot{\gamma}_{n+\frac{1}{2}} \tag{52}$$

The total tangent matrix is defined by

$$\mathbf{K}_{\mathrm{T}} = \frac{\partial \boldsymbol{f}_{R,n+\frac{1}{2}}}{\partial \boldsymbol{q}_{n+1}} \tag{53}$$

It can be observed that by updating the strains by integrated the strain velocity, see Eqs. (50)–(52), the obtained strains at $n+1$ are not equal to the strains (Eqs.(16)–(18)) determined from the total displacements and rotations at $n+1$. The similar concept is also applied to the acceleration quantities, see Eqs. (47)–(49). However, only these updated quantities by integrating from velocity guarantee energy conservation and also stability of the integration method.

5 NUMERICAL EXAMPLES

Two numerical examples are presented in this section. The first purpose is to verify that the proposed algorithm conserves the total energy of the system and remain stable even if a very large number of time steps are applied. The second purpose is to show that the proposed algorithm conserves the linear and angular momenta in the absence of applied external loads.

5.1 Simple beam

A simply-supported beam depicted in Figure 2 is subjected to a vertical concentrated load $P = 4.1$ MN at mid-span. The length of the beam is $L = 1$ m. The cross-section width and depth are $b = 0.20$ m and $h = 0.15$ m. The material parameters are: elastic modulus $E = 1$ GPa, Poisson's ratio $\nu = 0.3$, density $\rho = 7850$ kg/m^3. The number of elements in this example is 8. The time step size is $\Delta t = 10^{-4}$ s. The interest of this example is that, for this short beam, the shear effect is important.

The horizontal displacement at the right end and the vertical displacement at mid-span are depicted in Figures 3 and 4. Therefore, large displacements can be observed. In Figure 5, it is shown that the proposed energy-momentum method conserves the total energy of the system ,and that the solutions remain stable for one million time steps.

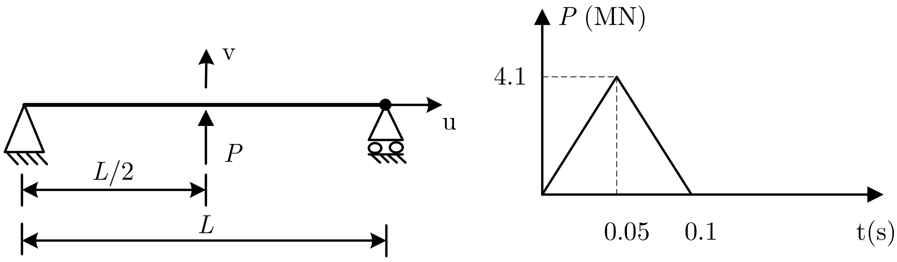

Figure 2: Geometry and loading of simple beam

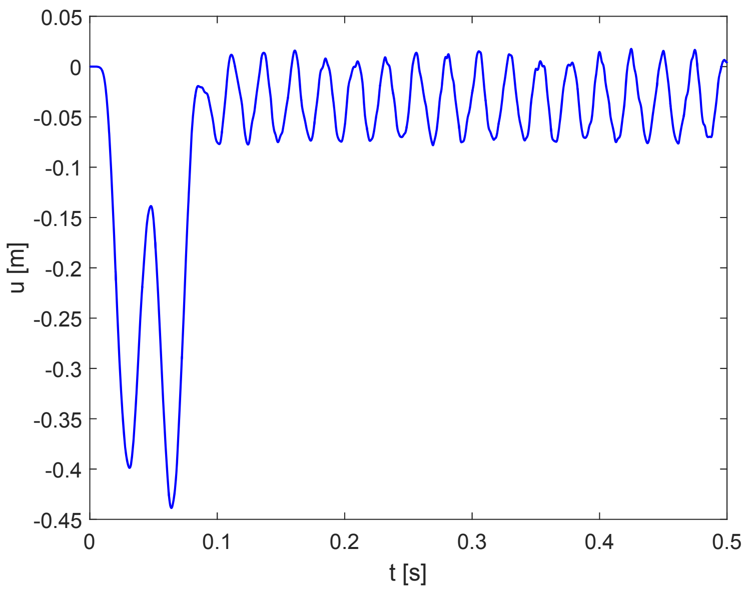

Figure 3: Horizontal displacement u [m] of simple beam

5.2 Free fly beam

Consider a free fly beam without support subjected to a triangle load $P = 3 \times 10^5$ N, see Figure 6. The length of the beam is $L = 3$ m, the cross-sectional area is $A = 0.02$ m^2 and the moment of inertia $I = 66.67 \times 10^{-8}$ m^4. The material properties are: elastic modulus $E = 200$

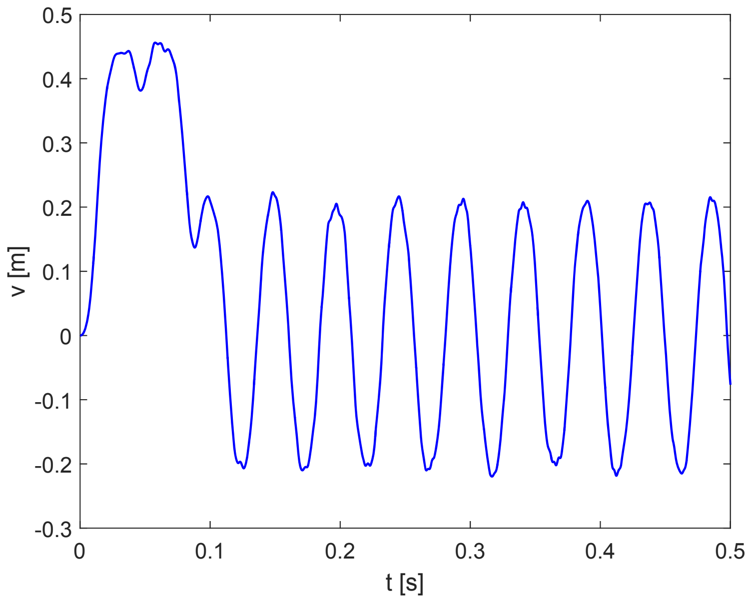

Figure 4: Vertical displacement v [m] of simple beam

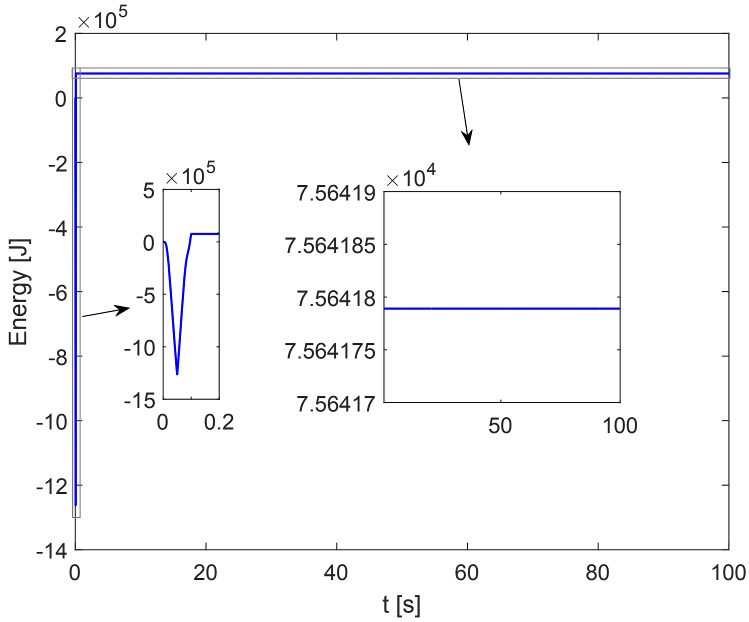

Figure 5: Energy [J] of simple beam

GPa, Poisson's ratio $\nu = 0.3$, density $\rho = 48831$ kg/m^3. The number of elements is 4 and the time step size is $\Delta t = 10^{-4}$ s.

In Figure 7, the vertical displacement at mid-span is depicted. It is shown that the beam moves in a long distance 2045 m at 100 s with the speed 73.62 km/h. In addition, the result in Figure 8 shows the conservation of the total energy even if one million time steps are applied.

The interest of this problem is to study the conservation of the linear and angular momenta after the time 0.4s because no forces and moments are then applied to the beam. Since only vertical loads are applied at the beginning, the linear momentum Lu in the horizontal direction should be zero. As shown in Figure 9, the linear momentum Lu is almost zeros with the maximum value of 2×10^{-5}. Moreover, Figures 9 and 10 show the conservation of linear momentum Lw in the vertical direction and angular momentum for one million time steps.

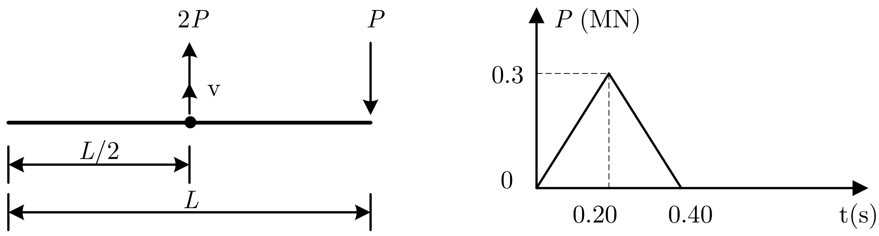

Figure 6: Geometry and loading of free fly beam

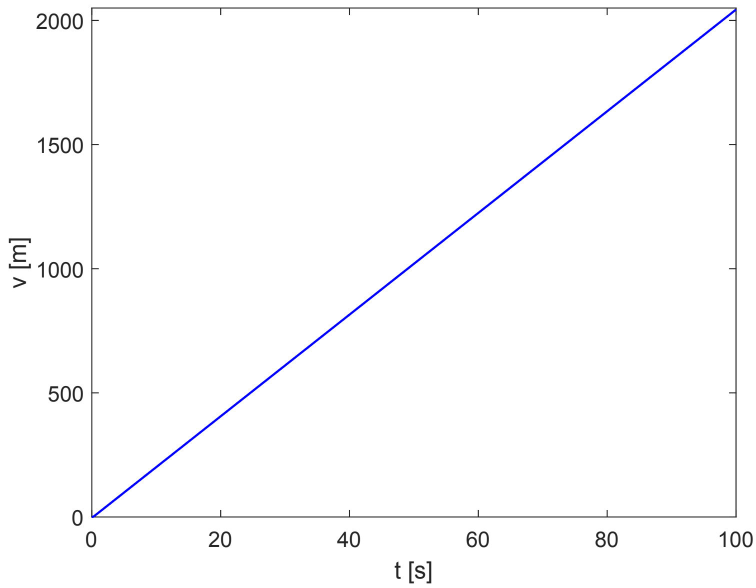

Figure 7: Vertical displacement [m] of free fly beam

6 CONCLUSION

This paper has presented an energy-momentum method for co-rotational planar shear flexible beams. The main idea is to use the classical midpoint rules for both the kinematic and strain quantities. The advantage of the proposed algorithm is that it conserves the total energy of the system and remains stable and accurate even if a very large number of time steps are applied. Besides, in the absence of applied external loads, the linear and angular momenta are constant. These characteristics have been proved numerically by using two numerical applications.

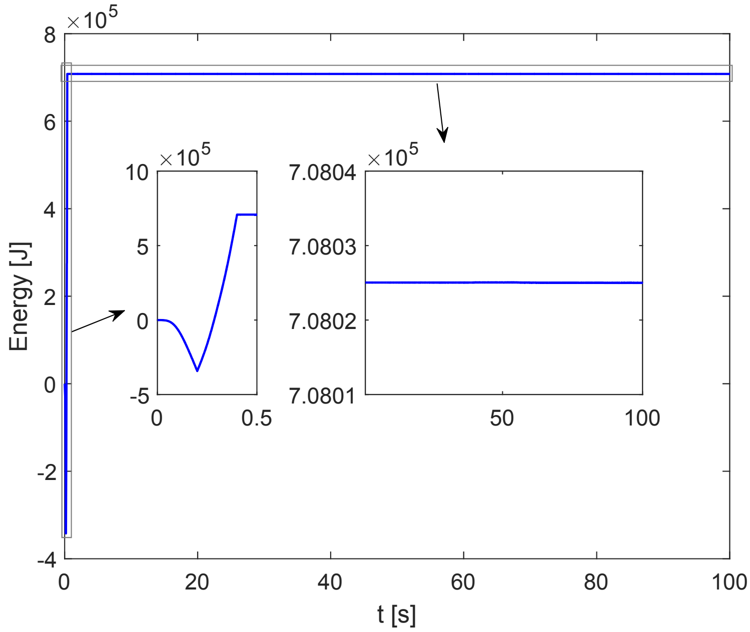

Figure 8: Energy [J] of free fly beam

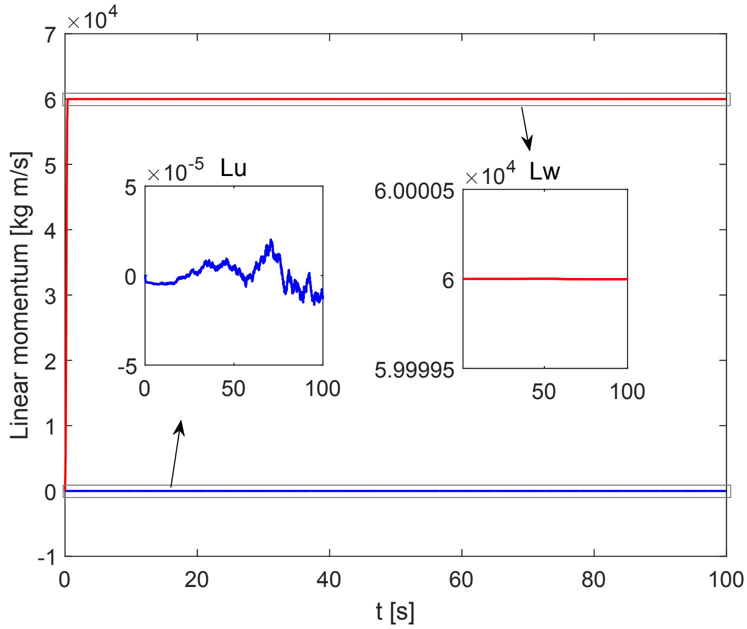

Figure 9: Linear momentum [kg m/s] of free fly beam

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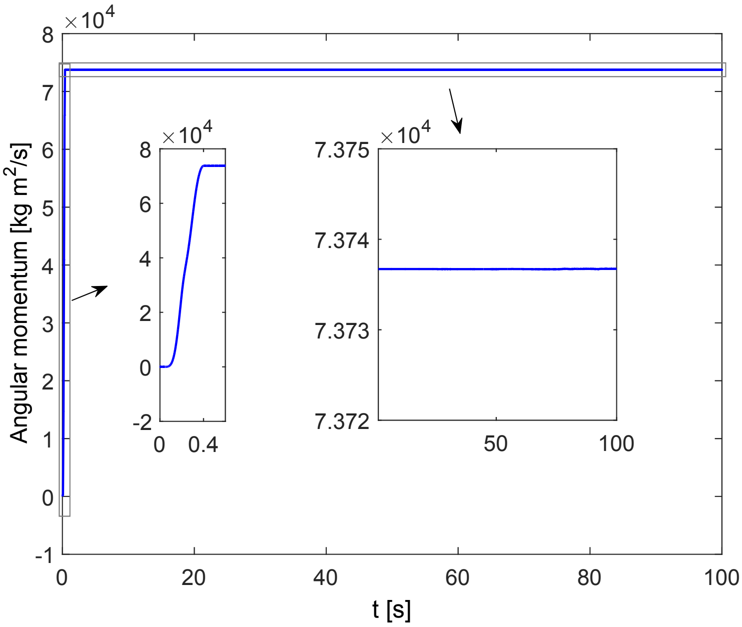

Figure 10: Angular momentum J [kg m^2/s] of free fly beam

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