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MODAL ANALYSIS PROCEDURE USING COMPLEX LEFT AND RIGHT EIGENVECTORS OF NON-PROPORTIONALLY DAMPED STRUCTURES

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Abstract. *In the general case of non-proportionally damped structural model the associated quadratic eigenvalue problem leads to complex eigenvalues and eigenvectors. The modal decomposition of the equations of motion is usually to be performed in complex space.*
In this paper are presented possible variants of a general method [2] - [5] for modal transformation of damped multi-degree-of-freedom-systems (MDOFS) with non-modal symmetric damping matrix. The assembly of a modal transformation matrix in real space is based either on the conjugated complex left eigenvectors, or on the right eigenvectors, or on a combination of the left and right eigenvectors of the system. The eigenvector normalization can be performed with respect to the general mass or to the general stiffness matrix. The equations of motion are stated in state-space formulation. The developed real-space modal transformation matrix is always built by a combination of two complex transformations. Analytically expressions for all presented variants of the modal transformation basis are developed be the aid of computer algebra software. Those formulas operate with the real and the imaginary parts of the eigenvectors and the associated eigenvalues. All variants of the suggested modal procedure retain the common advantages of the classic modal decomposition of the equations of motion.
The vibrations of a rotor blade of a wind turbine subjected to wind thrust loads have been calculated in two variants to demonstrate the performance of the presented modal analysis procedures. The initial computation of the complex eigenvalue solution of the FEM beam model and all subsequent computations are done by the aid of computer algebra software. The suggested procedures can be applied in structural systems containing different damping and energy-loss mechanism in various parts of the structure, described by non-proportional damping matrix.

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1 INTRODUCTION

The classic modal transformation of the equations of motion of multi-degree-of-freedom-systems (MDOFS) is applied to systems without damping. Fully real free frequencies and eigenvectors are the solution of the associated linear eigenvalue problem. The inclusion of damping in the equations of MDOFS leads to a quadratic eigenvalue problem with corresponding complex conjugate pairs of eigenvalues and eigenvectors. The modal decomposition of the equations has to be performed in complex space.

The damping can be account for in a simple way by use of the Rayleigh damping assumption – this leads to proportional damping matrix. In this special case the real eigenvectors of the undamped system are also eigenvectors of the system with proportional damping. The corresponding modal matrix, to be used for modal decomposition of the equations of motion, is fully real. In the general case of viscous damping, represented by non-proportional symmetric damping matrix, we have to deal with complex conjugate pairs of eigenvectors. This case is considered in the present paper.

There are presented three variants of a new modal transformation procedure for structural models with non-proportional symmetric damping matrix. The first one, briefly outlined in Sec. 2, is based on the complex right eigenvectors, normalized with respect to the mass matrix. A more detailed presentation of this procedure is given in [2] – [4]. The second variant, developed in a similar way to the first one, operates with the complex right eigenvectors too, but they are normalized with respect to the stiffness matrix. The procedure is described in details in Sec. 3.

In Sec. 4 is presented the third variant of the suggested procedure. Here we operate with both the right and the left complex eigenvectors of the MDOFS, the normalization is with respect to the stiffness matrix.

The main subject in all variants is to avoid the modal decomposition in complex arithmetic by assembling of a new modal transformation matrix in real space, belonging to the "state space" form of the equations of motions. The columns of the matrix are developed analytically by a combination of two complex transformations. Considering the proportional damping case which is also included, an analytical formula for the constant phase lag/lead of free vibrations is derived in both procedures presented in Sec. 2 and Sec. 3.

In all variants of the suggested procedure the complex eigenvectors and eigenvalues of the structural model should be computed first. In the presented example in Sec. 5 – vibration of a rotor blade of a wind generator - computer algebra software was applied to solve the eigenvalue problem. In real life applications of the presented method to high dimensional problems it must be available a reliable eigenmode solver for large complex eigenvalue computations. There are many literature references for large scaled problems with various solution strategies, see [10] – [12]. The author has used an implicitly restarted Arnoldi/Lanczos method [11], [12] to solve the complex eigenvalue problem in an application of the method to a fluid-structure-foundation interaction problem, see more details in [1] - [3].

In Sec. 5 the proposed modal analysis method, presented in Section 4, has been applied to a rotor blade beam structure with 54 DOF. The numerical example demonstrates the performance of the method for the general case of non-proportional damping. In this case the damping matrix of the system contains a stiffness proportional (Rayleigh) damping and aerodynamic (non-proportional) damping parts. In the second variant of the solution – with proportional damping matrix, the formula for the constant phase of the resonance modes is verified numerically.

1.1 Free vibrations of a viscously damped system

The equations of motion of a damped MDOFS are

$$\mathbf{M}\ddot{\mathbf{V}} + \mathbf{D}\dot{\mathbf{V}} + \mathbf{K}\mathbf{V} = \mathbf{p}(t) \tag{1.1}$$

where **M**, **D** and **K** are, respectively the (n x n) mass, damping and stiffness matrices, and **V**, $\dot{\mathbf{V}}$ are the (n x 1) displacement and velocity vectors and **p**(t) is the (n x 1) excitation vector.

In structural mechanics problems we consider the **M** and **K** matrices to be real, symmetric and positive definite, excluding the presence of rigid body modes. The matrix **D**, presenting a non-proportional damping**,** is assumed to be symmetric and non-negative.

With the assumed free vibration in the form

$$\mathbf{V} = \mathbf{X}e^{\lambda t}, \qquad \dot{\mathbf{V}} = \lambda \mathbf{X}e^{\lambda t}, \tag{1.2}$$

the associated quadratic eigenvalue problem is

$$\left(\lambda_j^2\, \mathbf{M} + \lambda_j \mathbf{D} + \mathbf{K}\right) \mathbf{X}_j = \mathbf{0} \qquad \forall\, (j = 1, \cdots, n) \tag{1.3}$$

In Eq. (1.3) the j[th] eigenvalue λ_j and the corresponding eigenmode $\mathbf{X}_j$ appear in complex conjugate pairs (index j omitted):

$$\lambda = \lambda_r + i\lambda_i\,, \qquad \bar{\lambda} = \lambda_r - i\lambda_i \tag{1.4a}$$
$$\mathbf{X} = \mathbf{X}_r + i\mathbf{X}_i\,, \qquad \bar{\mathbf{X}} = \mathbf{X}_r - i\mathbf{X}_i \tag{1.4b}$$

The dynamic equilibrium of a **viscously damped single oscillator** is governed by

$$m\ddot{v}(t) + c\dot{v}(t) + kv(t) = q(t) \qquad \text{or} \tag{1.5a}$$
$$\ddot{v}(t) + 2\eta\omega\dot{v}(t) + \omega^2 v(t) = p(t) \tag{1.5b}$$

where $\ddot{v}$ is acceleration,
$\dot{v}$ - velocity,
$\omega = \sqrt{\frac{k}{m}}$ - free vibration frequency,
$\eta = \frac{c}{2m\omega}$ - Lehr's damping ratio and $p(t) = \frac{q(t)}{m}$.

The exponential solution $x\, e^{\lambda t}$, introduced into the homogenous form of the differential equation (1.5b), yields the eigenvalue problem

$$\lambda^2 + 2\eta\omega\, \lambda + \omega^2 = 0 \tag{1.6}$$

The eigenvalue solution (assuming $\eta \ll 1$, subcritical damping) of Eq. (1.6) is a complex conjugate pair:

$$\lambda_{1/2} = \underbrace{-\eta\omega}_{\lambda_r} \pm i\underbrace{\omega\sqrt{1-\eta^2}}_{\lambda_i = \omega_D} = \lambda_r \pm i\lambda_i \qquad \text{where} \quad \omega_D = \omega\sqrt{1-\eta^2} \tag{1.7}$$

1.2 The constant phase lag problem

Introducing the single-oscillator-eigenvalues (1.7) into (1.4a), we can express the j[th] free vibration of the MDOFS as linear combination of the two complex conjugate eigenpairs (1.4b):

$$\mathbf{V} = \mathbf{X}\, e^{\lambda t} = \mathbf{X}\, e^{\left(-\eta\omega \pm i\omega\sqrt{1-\eta^2}\right)t} =$$

$$= e^{-\eta\omega t}[(\mathbf{X}_r + i\mathbf{X}_i)(cos\,\omega_D t + i \sin\omega_D t) + (\mathbf{X}_r - i\mathbf{X}_i)(cos\,\omega_D t - i \sin\omega_D t)]$$

$$= e^{-\eta\omega t}\left[\underbrace{2\mathbf{X}_r}_{\mathbf{F}\cos\varphi}\; cos\,\omega_D t - \underbrace{2\mathbf{X}_i}_{\mathbf{F}\sin\varphi}\; \sin\omega_D t\right] \tag{1.8}$$

The last relation leads to the real form of a damped free oscillation for every k[th] DOF:

$$V_k = e^{-\eta\omega t}[F_k\, cos(\omega_D t + \varphi_k)] \tag{1.9}$$

where $\varphi_k = arctan\frac{(X_i)_k}{(X_r)_k}$: phase lag/lead for the k[th] DOF (1.10)

Since the viscous damping is assumed to be non-proportional, the free vibration solution (1.9) represents non-synchronous damped oscillation (i.e. the phase φ_k is different for each k[th] DOF). In the case of proportionally damped system we have to deal with synchronous free oscillation – i.e. the phase φ_k is constant - the same for all DOF ($\varphi_k = 0$ for undamped systems) – see [6], [7], p.118.

The features, showed in Eq.(1.9), (1.10) are well known and used in modal analysis, see for example [6]. In Sec. 2 and 3 of the present paper the constant phase lag in the proportional damping case is expressed analytically. The developed formulas are in dependence of the normalization of the modal matrix.

2 MODAL ANALYSIS PROCEDURE BASED ON THE MASS NORMALIZED COMPLEX RIGHT EIGENVECTORS

The procedure, briefly reviewed here, has been presented in details in [2] – [4].

2.1 The single mass oscillator

The equation of motion of a damped single degree of freedom system (SDOFS) (1.5b) can be written in the extended form

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -\omega^2 \end{bmatrix}}_{\mathbf{m}}\underbrace{\begin{bmatrix} \dot{w}(t) \\ \dot{v}(t) \end{bmatrix}}_{\dot{\mathbf{q}}} + \underbrace{\begin{bmatrix} 2\eta\omega & \omega^2 \\ \omega^2 & 0 \end{bmatrix}}_{\mathbf{k}}\underbrace{\begin{bmatrix} w(t) \\ v(t) \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} p(t) \\ 0 \end{bmatrix}}_{\mathbf{p}(t)} \tag{2.1a}$$

$$\mathbf{m}\dot{\mathbf{q}} + \mathbf{k}\mathbf{q} = \mathbf{p} \tag{2.1b}$$

with the velocity

$$w = \dot{v} \tag{2.2}$$

The exponential approach $\mathbf{q}=\mathbf{x}e^{\lambda t}$, $\dot{\mathbf{q}}=\lambda\mathbf{x}e^{\lambda t}$, introduced into the homogenous form of the differential equation (2.1), leads to the quadratic eigenvalue problem

$$\dot{\mathbf{q}}+\underbrace{(\mathbf{m}^{-1}\mathbf{k})}_{\boldsymbol{a}}\,\mathbf{q}=\mathbf{0} \quad \rightarrow \quad (\mathbf{a}+\lambda_j\mathbf{e})\mathbf{r}_j=\mathbf{0} \quad (j=1{,}2) \tag{2.3}$$

The solution are two complex conjugate eigenvalues ($\eta<<1$, subcritical damped system):

$$\lambda_{1/2}=-\eta\omega\pm\sqrt{(\eta\omega)^2-\omega^2}=\underbrace{-\eta\omega}_{\lambda_r}\pm i\underbrace{\omega\sqrt{1-\eta^2}}_{\lambda_i}=\begin{cases}\lambda=\lambda_r+i\lambda_i\\ \bar{\lambda}=\lambda_r-i\lambda_i\end{cases} \tag{2.4a}$$

$$\text{where} \quad \omega=\sqrt{(\lambda_r)^2+(\lambda_i)^2}\ , \quad \eta=-\frac{\lambda_r}{\omega}, \tag{2.4b}$$

and two corresponding complex conjugate right eigenvectors

$$\mathbf{r}_j=\begin{bmatrix}\lambda_r\pm i\lambda_i\\ 1\end{bmatrix} \quad (j=1{,}2) \tag{2.5}$$

The eigenvectors are normalized with respect to the mass matrix

$$[\mathbf{r}_1\quad \mathbf{r}_2]^T\cdot\mathbf{m}\cdot[\mathbf{r}_1\quad \mathbf{r}_2]=\begin{bmatrix}g_1 & \\ & g_2\end{bmatrix} \quad \rightarrow \quad \varphi_j=\frac{\mathbf{r}_j}{\sqrt{g_j}} \quad (j=1{,}2) \tag{2.6}$$

and then combined into a modal matrix φ:

$$\varphi=[\varphi_1\quad \varphi_2] \tag{2.7}$$

Due to the mass normalization - Eq. (2.6), the orthogonality relationships (2.8), (2.9) can be derived:

$$\varphi^{\mathbf{T}}\mathbf{m}\,\varphi=\varphi^{\mathbf{T}}\begin{bmatrix}1 & 0\\ 0 & -\omega^2\end{bmatrix}\varphi=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad \leftrightarrow \quad \varphi^{-\mathbf{T}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\varphi^{-\mathbf{1}}=\begin{bmatrix}1 & 0\\ 0 & -\omega^2\end{bmatrix} \tag{2.8}$$

$$\varphi^{\mathbf{T}}\mathbf{k}\,\varphi=\varphi^{\mathbf{T}}\begin{bmatrix}2\eta\omega & \omega^2\\ \omega^2 & 0\end{bmatrix}\varphi=\begin{bmatrix}-\lambda & 0\\ 0 & -\bar{\lambda}\end{bmatrix} \leftrightarrow \quad \varphi^{-\mathbf{T}}\begin{bmatrix}-\lambda & 0\\ 0 & -\bar{\lambda}\end{bmatrix}\varphi^{-\mathbf{1}}=\begin{bmatrix}2\eta\omega & \omega^2\\ \omega^2 & 0\end{bmatrix} \tag{2.9}$$

The inverse of the complex modal matrix $\varphi(\omega,\eta)$ can be expressed analytically using computer algebra software:

$$\varphi^{-1}=\frac{1}{2\sqrt{1-\eta^2}}\begin{bmatrix}(-Z_1-iZ_2) & P-iQ\\ (-Z_1+iZ_2) & P+iQ\end{bmatrix} \tag{2.10}$$

where

$$Z_1=\sqrt{\sqrt{1-\eta^2}+(1-\eta^2)} \qquad Z_2=\sqrt{\sqrt{1-\eta^2}-(1-\eta^2)} \tag{2.11a}$$

$$P=\omega\left(\sqrt{1-\eta^2}\,Z_2-\eta Z_1\right) \qquad Q=\omega\left(\sqrt{1-\eta^2}\,Z_1+\eta Z_2\right) \tag{2.11b}$$

2.2 The damped multi-degree-of-freedom-system

The equations of motion (1.1) of damped MDOFS (n DOF) will be written in the state-space form:

$$\underbrace{\begin{bmatrix} \mathbf{M} & \\ & -\mathbf{K} \end{bmatrix}}_{\mathbf{M_G}} \underbrace{\begin{bmatrix} \dot{\mathbf{W}} \\ \dot{\mathbf{V}} \end{bmatrix}}_{\dot{\mathbf{Q}}} + \underbrace{\begin{bmatrix} \mathbf{D} & \mathbf{K} \\ \mathbf{K} & \end{bmatrix}}_{\mathbf{K_G}} \underbrace{\begin{bmatrix} \mathbf{W} \\ \mathbf{V} \end{bmatrix}}_{\mathbf{Q}} = \underbrace{\begin{bmatrix} \mathbf{p}(t) \\ \\ \end{bmatrix}}_{\mathbf{P}} \quad ; \qquad \dot{\mathbf{V}} = \mathbf{W} \tag{2.12a}$$

$$\mathbf{M_G}\,\dot{\mathbf{Q}} + \mathbf{K_G}\,\mathbf{Q} = \mathbf{P} \ , \tag{2.12b}$$

where $\mathbf{M_G}$ and $\mathbf{K_G}$ are, respectively the (2n x 2n) symmetric generalized mass and the generalized stiffness matrices.

With the exponential solution $\mathbf{V} = \mathbf{X} e^{\lambda t}$ the quadratic eigenvalue problem (1.3) can be written in the 2n-dimensional form

$$\left(\lambda \mathbf{M_G} + \mathbf{K_G}\right) \begin{bmatrix} \lambda \mathbf{X} \\ \mathbf{X} \end{bmatrix} = \mathbf{0} \ , \tag{2.13}$$

where

$$\lambda^{(j)} = \lambda_r^{(j)} + i\lambda_i^{(j)} \quad \rightarrow \begin{bmatrix} \lambda^{(j)}\mathbf{X}^{(j)} \\ \mathbf{X}^{(j)} \end{bmatrix}; \qquad \overline{\lambda}^{(j)} = \lambda_r^{(j)} - i\lambda_i^{(j)} \quad \rightarrow \begin{bmatrix} \overline{\lambda}^{(j)}\overline{\mathbf{X}}^{(j)} \\ \overline{\mathbf{X}}^{(j)} \end{bmatrix}; \quad \left(j = 1,2 \ldots n\right) \tag{2.14}$$

are the corresponding n complex conjugate eigenpairs.

Each j[th] eigenvector-pair $\mathbf{X}^{(j)}, \overline{\mathbf{X}}^{(j)}$ is normalized (index (j) omitted) with respect to the general mass matrix $\mathbf{M_G}$:

$$\begin{bmatrix} \lambda \mathbf{X} \\ \mathbf{X} \end{bmatrix}^T \mathbf{M_G} \begin{bmatrix} \lambda \mathbf{X} \\ \mathbf{X} \end{bmatrix} = A + iB \qquad \rightarrow \quad \mathbf{\Phi} = \frac{\mathbf{X}}{\sqrt{A+iB}} = \mathbf{\Phi_r} + i\,\mathbf{\Phi_i} \, , \tag{2.15a}$$

$$\begin{bmatrix} \overline{\lambda}\overline{\mathbf{X}} \\ \overline{\mathbf{X}} \end{bmatrix}^T \mathbf{M_G} \begin{bmatrix} \overline{\lambda}\overline{\mathbf{X}} \\ \overline{\mathbf{X}} \end{bmatrix} = A - iB \qquad \rightarrow \quad \overline{\mathbf{\Phi}} = \frac{\overline{\mathbf{X}}}{\sqrt{A-iB}} = \mathbf{\Phi_r} - i\,\mathbf{\Phi_i} \, , \tag{2.15b}$$

Due to the normalization (2.15) we have the orthogonality relationships (2.16), (2.17) – expressed in terms of the j[th] eigenvector-pair (index (j) omitted):

$$\begin{bmatrix} \lambda\mathbf{\Phi} & \overline{\lambda}\overline{\mathbf{\Phi}} \\ \mathbf{\Phi} & \overline{\mathbf{\Phi}} \end{bmatrix}^T \underbrace{\begin{bmatrix} \mathbf{M} & \\ & -\mathbf{K} \end{bmatrix}}_{\mathbf{M_G}} \begin{bmatrix} \lambda\mathbf{\Phi} & \overline{\lambda}\overline{\mathbf{\Phi}} \\ \mathbf{\Phi} & \overline{\mathbf{\Phi}} \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \tag{2.16}$$

$$\begin{bmatrix} \lambda\mathbf{\Phi} & \overline{\lambda}\overline{\mathbf{\Phi}} \\ \mathbf{\Phi} & \overline{\mathbf{\Phi}} \end{bmatrix}^T \underbrace{\begin{bmatrix} \mathbf{D} & \mathbf{K} \\ \mathbf{K} & \end{bmatrix}}_{\mathbf{K_G}} \begin{bmatrix} \lambda\mathbf{\Phi} & \overline{\lambda}\overline{\mathbf{\Phi}} \\ \mathbf{\Phi} & \overline{\mathbf{\Phi}} \end{bmatrix} = \begin{bmatrix} -\lambda & \\ & -\overline{\lambda} \end{bmatrix} \tag{2.17}$$

The (2n x 2n) complex modal matrix $\mathbf{\Phi_G}$ is make up of the n eigenvector-pairs (2.15):

$$\mathbf{\Phi_G} = \begin{bmatrix} \lambda^{(1)}\mathbf{\Phi}^{(1)} & \overline{\lambda}^{(1)}\overline{\mathbf{\Phi}}^{(1)} & \dots & \dots & \lambda^{(n)}\mathbf{\Phi}^{(n)} & \overline{\lambda}^{(n)}\overline{\mathbf{\Phi}}^{(n)} \\ \\ \mathbf{\Phi}^{(1)} & \overline{\mathbf{\Phi}}^{(1)} & \dots & \dots & \mathbf{\Phi}^{(n)} & \overline{\mathbf{\Phi}}^{(n)} \end{bmatrix} \tag{2.18}$$

By use of the orthogonality properties (2.16), (2.17) a modal decomposition of the equations of motion (2.12) can be performed:

$$\underbrace{\mathbf{\Phi}_G^T \begin{bmatrix} \mathbf{M} & \\ & -\mathbf{K} \end{bmatrix} \mathbf{\Phi_G}}_{\mathbf{E}} \dot{\mathbf{A}} + \underbrace{\mathbf{\Phi}_G^T \begin{bmatrix} \mathbf{D} & \mathbf{K} \\ \mathbf{K} & \end{bmatrix} \mathbf{\Phi_G}}_{diag\{-\lambda^{(j)}\}} \mathbf{A} = \mathbf{\Phi}_G^T \begin{bmatrix} \mathbf{p}(t) \\ \\ \end{bmatrix} \tag{2.19}$$

where

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{V} \end{bmatrix} = \mathbf{\Phi_G} \cdot \mathbf{A} = \mathbf{\Phi_G} \cdot \left[a^{(1)}\ b^{(1)} \quad \cdots \quad a^{(n)}\ b^{(n)} \right]^T \tag{2.20}$$

$a^{(j)}$, $b^{(j)}$: new complex variables, belonging to the j[th] eigenpair

In regard to Eqs.(2.8), (2.9) and (2.4b), the differential equations (2.19) can be transformed **in pairs** into the real form of SDOFS-equation (index (j) omitted):

$$\underbrace{\varphi^{-\mathbf{T}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \varphi^{-\mathbf{1}}}_{} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \underbrace{\varphi^{-\mathbf{T}} \begin{bmatrix} -\lambda & 0 \\ 0 & -\overline{\lambda} \end{bmatrix} \varphi^{-\mathbf{1}}}_{} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\varphi^{-\mathbf{T}} \begin{bmatrix} \lambda\mathbf{\Phi}^T\mathbf{p}(t) \\ \overline{\lambda}\overline{\mathbf{\Phi}}^T\mathbf{p}(t) \end{bmatrix}}_{}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -\omega^2 \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 2\eta\omega & \omega^2 \\ \omega^2 & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix} \tag{2.21}$$

where $x^{(j)}$, $y^{(j)}$ are real modal coordinates for each j[th] eigenpair.

The matrix $[\varphi(\omega,\eta)]^{-1}$ of the corresponding j[th] eigenvalue pair $\lambda^{(j)} = \lambda_r^{(j)} \pm i\lambda_i^{(j)}$ can be computed by Eqs. (2.4), (2.5), (2.10), (2.11).

The new (2n x 2n) transformation basis **Y** in real space can be defined by combination of two complex transformations (2.19), (2.21):

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{V} \end{bmatrix} = \mathbf{\Phi_G} \cdot \underbrace{\begin{bmatrix} \left(\varphi^{(\mathbf{1})}\right)^{-\mathbf{1}} & & \\ & \dots & \\ & & \left(\varphi^{(\mathbf{n})}\right)^{-\mathbf{1}} \end{bmatrix}}_{\mathbf{\Psi}^{-1}} \cdot \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ \dots \\ x_n \\ y_n \end{bmatrix}}_{\mathbf{X}} = \underbrace{\mathbf{\Phi_G} \cdot \mathbf{\Psi}^{-1}}_{\mathbf{Y}} \cdot \mathbf{X} = \mathbf{Y} \cdot \mathbf{X} \tag{2.22}$$

By the aid of the relationship (2.22) the equations of motion (2.12) will be transformed into n real uncoupled SDOFS block equations as follows:

$$\underbrace{\mathbf{Y}^{\mathrm{T}}\cdot\begin{bmatrix}\mathbf{M} & \\ & -\mathbf{K}\end{bmatrix}\cdot\mathbf{Y}}_{\begin{bmatrix}1 & & & & \\ & -\omega_1^2 & & & \\ & & \cdots & & \\ & & & 1 & \\ & & & & -\omega_n^2\end{bmatrix}}\cdot\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{y}_1\\ \cdots\\ \dot{x}_n\\ \dot{y}_n\end{bmatrix}}_{\dot{\mathbf{X}}}+\underbrace{\mathbf{Y}^{\mathrm{T}}\cdot\begin{bmatrix}\mathbf{D} & \mathbf{K}\\ \mathbf{K} & \end{bmatrix}\cdot\mathbf{Y}}_{\begin{bmatrix}2\eta_1\,\omega_1 & \omega_1^2 & & & \\ \omega_1^2 & 0 & & & \\ & & \cdots & & \\ & & & 2\eta_n\,\omega_n & \omega_n^2\\ & & & \omega_n^2 & 0\end{bmatrix}}\cdot\underbrace{\begin{bmatrix}x_1\\ y_1\\ \cdots\\ x_n\\ y_n\end{bmatrix}}_{\mathbf{X}}=\underbrace{\mathbf{Y}^{\mathrm{T}}\cdot\begin{bmatrix}\mathbf{p}\\ \end{bmatrix}}_{\begin{bmatrix}g_1\\ h_1\\ \cdots\\ g_n\\ h_n\end{bmatrix}} \tag{2.23}$$

It can be shown that the $\mathbf{Y}$-matrix and all „load“-vectors $[g(t) \quad h(t)]^T$ in Eq. (2.23) are purely real. After component multiplication of the analytically expressed terms of $\mathbf{\Phi}_G$ and of $\mathbf{\Psi}^{-1}$ all imaginary parts cancel each other, see details in [3], [4].

To each j$^{\text{th}}$ eigenvector-pair in the real (2n x 2n) matrix $\mathbf{Y}$ are belonging two columns, denoted like

$$\mathbf{Y}=\begin{bmatrix}\cdots & \mathbf{Y}_{\mathrm{x}}^{(\mathrm{j})\mathbf{W}} & \mathbf{Y}_{\mathrm{y}}^{(\mathrm{j})\mathbf{W}} & \cdots\\ \cdots & \mathbf{Y}_{\mathrm{x}}^{(\mathrm{j})\mathbf{V}} & \mathbf{Y}_{\mathrm{y}}^{(\mathrm{j})\mathbf{V}} & \cdots\end{bmatrix}$$

Their components can be expressed analytically using Eqs.(2.10), (2.11):

$$\begin{aligned}
\mathbf{Y}_{\mathrm{x}}^{(\mathrm{j})\mathbf{W}} &= \frac{1}{\sqrt{1-\eta^2}}\left\{\left(Z_2\,\omega\sqrt{1-\eta^2}+\eta\omega Z_1\right)\mathbf{\Phi}_{\mathbf{r}}+\left(Z_1\,\omega\sqrt{1-\eta^2}-\eta\omega Z_2\right)\mathbf{\Phi}_{\mathbf{i}}\right\}\\
\mathbf{Y}_{\mathrm{y}}^{(\mathrm{j})\mathbf{W}} &= \frac{1}{\sqrt{1-\eta^2}}\left(\omega^2 Z_1\mathbf{\Phi}_{\mathbf{r}}-\omega^2 Z_2\mathbf{\Phi}_{\mathbf{i}}\right)\\
\mathbf{Y}_{\mathrm{x}}^{(\mathrm{j})\mathbf{V}} &= \frac{1}{\sqrt{1-\eta^2}}\left(-Z_1\mathbf{\Phi}_{\mathbf{r}}+Z_2\mathbf{\Phi}_{\mathbf{i}}\right)\\
\mathbf{Y}_{\mathrm{y}}^{(\mathrm{j})\mathbf{V}} &= \frac{1}{\sqrt{1-\eta^2}}\left\{\left(Z_2\,\omega\sqrt{1-\eta^2}-\eta\omega Z_1\right)\mathbf{\Phi}_{\mathbf{r}}+\left(Z_1\,\omega\sqrt{1-\eta^2}+\eta\omega Z_2\right)\mathbf{\Phi}_{\mathbf{i}}\right\}
\end{aligned} \tag{2.24a-d}$$

The two components of the associated “load” vector, Eq. (2.23), are purely real too:

$$g(t)=\frac{\omega}{\sqrt{1-\eta^2}}\left\{\left(Z_2\sqrt{1-\eta^2}+Z_1\eta\right)\mathbf{\Phi}_{\mathbf{r}}^{\mathbf{T}}+\left(Z_1\sqrt{1-\eta^2}-Z_2\,\eta\right)\mathbf{\Phi}_{\mathbf{i}}^{\mathbf{T}}\right\}\mathbf{p}(t) \tag{2.25a}$$

$$h(t)=\frac{\omega^2}{\sqrt{1-\eta^2}}\left\{Z_1\mathbf{\Phi}_{\mathbf{r}}^{\mathbf{T}}-Z_2\,\mathbf{\Phi}_{\mathbf{i}}^{\mathbf{T}}\right\}\mathbf{p}(t) \tag{2.25b}$$

In [5] has been shown that in the case of proportional damping all coordinates of $\mathbf{Y}_{\mathrm{y}}^{(\mathrm{j})\mathbf{W}}$ and $\mathbf{Y}_{\mathrm{x}}^{(\mathrm{j})\mathbf{V}}$ have to be equal to zero:

$$\mathbf{Y}_{\mathbf{y}}^{\mathbf{W}}=\frac{1}{\sqrt{1-\eta^2}}\left(\omega^2 Z_1\mathbf{\Phi}_{\mathbf{r}}-\omega^2 Z_2\mathbf{\Phi}_{\mathbf{i}}\right)=\mathbf{0} \quad\rightarrow\quad -Z_1\mathbf{\Phi}_{\mathbf{r}}+Z_2\mathbf{\Phi}_{\mathbf{i}}=\mathbf{0} \tag{2.26}$$

The relationship (2.26) leads to

$$\rightarrow\quad \frac{\Phi_{i(k)}}{\Phi_{r(k)}}=\frac{Z_1}{Z_2}=\frac{\sqrt{\sqrt{1-\eta^2}+(1-\eta^2)}}{\sqrt{\sqrt{1-\eta^2}-(1-\eta^2)}}=\frac{\eta}{1-\sqrt{1-\eta^2}}=const. \tag{2.27}$$

for all kth DOF of the jth eigenmode pair $(\mathbf{\Phi_r} \pm i\mathbf{\Phi_i})$ with corresponding eigenvalue $(\lambda_r \pm i\lambda_i)$. This is the analytical proof of the statement of a constant phase lag/lead, see Eq. (1.9), (1.10). In the case of proportionally damped system each free vibration is a synchronous motion of all DOF. This special case is considered more detailed in [5], Sec. 3.

2.3 Solution of the modal equations and back transformation

Each SDOFS block equation in (2.23) can be solved eliminating the modal coordinate $x^{(j)}$ to obtain the usual form of the SDOFS equation of motion (index (j) omitted):

$$x = \dot{y} + \frac{1}{\omega^2} h(t) \tag{2.28a}$$

$$\ddot{y} + 2\eta\omega\,\dot{y} + \omega^2 y = g(t) - \frac{2\eta}{\omega} h(t) - \frac{1}{\omega^2} \dot{h}(t) \tag{2.28b}$$

A usual step-by-step integration of Eq. (2.28b) yields the modal response $y^{(j)}(t)$. The final time series of the original n DOFs are calculated by superposition of the modal coordinates $x^{(j)}$, $y^{(j)}$(assembled in **X**) in accordance to Eq. (2.22).

3 MODAL ANALYSIS PROCEDURE BASED ON THE STIFFNESS NORMALIZED COMPLEX RIGHT EIGENVECTORS

The procedure presented here, is a new variant of the method from Sec. 2. For this reason only the differences in comparison to the first variant are described.

3.1 The single mass oscillator

We operate with the same two corresponding complex conjugate eigenvectors $\varphi_{1/2}$, but now they are normalized with respect to the stiffness matrix **k**, see Eq. (2.1), (2.6):

$$[\mathbf{r}_1 \quad \mathbf{r}_2]^T \cdot \mathbf{k} \cdot [\mathbf{r}_1 \quad \mathbf{r}_2] = \begin{bmatrix} h_{11} & \\ & h_{22} \end{bmatrix} \quad \rightarrow \quad \varphi_j = \frac{\mathbf{r}_j}{\sqrt{h_{jj}}} \qquad (j = 1,2) \tag{3.1}$$

The normalization with respect to **k** leads now to the orthogonality relationships (3.2), (3.3):

$$\varphi^{\mathbf{T}}\mathbf{m}\,\varphi = \varphi^{\mathbf{T}}\begin{bmatrix} 1 & 0 \\ 0 & -\omega^2 \end{bmatrix}\varphi = \begin{bmatrix} -\frac{1}{\lambda} & 0 \\ 0 & -\frac{1}{\bar{\lambda}} \end{bmatrix} \quad \leftrightarrow \quad \varphi^{-\mathbf{T}}\begin{bmatrix} -\frac{1}{\lambda} & 0 \\ 0 & -\frac{1}{\bar{\lambda}} \end{bmatrix}\varphi^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -\omega^2 \end{bmatrix} \tag{3.2}$$

$$\varphi^{\mathbf{T}}\,\mathbf{k}\,\varphi = \varphi^{\mathbf{T}}\begin{bmatrix} 2\eta\omega & \omega^2 \\ \omega^2 & 0 \end{bmatrix}\varphi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \leftrightarrow \quad \varphi^{-\mathbf{T}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\varphi^{-1} = \begin{bmatrix} 2\eta\omega & \omega^2 \\ \omega^2 & 0 \end{bmatrix} \tag{3.3}$$

The analytical expression for the inverse of the complex modal matrix $\varphi(\omega,\eta)$ is now:

$$\varphi^{-1} = \frac{1}{2\sqrt{1-\eta^2}}\begin{bmatrix} W_2 - iW_1 & Q + iP \\ W_2 + iW_1 & Q - iP \end{bmatrix} \tag{3.4}$$

where

$$W_1 = \sqrt{\omega\sqrt{1-\eta^2}\left(1-2\eta\sqrt{1-\eta^2}\right)}\,, \quad W_2 = \sqrt{\omega\sqrt{1-\eta^2}\left(1+2\eta\sqrt{1-\eta^2}\right)} \tag{3.5a}$$

$$P = \omega\left(W_2\sqrt{1-\eta^2}-\eta W_1\right)\,, \qquad Q = \omega\left(W_1\sqrt{1-\eta^2}+\eta W_2\right) \tag{3.5b}$$

3.2 The multi-degree-of-freedom-system with damping

The state-space form of equations of motion and the associated eigenvalue problem – Eq. (2.12) - (2.14) remain. The difference now is the normalization of the eigenvectors with respect to the general stiffness matrix $\mathbf{K}_G$**:**

$$\begin{bmatrix}\lambda\mathbf{X}\\ \mathbf{X}\end{bmatrix}^T \mathbf{K}_G \begin{bmatrix}\lambda\mathbf{X}\\ \mathbf{X}\end{bmatrix} = A+iB \qquad \to \quad \mathbf{\Phi} = \frac{\mathbf{X}}{\sqrt{A+iB}} = \mathbf{\Phi}_r + i\mathbf{\Phi}_i\,, \tag{3.6a}$$

$$\begin{bmatrix}\bar{\lambda}\bar{\mathbf{X}}\\ \bar{\mathbf{X}}\end{bmatrix}^T \mathbf{K}_G \begin{bmatrix}\bar{\lambda}\bar{\mathbf{X}}\\ \bar{\mathbf{X}}\end{bmatrix} = A-iB \qquad \to \quad \bar{\mathbf{\Phi}} = \frac{\bar{\mathbf{X}}}{\sqrt{A-iB}} = \mathbf{\Phi}_r - i\mathbf{\Phi}_i\,, \tag{3.6b}$$

Due to the normalization (3.6) we have the orthogonality relationships (3.7), (3.8) – expressed in terms of the j[th] eigenvector-pair (index (j) omitted):

$$\begin{bmatrix}\lambda\mathbf{\Phi} & \bar{\lambda}\bar{\mathbf{\Phi}}\\ \mathbf{\Phi} & \bar{\mathbf{\Phi}}\end{bmatrix}^T \underbrace{\begin{bmatrix}\mathbf{M} & \\ & -\mathbf{K}\end{bmatrix}}_{\mathbf{M}_G} \begin{bmatrix}\lambda\mathbf{\Phi} & \bar{\lambda}\bar{\mathbf{\Phi}}\\ \mathbf{\Phi} & \bar{\mathbf{\Phi}}\end{bmatrix} = \begin{bmatrix}-\frac{1}{\lambda} & \\ & -\frac{1}{\bar{\lambda}}\end{bmatrix} \tag{3.7}$$

$$\begin{bmatrix}\lambda\mathbf{\Phi} & \bar{\lambda}\bar{\mathbf{\Phi}}\\ \mathbf{\Phi} & \bar{\mathbf{\Phi}}\end{bmatrix}^T \underbrace{\begin{bmatrix}\mathbf{D} & \mathbf{K}\\ \mathbf{K} & \end{bmatrix}}_{\mathbf{K}_G} \begin{bmatrix}\lambda\mathbf{\Phi} & \bar{\lambda}\bar{\mathbf{\Phi}}\\ \mathbf{\Phi} & \bar{\mathbf{\Phi}}\end{bmatrix} = \begin{bmatrix}1 & \\ & 1\end{bmatrix} \tag{3.8}$$

The modal matrix $\mathbf{\Phi}_G$ is build up according to Eq. (2.18). By the aid of $\mathbf{\Phi}_G$ the modal decomposition of the equations of motions is performed to:

$$\underbrace{\mathbf{\Phi}_G^T\begin{bmatrix}\mathbf{M} & \\ & -\mathbf{K}\end{bmatrix}\mathbf{\Phi}_G}_{\begin{bmatrix}-\frac{1}{\lambda^{(1)}} & & \\ & \cdots & \\ & & -\frac{1}{\bar{\lambda}^{(n)}}\end{bmatrix}}\,\dot{\mathbf{A}} + \underbrace{\mathbf{\Phi}_G^T\begin{bmatrix}\mathbf{D} & \mathbf{K}\\ \mathbf{K} & \end{bmatrix}\mathbf{\Phi}_G}_{\begin{bmatrix}1 & & \\ & \cdots & \\ & & 1\end{bmatrix}}\,\mathbf{A} = \mathbf{\Phi}_G^T\begin{bmatrix}\mathbf{p}(t)\\ \end{bmatrix} \tag{3.9}$$

In regard to Eqs. (3.2), (3.3) the complex differential equations (3.9) can be transformed **in pairs** into the real form of SDOFS-equation, corresponding to the j[th] eigenpair (index (j) omitted):

$$\underbrace{\varphi^{-\mathrm{T}}\begin{bmatrix} -\frac{1}{\lambda} & \\ & -\frac{1}{\bar{\lambda}} \end{bmatrix}\varphi^{-1}}_{} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \underbrace{\varphi^{-\mathrm{T}}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\varphi^{-1}}_{} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\varphi^{-\mathrm{T}}\begin{bmatrix} \lambda \mathbf{\Phi}^{T}\,\mathbf{p}(t) \\ \bar{\lambda}\bar{\mathbf{\Phi}}^{T}\,\mathbf{p}(t) \end{bmatrix}}_{}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -\omega^2 \end{bmatrix} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 2\eta\omega & \omega^2 \\ \omega^2 & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix} \tag{3.10}$$

The matrix $[\varphi(\omega,\eta)]^{-1}$ in (3.10) for the j$^{\text{th}}$ eigenvalue pair $\lambda^{(j)} = \lambda_r^{(j)} \pm i\lambda_i^{(j)}$ can be computed by Eqs. (3.4),(3.5).

The (2n x 2n) transformation basis **Y** is now defined formally equal to (2.22), but here by use of the complex transformations (3.9), (3.10):

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{V} \end{bmatrix} = \mathbf{\Phi_G} \underbrace{\begin{bmatrix} \left(\varphi^{(1)}\right)^{-1} & & \\ & \cdots & \\ & & \left(\varphi^{(n)}\right)^{-1} \end{bmatrix}}_{\mathbf{\Psi}^{-1}} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ \cdots \\ x_n \\ y_n \end{bmatrix}}_{\mathbf{X}} = \underbrace{(\mathbf{\Phi_G \Psi^{-1}})}_{\mathbf{Y}} \mathbf{X} = \mathbf{Y\,X} \tag{3.11}$$

By the aid of the modal transformation basis **Y** from (3.11) the modal form of the equations of motion (2.23) remains.

In this case the components of two columns in the real matrix **Y**, belonging to the j$^{\text{th}}$ eigenvector-pair, are - see also Eq. (3.5a-b):

$$\mathbf{Y}_{\mathrm{x}}^{(\mathrm{j})\mathbf{W}} = \frac{\omega}{\sqrt{1-\eta^2}}\left[\left(-\eta\, W_2 + \sqrt{1-\eta^2}\, W_1\right)\mathbf{\Phi_r} - \left(\eta\, W_1 + \sqrt{1-\eta^2}\, W_2\right)\mathbf{\Phi_i}\right]$$

$$\mathbf{Y}_{y}^{(\mathrm{j})\mathbf{W}} = \frac{\omega^2}{\sqrt{1-\eta^2}}\left[\left(-\eta\, Q - \sqrt{1-\eta^2}\, P\right)\mathbf{\Phi_r} + \left(\eta\, P - \sqrt{1-\eta^2}\, Q\right)\mathbf{\Phi_i}\right] \tag{3.12a-d}$$

$$\mathbf{Y}_{x}^{(\mathrm{j})\mathbf{V}} = \frac{1}{\sqrt{1-\eta^2}}[W_2\mathbf{\Phi_r} + W_1\mathbf{\Phi_i}]$$

$$\mathbf{Y}_{y}^{(\mathrm{j})\mathbf{V}} = \frac{\omega}{\sqrt{1-\eta^2}}[Q\mathbf{\Phi_r} - P\mathbf{\Phi_i}]$$

The two components of the associated "load" vector are now – by use of Eq.(3.5a-b):

$$g(t) = \frac{\omega}{\sqrt{1-\eta^2}}\left[\left(-\eta\, W_2 + \sqrt{1-\eta^2}\, W_1\right)\mathbf{\Phi_r^T} - \left(\eta\, W_1 + \sqrt{1-\eta^2}\, W_2\right)\mathbf{\Phi_i^T}\right]\mathbf{p}(t)$$

$$h(t) = \frac{\omega^2}{\sqrt{1-\eta^2}}\left[\left(-\eta\, Q - \sqrt{1-\eta^2}\, P\right)\mathbf{\Phi_r^T} + \left(\eta\, P - \sqrt{1-\eta^2}\, Q\right)\mathbf{\Phi_i^T}\right]\mathbf{p}(t) \tag{3.13a-b}$$

The numerical solution of each SDOFS block equation and the final back transformation to the original DOFs remain the same like in Sec. 2 – in accordance to Eqs. (2.28), (3.11).

The phase angle $\varphi_k = arctan\frac{\Phi_{i(k)}}{\Phi_{r(k)}}$ in the special case of proportional damping can be determine from the restriction that all coordinates of $\mathbf{Y}_{\mathbf{x}}^{(j)V}$ have to be equal to zero, see [5]:

$$\mathbf{Y}_x^V = \frac{1}{\sqrt{1-\eta^2}}(W_2\mathbf{\Phi_r} + W_1\mathbf{\Phi_i}) = \mathbf{0} \tag{3.14}$$

The relationship leads together with (3.5a,b) to

$$\rightarrow \quad \frac{\Phi_{i(k)}}{\Phi_{r(k)}} = -\frac{W_2}{W_1} = \frac{1+2\eta\sqrt{1-\eta^2}}{2\eta^2-1} = const. \tag{3.15}$$

Comparing this result to Eq.(2.27), the constant phase lag depends obviously on the kind of the eigenmode normalization.

4 MODAL ANALYSIS PROCEDURE BASED ON BOTH THE COMPLEX RIGHT AND LEFT EIGENVECTORS

The procedure described in this section has been presented in [3] in slightly different form. The variant here will be developed without normalization of the eigenvectors of the SDOFS.

4.1 The single mass oscillator

The form (2.3) of the general eigenvalue problem and his solution – Eq. (2.4), (2.5), remain the same. The eigenvectors $\mathbf{r_j}$ $(j=1,2)$ are **right eigenvectors** of the matrix $\mathbf{a}=\mathbf{m}^{-1}\mathbf{k}$ **.**

In order to determine the left eigenvectors of the matrix $\mathbf{a}$ the substitution

$$\underbrace{\begin{bmatrix} f_w \\ f_v \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} 2\eta\omega & \omega^2 \\ \omega^2 & 0 \end{bmatrix}}_{\mathbf{k}} \underbrace{\begin{bmatrix} w \\ v \end{bmatrix}}_{\mathbf{q}} \tag{4.1a}$$

$$\mathbf{f}=\mathbf{k}\mathbf{q}, \qquad \dot{\mathbf{f}}=\mathbf{k}\dot{\mathbf{q}} \tag{4.1b}$$

has been introduced. The equation of motion (2.1b) is transformed to

$$\underbrace{\mathbf{k}\dot{\mathbf{q}}} \quad + \quad \mathbf{k}(\mathbf{m}^{-1}\underbrace{\mathbf{k})\mathbf{q}} \quad = \quad \mathbf{k}\mathbf{m}^{-1}\mathbf{p}$$

$$\dot{\mathbf{f}} \quad + \quad \mathbf{k}\mathbf{m}^{-1}\mathbf{f} \quad = \quad \mathbf{k}\mathbf{m}^{-1}\mathbf{p}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{e}} \underbrace{\begin{bmatrix} \dot{f}_w \\ \dot{f}_v \end{bmatrix}}_{\dot{\mathbf{f}}} + \underbrace{\begin{bmatrix} 2\eta\omega & -1 \\ \omega^2 & 0 \end{bmatrix}}_{\mathbf{k}\mathbf{m}^{-1}} \underbrace{\begin{bmatrix} f_w \\ f_v \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} 2\eta\omega & -1 \\ \omega^2 & 0 \end{bmatrix}}_{\mathbf{k}\mathbf{m}^{-1}} \underbrace{\begin{bmatrix} p(t) \\ 0 \end{bmatrix}}_{\mathbf{p}(t)} \tag{4.2}$$

The corresponding eigenvalue problem is formulated assuming

$$\mathbf{f}=\mathbf{x}e^{\lambda t}, \qquad \dot{\mathbf{f}}=\lambda\mathbf{x}e^{\lambda t} \tag{4.3}$$

$$\rightarrow \quad \Big(\underbrace{\mathbf{k\,m}^{-1}}_{\mathbf{a}^{\mathrm{T}}} + \lambda\,\mathbf{e}\Big)\mathbf{x} = \mathbf{0} \qquad \rightarrow \quad \left(\mathbf{a}^{\mathrm{T}} + \lambda\,\mathbf{e}\right)\mathbf{x} = \mathbf{0} \tag{4.4}$$

where

$$\mathbf{a}^{\mathrm{T}} = \left(\mathbf{m}^{-1}\mathbf{k}\right)^{\mathrm{T}} = \mathbf{k}^{\mathrm{T}}\left(\mathbf{m}^{-1}\right)^{\mathrm{T}} = \mathbf{k\,m}^{-1} = \begin{bmatrix} 2\eta\omega & -1 \\ \omega^2 & 0 \end{bmatrix} \tag{4.5}$$

due to the symmetry of the matrices **k** and **m**. The two eigenvalues $\lambda_{1/2}$ of (4.4) remain the same - Eq. (2.4), but the corresponding complex conjugate eigenvectors are now

$$\mathbf{x}_{1/2} = \mathbf{x}_r \pm i\mathbf{x}_i = \begin{bmatrix} \frac{\eta \mp i\sqrt{1-\eta^2}}{\omega} \\ 1 \end{bmatrix} \tag{4.6}$$

The formulation

$$\mathbf{x}^{\mathrm{T}}\left(\mathbf{a} + \lambda\mathbf{e}\right) = \mathbf{0} \qquad \rightarrow \quad \left(\mathbf{a}^{\mathrm{T}} + \lambda\,\mathbf{e}\right)\mathbf{x} = \mathbf{0} \quad \rightarrow \quad \left(\mathbf{a}^{\mathrm{T}} + \lambda_j\,\mathbf{e}\right)\mathbf{l_j} = \mathbf{0} \tag{4.7}$$

shows that **x** represents here the **left eigenvectors** of the matrix $\mathbf{a} = \mathbf{m}^{-1}\mathbf{k}$ (respectively, the right eigenvectors of the matrix $\mathbf{a}^{\mathrm{T}} = \mathbf{k\,m}^{-1}$), i.e. $\mathbf{x_j} = \mathbf{l_j}$ (j=1,2) .

In this variant the modal matrix is defined without normalization by

$$\varphi^{\mathrm{L}} = [\mathbf{l}_1 \quad \mathbf{l}_2] = \begin{bmatrix} \frac{\eta - i\sqrt{1-\eta^2}}{\omega} & \frac{\eta + i\sqrt{1-\eta^2}}{\omega} \\ 1 & 1 \end{bmatrix} \tag{4.8}$$

The eigenvalue problem in the „left“ formulation (4.7) may be rewritten using the φ^{L} modal matrix as:

$$\mathbf{a}^{\mathbf{T}}\varphi^{\mathrm{L}} + \varphi^{\mathrm{L}}\boldsymbol{\lambda} = \mathbf{0} \tag{4.9}$$

where $$\boldsymbol{\lambda} = \begin{bmatrix} \lambda & \\ & \bar{\lambda} \end{bmatrix} \tag{4.10}$$

represents the spectral matrix of $\mathbf{a}^{\mathrm{T}}$. From Eq. (4.9) the diagonalization of the $\mathbf{a}^{\mathrm{T}}$ matrix and the inverse relation can be developed:

$$\mathbf{a}^{\mathrm{T}} \;=\; \varphi^{\mathrm{L}}\cdot(-\boldsymbol{\lambda})\cdot\left(\varphi^{\mathrm{L}}\right)^{-1} \qquad \leftrightarrow \qquad -\boldsymbol{\lambda} \;=\; \left(\varphi^{\mathrm{L}}\right)^{-1}\cdot\mathbf{a}^{\mathrm{T}}\cdot\left(\varphi^{\mathrm{L}}\right) \tag{4.11}$$

The inverse matrix $\left(\varphi^{\mathrm{L}}\right)^{-1}$ is calculated now to:

$$(\varphi^{\mathrm{L}})^{-1} = \frac{1}{2\sqrt{1-\eta^2}}\begin{bmatrix} i\omega & \sqrt{1-\eta^2} - i\eta \\ -i\omega & \sqrt{1-\eta^2} + i\eta \end{bmatrix} \tag{4.12}$$

4.2 Modal decomposition of a multi-degree-of-freedom-system with damping

The state-space form of equations of motion (2.12) is transformed in two variants using the substitution

$$\mathbf{F} = \mathbf{K_G}\,\mathbf{Q}\,, \qquad \dot{\mathbf{F}} = \mathbf{K_G}\,\dot{\mathbf{Q}} \tag{4.13}$$

1) $$\dot{\mathbf{Q}} + \underbrace{\mathbf{M_G^{-1}} \cdot \mathbf{K_G}}_{\mathbf{A}} \cdot \mathbf{Q} = \mathbf{M_G^{-1}} \cdot \mathbf{P} \tag{4.14a}$$

2) $$\begin{aligned} \underbrace{\mathbf{K_G}\,\dot{\mathbf{Q}}} \quad &+ \quad \mathbf{K_G}\,\mathbf{M_G^{-1}}\underbrace{\mathbf{K_G}\,\mathbf{Q}} \quad = \quad \mathbf{K_G}\,\mathbf{M_G^{-1}}\,\mathbf{P} \\ \dot{\mathbf{F}} \quad &+ \quad \mathbf{K_G}\,\mathbf{M_G^{-1}}\,\mathbf{F} \quad = \quad \mathbf{K_G}\,\mathbf{M_G^{-1}}\,\mathbf{P} \end{aligned} \tag{4.14b}$$

Because of the symmetry of $\mathbf{M_G}$ and $\mathbf{K_G}$ we receive the relationship

$$\mathbf{K_G}\,\mathbf{M_G^{-1}} = \mathbf{K_G^T}\left(\mathbf{M_G^{-1}}\right)^{\mathbf{T}} = \left(\underbrace{\mathbf{M_G^{-1}}\mathbf{K_G}}_{\mathbf{A}}\right)^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}} \tag{4.15}$$

Here we need the right and the left eigenvectors of the matrix **A**, to be calculated from the form (4.14a) resp.(4.14b):

$$\left(\mathbf{A} + \lambda_j \mathbf{E}\right)\mathbf{R}_j = \mathbf{0} \qquad \text{resp.} \qquad \left(\mathbf{A}^T + \lambda_j \mathbf{E}\right)\mathbf{L}_j = \mathbf{0} \tag{4.16a-b}$$

The formulations (4.14a) and (4.14b) yield directly the relationship between an arbitrary j^{th} right and left eigenvector

$$\mathbf{L_j} = \mathbf{K_G}\;\mathbf{R_j} \tag{4.17}$$

The right and the left modal matrix, respectively, are complete sets of the corresponding n eigenpairs

$$\mathbf{R} = \left[\mathbf{R_1}\;\overline{\mathbf{R}}_\mathbf{1} \qquad \ldots \quad \mathbf{R_n}\;\overline{\mathbf{R}}_\mathbf{n}\right] \tag{4.18a}$$

$$\mathbf{L} = \left[\mathbf{L_1}\;\overline{\mathbf{L}}_\mathbf{1} \qquad \ldots \quad \mathbf{L_n}\;\overline{\mathbf{L}}_\mathbf{n}\right] \tag{4.18b}$$

The orthogonality property of the eigenvectors leads to

$$\mathbf{R^T L} = \mathbf{R^T K_G R} = \begin{bmatrix} \gamma_{11} & & & & \\ & \overline{\gamma}_{11} & & & \\ & & \cdots & & \\ & & & \gamma_{nn} & \\ & & & & \overline{\gamma}_{nn} \end{bmatrix} \tag{4.19}$$

Using the main diagonal components γ_{kk} from Eq. (4.19) the modal matrices $\mathbf{R}$, $\mathbf{L}$ are normalized with respect to the general stiffness matrix $\mathbf{K_G}$:

$$\boldsymbol{\Phi}_k^{\mathbf{R}} = \frac{1}{\sqrt{\gamma_{kk}}}\mathbf{R_k} \qquad \rightarrow \quad \boldsymbol{\Phi}^{\mathbf{R}} = \left[\boldsymbol{\Phi}_\mathbf{1}^{\mathbf{R}}\;\;\overline{\boldsymbol{\Phi}}_\mathbf{1}^{\mathbf{R}} \qquad \ldots \quad \boldsymbol{\Phi}_\mathbf{n}^{\mathbf{R}}\;\;\overline{\boldsymbol{\Phi}}_\mathbf{n}^{\mathbf{R}}\right] \tag{4.20a}$$

$$\mathbf{\Phi}_k^{\mathbf{L}} = \frac{1}{\sqrt{\gamma_{kk}}} \mathbf{L_k} \quad \rightarrow \quad \mathbf{\Phi}^{\mathbf{L}} = \left[\mathbf{\Phi}_1^{\mathbf{L}} \;\; \overline{\mathbf{\Phi}}_1^{\mathbf{L}} \quad \dots \quad \mathbf{\Phi}_n^{\mathbf{L}} \;\; \overline{\mathbf{\Phi}}_n^{\mathbf{L}} \right] \tag{4.20b}$$

Due to (4.19) and the chosen type of normalization (4.20) we receive the relationships

$$\left(\mathbf{\Phi}^{\mathbf{L}}\right)^{\mathbf{T}} \cdot \mathbf{\Phi}^{\mathbf{R}} = \left(\mathbf{\Phi}^{\mathbf{R}}\right)^{\mathbf{T}} \cdot \mathbf{\Phi}^{\mathbf{L}} = \mathbf{E} \tag{4.21}$$

$$\mathbf{\Phi}^{\mathbf{R}} = \left[\left(\mathbf{\Phi}^{\mathbf{L}}\right)^{\mathbf{T}}\right]^{-1} = \left(\mathbf{\Phi}^{\mathbf{L}}\right)^{-\mathbf{T}} \quad \leftrightarrow \quad \left(\mathbf{\Phi}^{\mathbf{R}}\right)^{\mathbf{T}} = \left(\mathbf{\Phi}^{\mathbf{L}}\right)^{-1} \tag{4.22}$$

where $\mathbf{E}$ is a (2n x 2n) identity matrix.

The eigenvalue problem (4.16b) may be rewritten using the $\mathbf{\Phi}^{\mathbf{L}}$ - modal matrix:

$$\left(\mathbf{A}^{\mathbf{T}} + \lambda_k \mathbf{E}\right)\mathbf{\Phi}_k^{\mathbf{L}} = \mathbf{0} \quad \rightarrow \quad \mathbf{A}^{\mathbf{T}} \cdot \mathbf{\Phi}^{\mathbf{L}} + \mathbf{\Phi}^{\mathbf{L}} \cdot \mathbf{\Lambda} = \mathbf{0} \tag{4.23}$$

where

$\mathbf{\Lambda} = diag\{\lambda_j\} \quad (j = 1, \dots, 2n)$: spectral matrix

From Eq. (4.23) can be derived the diagonalization of the $\mathbf{A}^{\mathbf{T}}$-matrix and the associated inverse relationship:

$$\mathbf{A}^{\mathbf{T}} = \mathbf{\Phi}^{\mathbf{L}} \cdot (-\mathbf{\Lambda}) \cdot \left(\mathbf{\Phi}^{\mathbf{L}}\right)^{-1} \quad \leftrightarrow \quad -\mathbf{\Lambda} = \left(\mathbf{\Phi}^{\mathbf{L}}\right)^{-1} \cdot \mathbf{A}^{\mathbf{T}} \cdot \mathbf{\Phi}^{\mathbf{L}} \tag{4.24}$$

The modal decomposition of the system equations (4.14b) is based on the modal superposition relationship

$$\mathbf{F} = \mathbf{\Phi}^{\mathbf{L}} \cdot \underbrace{\left[a_1 \quad b_1 \quad \dots \quad a_n \quad b_n\right]^T}_{\mathbf{B}^{\mathbf{L}}} = \mathbf{\Phi}^{\mathbf{L}} \cdot \mathbf{B}^{\mathbf{L}} \tag{4.25a}$$

where

$$\mathbf{B}^{\mathbf{L}} = \left[a_1 \quad b_1 \quad \dots \quad a_n \quad b_n\right]^T \quad \text{: new modal complex coordinates} \tag{4.25b}$$

The equations of motion (4.14b) are transformed into a set of 2n uncoupled complex equations:

$$\underbrace{\left(\mathbf{\Phi}^{\mathbf{L}}\right)^{-1} \mathbf{E} \mathbf{\Phi}^{\mathbf{L}}}_{\begin{bmatrix} 1 & & \\ & \dots & \\ & & 1 \end{bmatrix}} \dot{\mathbf{B}}^{\mathbf{L}} + \underbrace{\left(\mathbf{\Phi}^{\mathbf{L}}\right)^{-1} \overbrace{\left(\mathbf{K_G} \mathbf{M_G^{-1}}\right)}^{\mathbf{A}^{\mathbf{T}}} \mathbf{\Phi}^{\mathbf{L}}}_{-\mathbf{\Lambda} = \begin{bmatrix} -\lambda_1 & & & & \\ & -\bar{\lambda}_1 & & & \\ & & \dots & & \\ & & & -\lambda_n & \\ & & & & -\bar{\lambda}_n \end{bmatrix}} \mathbf{B}^{\mathbf{L}} = \underbrace{\left(\mathbf{\Phi}^{\mathbf{L}}\right)^{-1} \mathbf{A}^{\mathbf{T}} \mathbf{P}}_{\mathbf{P}^{\mathbf{L}} = \begin{bmatrix} p_{a1}^L \\ p_{b1}^L \\ \dots \\ p_{an}^L \\ p_{bn}^L \end{bmatrix}} \tag{4.26}$$

Each j-th pair of the n uncoupled equations (4.26) can now be transformed using Eq. (4.11) (index (j) omitted):

$$\underbrace{\varphi^{\mathrm{L}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\left(\varphi^{\mathrm{L}}\right)^{-1}}_{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}} \cdot \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}^{L} + \underbrace{\varphi^{\mathrm{L}}\begin{bmatrix}-\lambda & 0\\ 0 & -\bar{\lambda}\end{bmatrix}\left(\varphi^{\mathrm{L}}\right)^{-1}}_{\left(\mathbf{a}^{T}\right)\ \begin{bmatrix}2\eta\omega & -1\\ \omega^{2} & \end{bmatrix}} \cdot \begin{bmatrix}x\\ y\end{bmatrix}^{L} = \underbrace{\varphi^{\mathrm{L}}\begin{bmatrix}p_a^L\\ p_b^L\end{bmatrix}}_{\begin{bmatrix}g_L\\ h_L\end{bmatrix}} \qquad (4.27)$$

The free vibration frequency $\omega^{(j)}$ and the modal damping ratio $\eta^{(j)}$ for the j[th] eigenpair $\left(\lambda^{(j)} = \lambda_r + i\lambda_i\ ,\ \ \bar{\lambda}^{(j)} = \lambda_r - i\lambda_i\right)$ are to be computed according to Eq. (2.4). The corresponding matrices $\varphi^{\mathrm{L}}, (\varphi^{\mathrm{L}})^{-1}$ can be evaluated by Eq. (4.8), (4.12).

A purely real transformation basis $\mathbf{Y_L}$ will be built up by combination of the complex transformations (4.26) and (4.27):

$$\mathbf{F} = \mathbf{K}_G\begin{bmatrix}\mathbf{W}\\ \mathbf{V}\end{bmatrix} = \mathbf{\Phi}^{\mathbf{L}} \underbrace{\begin{bmatrix}\left(\varphi^{\mathrm{L}(1)}\right)^{-1} & & \\ & \cdots & \\ & & \left(\varphi^{\mathrm{L}(n)}\right)^{-1}\end{bmatrix}}_{(\mathbf{\Psi}_{\mathrm{L}})^{-1}} \begin{bmatrix}x_1\\ y_1\\ \cdots\\ x_n\\ y_n\end{bmatrix}_{\mathrm{L}} = \underbrace{\mathbf{\Phi}^{\mathbf{L}}\,(\mathbf{\Psi}_{\mathrm{L}})^{-1}}_{\mathbf{Y_L}}\mathbf{X_L} = \mathbf{Y_L}\,\mathbf{X_L} \qquad (4.28)$$

In the product of the two complex matrices $\mathbf{\Phi}^{\mathbf{L}} \cdot \left(\mathbf{\Psi_L}\right)^{-1}$, Eq.(4.28), the imaginary parts cancel each other, the resulting transformation matrix $\mathbf{Y_L}$ is purely real – see Eq. (4.30). The same applies to all "load" vectors $\left[g_L \quad h_L\right]^T$ in Eq. (4.27).

Finally the equations of motion (4.14b) can be uncoupled by means of the transformation basis $\mathbf{Y_L}$ into n SDOFS block equations in real arithmetic:

$$\underbrace{\left(\mathbf{Y_L}\right)^{-1}\cdot\mathbf{E}\cdot\mathbf{Y_L}}_{\begin{bmatrix}1 & & & \\ & 1 & & \\ & & \cdots & \\ & & & 1\end{bmatrix}} \cdot \dot{\mathbf{X}}_{\mathbf{L}}(t) + \underbrace{\left(\mathbf{Y_L}\right)^{-1}\cdot\mathbf{A^T}\cdot\mathbf{Y_L}}_{\begin{bmatrix}2\eta^{(1)}\omega^{(1)} & -1 & & & \\ \left(\omega^{(1)}\right)^2 & 0 & & & \\ & & \cdots & & \\ & & & 2\eta^{(n)}\omega^{(n)} & -1\\ & & & \left(\omega^{(n)}\right)^2 & 0\end{bmatrix}} \cdot \underbrace{\mathbf{X_L}(t)}_{\begin{bmatrix}x_1\\ y_1\\ \cdots\\ x_n\\ y_n\end{bmatrix}_L} = \underbrace{\left(\mathbf{Y_L}\right)^{-1}\cdot\mathbf{A^T}\cdot\mathbf{P}(t)}_{\begin{bmatrix}g_L^{(1)}\\ h_L^{(1)}\\ \cdots\\ g_L^{(n)}\\ h_L^{(n)}\end{bmatrix}} \qquad (4.29)$$

The components of $\mathbf{Y_L}$, which belong to the j[th] eigenvector-pair, are the following two columns - see (4.28), (4.12):

$$\begin{bmatrix}\cdots & (\mathbf{Y}_x)^{(j)} & \left(\mathbf{Y}_y\right)^{(j)} & \cdots\end{bmatrix} = [\mathbf{\Phi}_{\mathrm{r}}^{\mathbf{L}} + i\mathbf{\Phi}_{\mathrm{i}}^{\mathbf{L}} \quad \mathbf{\Phi}_{\mathrm{r}}^{\mathbf{L}} - i\mathbf{\Phi}_{\mathrm{i}}^{\mathbf{L}}] \cdot \frac{1}{2\sqrt{1-\eta^2}}\begin{bmatrix}i\omega & \sqrt{1-\eta^2} - i\eta\\ -i\omega & \sqrt{1-\eta^2} + i\eta\end{bmatrix}$$

$$= \frac{1}{\sqrt{1-\eta^2}}\begin{bmatrix}\cdots & -\omega\,\mathbf{\Phi}_{\mathrm{i}}^{\mathbf{L}} & \left(\mathbf{\Phi}_r^{\mathbf{L}}\sqrt{1-\eta^2} + \mathbf{\Phi}_{\mathrm{i}}^{\mathbf{L}}\,\eta\right) & \cdots\end{bmatrix} \qquad (4.30)$$

The associated two components of the "load" vector from (4.29) are calculated - with regard to (4.8) - to be fully real too:

$$\begin{bmatrix} \cdots \\ g_L \\ h_L \\ \cdots \end{bmatrix} = (\mathbf{Y_L})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P} = \mathbf{\Psi_L}\underbrace{(\mathbf{\Phi^L})^{-1}}_{(\mathbf{\Phi^R})^{\mathrm{T}}}\mathbf{A}^{\mathrm{T}}\mathbf{P} = \begin{bmatrix} \cdots \\ \frac{2}{\omega}\left(\eta\,(\mathbf{\Phi_r^R})^{\mathrm{T}} + \sqrt{1-\eta^2}\left(\mathbf{\Phi_i^R}\right)^{\mathrm{T}}\right) \\ 2(\mathbf{\Phi_r^R})^{\mathrm{T}} \\ \cdots \end{bmatrix}\mathbf{A}^{\mathrm{T}}\mathbf{P} \tag{4.31}$$

4.3 Solution of the modal equations and back transformation

The solution of each j[th] SDOFS block equation in (4.29) is performed eliminating first the modal coordinate $x^{(j)}$ (index (j) omitted):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 2\eta\omega & -1 \\ \omega^2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g_L(t) \\ h_L(t) \end{bmatrix} \tag{4.32}$$

The second equation

$$x = -\frac{\dot{y}}{\omega^2} + \frac{h_L}{\omega^2} \tag{4.33a}$$

should be introduced into the first one:

$$\ddot{y} + 2\eta\omega\,\dot{y} + \omega^2\,y = -\omega^2\,g_L + 2\eta\omega\,h_L + \dot{h}_L \tag{4.33b}$$

The modal response $y(t)$ is easy determined by step-by-step integration of Eq. (4.33b), then the $x(t)$ according to (4.33a). The final time series of the original n DOFs are calculated by superposition of the modal coordinates in accordance to Eq. (4.28):

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{V} \end{bmatrix} = (\mathbf{K}_G)^{-1}\,\mathbf{Y_L}\,\mathbf{X_L} \tag{4.34}$$

5 NUMERICAL EXAMPLE

5.1 Structural system, stiffness and geometry data

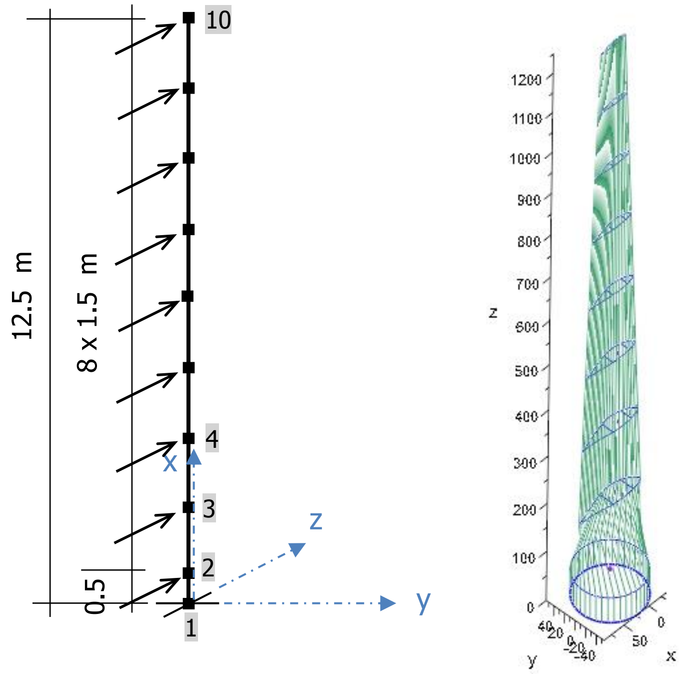


Fig. 1 Rotor blade beam model subjected to wind loads

The stiffness data of the blade thin wall cross sections have been calculated in [14]. The generic aerodynamic blade geometry has been derived from real blade data.

The finite element solution is based on the numerical integration of the system of differential equations for the Bernoulli-beam. The reference axis of the beam model coincides with the centre of the circular-section at the root – it is the real rotational axis of the rotor blade. Thereby the differential equations and all cross section stiffness data are referred to this axis, accounting for the eccentric mass application.

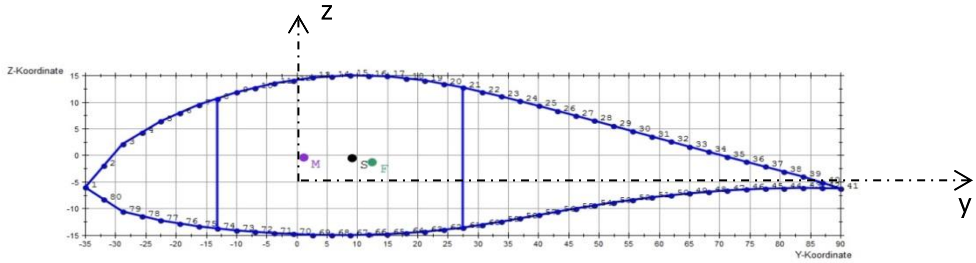


Fig. 2 Rotor blade sections at 2.0 m – thin wall cross section model

5.2 Wind loads

The wind loads are calculated according to the formula for the aerodynamic lift force per unit length of an aerofoil, see [13] p.59:

$$L = \frac{1}{2}\rho \cdot c(r) \cdot W^2 \cdot C_L \tag{5.1}$$

where: W : air velocity relative to the aerofoil
ρ : air density = 1.225 [kg/m³]
$c(r)$: chord of the aerofoil
C_L : lift coefficient $C_L = 2\pi\,\alpha = 2\pi\left(\frac{\pi}{180}6.0\right) = 0.658$,
the flow angle α is assumed to be 6.0 [deg]

The air velocity W is the vector sum of the rotational speed Ω (assumed to reach 60 rpm in the initial four seconds) and the wind speed u, incident on the aerofoil in accordance with the Betz-theory:

$$W = \sqrt{(\Omega\, r)^2 + \left(\frac{2}{3}u\right)^2} \qquad \text{where } \Omega = \left(\frac{60}{30}\pi\right) \text{ in [rad/s]} \tag{5.2}$$

The wind speed time series $u(t)$, used for calculation of the wind thrust force, is shown in fig.3 :

Fig. 3 Wind speed time series

The resulting wind thrust loads $T(t)$ per unit length along the x-axis of the rotor blade can be determined as function of the wind speed $u(t)$. In the structural model the wind thrust loads are acting as summarized nodal forces.

Fig. 4 Wind thrust function acting on the rotor blade at 12.5 m

5.3 Relationships and data for the damping approach

Starting point of the computation are the equations of motion

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{W}} \\ \dot{\mathbf{V}} \end{bmatrix} + \begin{bmatrix} \mathbf{D} & \mathbf{K} \\ \mathbf{K} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{V} \end{bmatrix} = \begin{bmatrix} \mathbf{P}(t) \\ \mathbf{0} \end{bmatrix} \tag{5.3}$$

where $\mathbf{P}(t)$ is the nodal force vector, representing the wind thrust according to Sec. 5.2.

The system equations (5.3) will be solved applying the proposed modal analysis method in Sec. 3 for two cases: non-proportional and proportional damping.

The lowest four free-vibration frequencies and associated periods for the undamped system are calculated to

$$\begin{aligned} f_1 &= 2.643\ [s^{-1}] & T_1 &= 0.378\ [s] \\ f_2 &= 4.622\ [s^{-1}] & T_2 &= 0.216\ [s] \\ f_3 &= 7.942\ [s^{-1}] & T_3 &= 0.126\ [s] \\ f_4 &= 16.650\ [s^{-1}] & T_4 &= 0.060\ [s] \end{aligned} \tag{5.4}$$

Stiffness proportional damping as a special case of Rayleigh damping has been assumed:

$$\mathbf{D_p} = \beta\, \mathbf{K} \tag{5.5a}$$

where $$\beta = \frac{2\eta}{\omega_1} = \frac{\eta\, T_1}{\pi} = 0.000964[s] \tag{5.5b}$$

In Eq. (5.5b) the damping ratio $\eta = 0.008$ for the first natural period T_1 has been taken in accordance with [13] p. 249.

The non-proportional symmetric damping matrix $\mathbf{D_{np}}$ is build adding to the $\mathbf{D_p}$-matrix a new matrix $\mathbf{D_a}$, which represents the aerodynamic damping. The formulation is based on a simple expression for the aerodynamic damping coefficient $c_d(r)$ per unit length, given in [13], p. 247:

$$c_d(r) = \frac{1}{2}\rho \cdot \Omega r \cdot c(r) \cdot \frac{dC_L}{d\alpha} \quad \left[\frac{kg}{s}\frac{1}{m}\right], \quad \text{where} \quad \frac{dC_L}{d\alpha} = 2\pi \tag{5.6}$$

In accordance with Eq. (5.1), (5.2), the corresponding damping coefficients $c_d(r)$ along the x-axis of the rotor blade are calculated to

r [m]	$c(r)$ [m]	$c_d(r)$ [kg/s.m]
0	1.1	0
0.5	1.1	13.2993
2.0	1.25	60.4513
3.5	1.15	97.3266
5.0	1.05	126.948
6.5	0.95	149.315
8.0	0.85	164.428
9.5	0.75	172.286
11.0	0.65	172.891
12.5	0.55	166.241

The coefficients $c_d(r)$, which represent the aerodynamic damping, are active for vibration in z-direction of the cross-section coordinate system, see Fig. 2. The associate symmetric damping matrix for the Bernoulli-beam element is derived by analogy with the method used to derive the finite element mass matrix, see [15]. Finally the symmetric system damping matrix, $\mathbf{D_{np}}$, is assembled in a finite-element manner, including structural (proportional) and aerodynamic damping:

$$\mathbf{D_{np}} = \mathbf{D_p} + \mathbf{D_a} \tag{5.7}$$

5.4 Non-proportional damped system

We use here the matrix $\mathbf{D_{np}}$ – Eq.(5.7). The vector of the first ten complex conjugate eigenvalue pairs of the matrix $\mathbf{A} = \mathbf{M_G^{-1}} \cdot \mathbf{K_G}$, see Eq.(4.16), is

$$\begin{pmatrix} -5.56181 + 15.7652\,i \\ -5.56181 - 15.7652\,i \\ -0.40981 + 29.0336\,i \\ -0.40981 - 29.0336\,i \\ -6.33469 + 49.2454\,i \\ -6.33469 - 49.2454\,i \\ -9.53814 + 104.542\,i \\ -9.53814 - 104.542\,i \\ -5.43041 + 105.219\,i \\ -5.43041 - 105.219\,i \\ -20.7608 + 185.185\,i \\ -20.7608 - 185.185\,i \\ -22.1068 + 207.402\,i \\ -22.1068 - 207.402\,i \\ -27.8796 + 238.91\,i \\ -27.8796 - 238.91\,i \\ -45.8047 + 292.379\,i \\ -45.8047 - 292.379\,i \\ -63.5216 + 353.962\,i \\ -63.5216 - 353.962\,i \end{pmatrix} \tag{5.8}$$

The number of modes considered in the modal transformation is limited to the first four eigenvector pairs. The structural system in Fig. 1 has 54 DOF. The corresponding (108x8) modal matrix $\mathbf{\Phi_G}$ with stiffness normalized eigenvectors – Eq.(3.6a,b), is computed to (only the first ten rows are printed)

$$\begin{pmatrix}
-3.49\,10^{-7}-5.71\,10^{-7}i & -3.49\,10^{-7}+5.71\,10^{-7}i & 9.6\,10^{-5}+1.89\,10^{-8}i & 9.6\,10^{-5}-1.89\,10^{-8}i & -3.85\,10^{-8}+5.43\,10^{-7}i & -3.85\,10^{-8}-5.43\,10^{-7}i & -3.83\,10^{-6}-2.72\,10^{-5}i & -3.83\,10^{-6}+2.72\,10^{-5}i \\
-4.4\,10^{-7}-7.21\,10^{-7}i & -4.4\,10^{-7}+7.21\,10^{-7}i & 1.21\,10^{-4}+2.39\,10^{-8}i & 1.21\,10^{-4}-2.39\,10^{-8}i & -4.74\,10^{-8}+6.9\,10^{-7}i & -4.74\,10^{-8}-6.9\,10^{-7}i & -5.13\,10^{-6}-3.48\,10^{-5}i & -5.13\,10^{-6}+3.48\,10^{-5}i \\
3.38\,10^{-5}+5.85\,10^{-6}i & 3.38\,10^{-5}-5.85\,10^{-6}i & 2.66\,10^{-6}+7.94\,10^{-7}i & 2.66\,10^{-6}-7.94\,10^{-7}i & -8.14\,10^{-5}+2.24\,10^{-6}i & -8.14\,10^{-5}-2.24\,10^{-6}i & 1.53\,10^{-4}-5.7\,10^{-6}i & 1.53\,10^{-4}+5.7\,10^{-6}i \\
4.53\,10^{-6}+4.29\,10^{-6}i & 4.53\,10^{-6}-4.29\,10^{-6}i & 8.18\,10^{-7}+3.09\,10^{-7}i & 8.18\,10^{-7}-3.09\,10^{-7}i & -3.35\,10^{-5}-1.27\,10^{-6}i & -3.35\,10^{-5}+1.27\,10^{-6}i & 1.21\,10^{-4}+9.28\,10^{-7}i & 1.21\,10^{-4}-9.28\,10^{-7}i \\
-1.34\,10^{-4}-2.3\,10^{-5}i & -1.34\,10^{-4}+2.3\,10^{-5}i & -1.05\,10^{-5}-3.13\,10^{-6}i & -1.05\,10^{-5}+3.13\,10^{-6}i & 3.19\,10^{-4}-9.05\,10^{-6}i & 3.19\,10^{-4}+9.05\,10^{-6}i & -5.92\,10^{-4}+2.25\,10^{-5}i & -5.92\,10^{-4}-2.25\,10^{-5}i \\
-1.74\,10^{-6}-2.85\,10^{-6}i & -1.74\,10^{-6}+2.85\,10^{-6}i & 4.8\,10^{-4}+9.48\,10^{-8}i & 4.8\,10^{-4}-9.48\,10^{-8}i & -2.36\,10^{-7}+2.72\,10^{-6}i & -2.36\,10^{-7}-2.72\,10^{-6}i & -2.0\,10^{-5}-1.36\,10^{-4}i & -2.0\,10^{-5}+1.36\,10^{-4}i \\
-6.15\,10^{-5}-1.66\,10^{-5}i & -6.15\,10^{-5}+1.66\,10^{-5}i & 0.00129-1.0\,10^{-6}i & 0.00129+1.0\,10^{-6}i & 1.13\,10^{-4}+1.86\,10^{-6}i & 1.13\,10^{-4}-1.86\,10^{-6}i & -2.22\,10^{-4}-2.9\,10^{-4}i & -2.22\,10^{-4}+2.9\,10^{-4}i \\
-1.59\,10^{-4}-8.49\,10^{-5}i & -1.59\,10^{-4}+8.49\,10^{-5}i & 0.011-4.88\,10^{-7}i & 0.011+4.88\,10^{-7}i & 2.38\,10^{-4}+4.97\,10^{-5}i & 2.38\,10^{-4}-4.97\,10^{-5}i & -8.16\,10^{-4}-0.00269i & -8.16\,10^{-4}+0.00269i \\
0.00737+0.00119i & 0.00737-0.00119i & 1.58\,10^{-4}+1.66\,10^{-4}i & 1.58\,10^{-4}-1.66\,10^{-4}i & -0.0157+5.98\,10^{-4}i & -0.0157-5.98\,10^{-4}i & 0.026-0.00105i & 0.026+0.00105i \\
2.0\,10^{-4}+1.9\,10^{-4}i & 2.0\,10^{-4}-1.9\,10^{-4}i & 3.59\,10^{-5}+1.36\,10^{-5}i & 3.59\,10^{-5}-1.36\,10^{-5}i & -0.00147-5.38\,10^{-5}i & -0.00147+5.38\,10^{-5}i & 0.00524+3.57\,10^{-5}i & 0.00524-3.57\,10^{-5}i
\end{pmatrix} \tag{5.9}$$

The matrix $\mathbf{\Psi}^{-1}$ is now calculated in the case of four involved eigenmodes according to Eq. (3.11), (3.4):

$$\mathbf{\Psi}^{-1} = \begin{bmatrix} \left(\varphi^{(1)}\right)^{-1} & & & \\ & \left(\varphi^{(2)}\right)^{-1} & & \\ & & \left(\varphi^{(3)}\right)^{-1} & \\ & & & \left(\varphi^{(4)}\right)^{-1} \end{bmatrix} =$$

$$\begin{pmatrix}
2.6857-1.2849i & 35.194+35.194i & 0 & 0 & 0 & 0 & 0 & 0 \\
2.6857+1.2849i & 35.194-35.194i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2.7322-2.6561i & 78.236+78.236i & 0 & 0 & 0 & 0 \\
0 & 0 & 2.7322+2.6561i & 78.236-78.236i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3.9601-3.0574i & 175.65+175.65i & 0 & 0 \\
0 & 0 & 0 & 0 & 3.9601+3.0574i & 175.65-175.65i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5.5787-4.6458i & 538.89+538.89i \\
0 & 0 & 0 & 0 & 0 & 0 & 5.5787+4.6458i & 538.89-538.89i
\end{pmatrix} \tag{5.10}$$

Finally the (108x8) real transformation matrix $\mathbf{Y}$ is computed according to (3.11) – here only the first ten rows:

$$\mathbf{Y} = \begin{pmatrix}
0 & 0.000002 & 0.000022 & 0 & 0 & -0.000009 & 0 & 0.001258 \\
0 & 0.000003 & 0.000028 & 0 & 0 & -0.000012 & 0 & 0.001609 \\
0.000008 & -0.000017 & 0.000001 & -0.000005 & -0.000028 & 0.000027 & 0.000067 & -0.000011 \\
0.000001 & -0.000018 & 0 & -0.000002 & -0.000011 & 0.000049 & 0.000053 & -0.000259 \\
-0.000033 & 0.000066 & -0.000002 & 0.000021 & 0.00011 & -0.000103 & -0.000259 & 0.000027 \\
0 & 0.000012 & 0.000111 & 0 & 0 & -0.000047 & -0.000001 & 0.00628 \\
-0.000015 & 0.000058 & 0.000297 & 0.000009 & 0.000039 & -0.000124 & -0.00008 & 0.013754 \\
-0.000035 & 0.000337 & 0.00255 & 0.00002 & 0.000079 & -0.001052 & -0.000202 & 0.125274 \\
0.001819 & -0.003317 & 0.000036 & -0.001111 & -0.005438 & 0.002459 & 0.011388 & 0.001937 \\
0.000038 & -0.000796 & 0.000008 & -0.000091 & -0.000501 & 0.002123 & 0.002276 & -0.011014
\end{pmatrix}$$

(5.11)

After the modal transformation the time-dependent "load" vector is calculated to be a function of the wind speed time series, see Fig. 3 :

$$\begin{bmatrix} g_1(t) \\ h_1(t) \\ \cdots \\ g_3(t) \\ h_3(t) \end{bmatrix} = \mathbf{Y}^{\mathrm{T}} \cdot \left[\mathbf{p} \right] = \begin{pmatrix} 5.788\, u^2 \\ 1.359\, u^2 \\ 0.1244\, u^2 \\ -1.915\, u^2 \\ 1.734\, u^2 \\ -32.53\, u^2 \\ 0.9771\, u^2 \\ -15.46\, u^2 \end{pmatrix} \qquad (5.12)$$

The resultant four uncoupled SDOFS block equations from type of Eq. (3.10), prepared in the form (2.23), are solved by step-by-step integration:

$$\begin{bmatrix} 1 & & & & \\ & -\omega_1^2 & & & \\ & & \cdots & & \\ & & & 1 & \\ & & & & -\omega_n^2 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \cdots \\ \dot{x}_n \\ \dot{y}_n \end{bmatrix}}_{\dot{\mathbf{X}}} + \begin{bmatrix} 2\eta_1\,\omega_1 & \omega_1^2 & & & \\ \omega_1^2 & 0 & & & \\ & & \cdots & & \\ & & & 2\eta_n\,\omega_n & \omega_n^2 \\ & & & \omega_n^2 & 0 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ \cdots \\ x_n \\ y_n \end{bmatrix}}_{\mathbf{X}} = \begin{bmatrix} g_1 \\ h_1 \\ \cdots \\ g_n \\ h_n \end{bmatrix}, \qquad (n = 4)$$

(5.13)

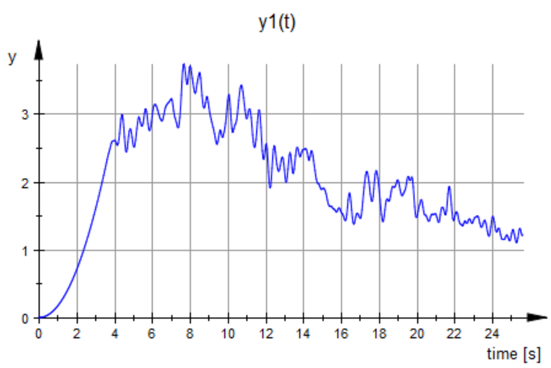
where $[\omega_i] = (\,16.7175 \;\; 29.0365 \;\; 49.6511 \;\; 104.976\,)$

$[\eta_i] = (\,0.332693 \;\; 0.0141136 \;\; 0.127584 \;\; 0.0908603\,)$ (5.14a,b)

The effect of the implied additional aerodynamic damping results evidently in the large damping ratio $\eta_i = 0.33269$ for the first free vibration.

The vibration-response has been determined in the time 0… 25.6 s, the time step length for the applied Newmark integration method is 0.03665 s.

The time response of the modal coordinates $y_j(t), x_j(t),\ (j = 1,2,3)$ are shown in the figures 5a-c:



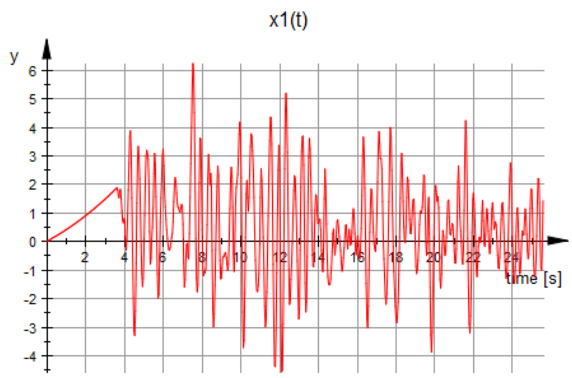


Fig. 5a Time response of the modal coordinates $y_1(t), x_1(t)$ for the case “non-proportional damping”

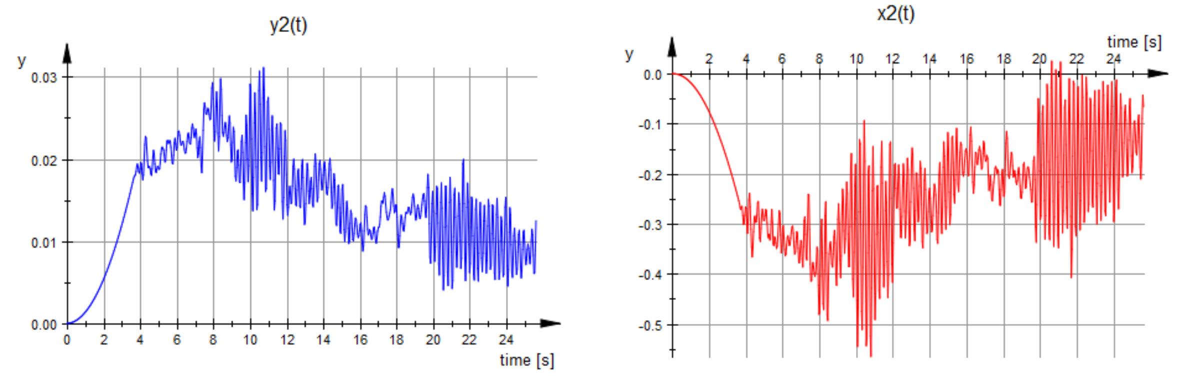


Fig. 5b Time response of the modal coordinates $y_2(t), x_2(t)$ for the case “non-proportional damping”

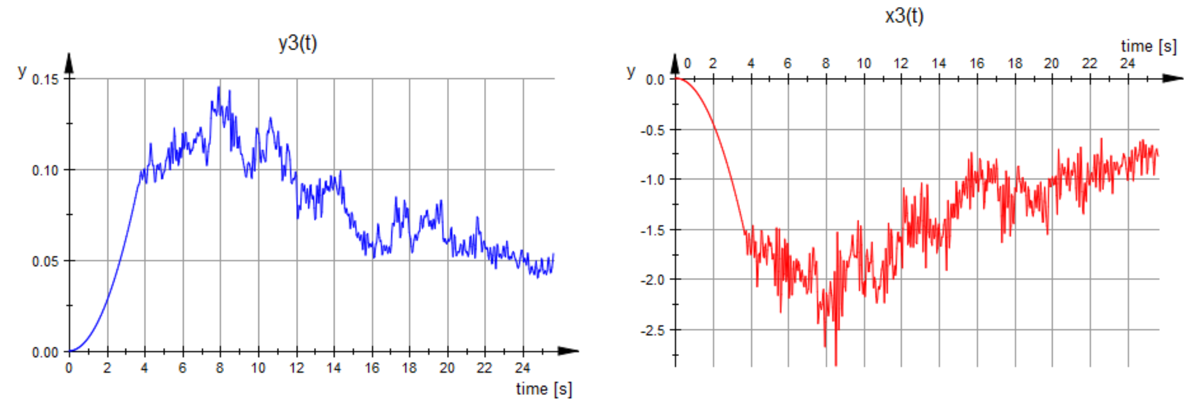


Fig. 5c Time response of the modal coordinates $y_3(t), x_3(t)$ for the case “non-proportional damping”

By a back transformation according to Eq. (3.11) the total response $\mathbf{V}(t)$ is obtained - see Figs. 6, 7 (vibration components at the rotor blade tip node #10).

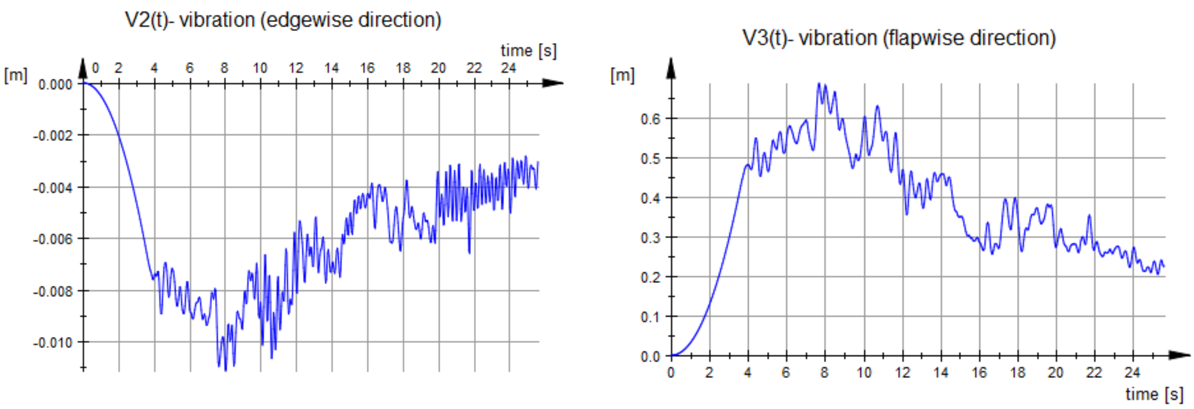


Fig.6 Total vibrations $u_2(t), u_3(t)$ $[m]$ (y- and z-direction, see fig.2) at the rotor blade tip - node #10

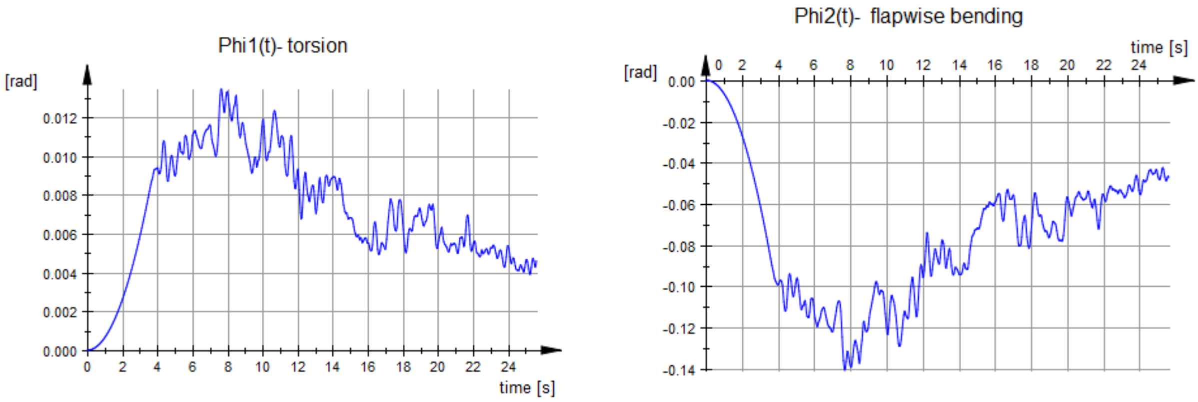


Fig.7 Total rotation $\varphi_1(t),\ \varphi_2(t)\ [rad]$ at the rotor blade tip (about x- and y-axis at node #10)

The vibration responses, computed by direct step-by-step integration of the equations (5.3), are practically identical to those in Fig. 6,7.

5.5 Proportional damped system

In this case we use the derived symmetric damping matrix $\mathbf{D_p}$ – Eq.(5.7). The first ten lowest complex conjugate eigenvalue pairs, resulting from Eq. (2.13), are now:

$$\begin{pmatrix} -0.132832+16.6035\,i \\ -0.132832-16.6035\,i \\ -0.406268+29.0352\,i \\ -0.406268-29.0352\,i \\ -1.19966+49.8844\,i \\ -1.19966-49.8844\,i \\ -5.27314+104.483\,i \\ -5.27314-104.483\,i \\ -5.39463+105.676\,i \\ -5.39463-105.676\,i \\ -16.7361+185.622\,i \\ -16.7361-185.622\,i \\ -20.9751+207.591\,i \\ -20.9751-207.591\,i \\ -27.8753+238.91\,i \\ -27.8753-238.91\,i \\ -42.2056+292.945\,i \\ -42.2056-292.945\,i \\ -62.3277+354.226\,i \\ -62.3277-354.226\,i \end{pmatrix} \qquad (5.15)$$

The corresponding (108x8) $\mathbf{\Phi_G}$ modal matrix – Eq. (2.18), comprises the first four complex conjugate eigenvector pairs, normalized with respect to the stiffness matrix – see Eq.(3.6a,b). In order to verify the derived relationship for the constant phase lag, see (3.15), we compute this ratio for all components of the involved $(\mathbf{\Phi_r} \pm i\mathbf{\Phi_i})^{(j)}\ (j = 1, \dots 4)$ eigenvectors (for instance the first ten rows only):

$$\begin{pmatrix} -1.01613 & -1.02838 & -1.04928 & -1.1063 \\ -1.01613 & -1.02838 & -1.04928 & -1.1063 \\ -1.01613 & -1.02838 & -1.04928 & -1.1063 \\ -1.01613 & -1.02838 & -1.04928 & -1.1063 \\ -1.01613 & -1.02838 & -1.04928 & -1.1063 \\ -1.01613 & -1.02838 & -1.04928 & -1.1063 \\ -1.01613 & -1.02838 & -1.04928 & -1.1063 \\ -1.01613 & -1.02838 & -1.04928 & -1.1063 \\ -1.01613 & -1.02838 & -1.04928 & -1.1063 \\ -1.01613 & -1.02838 & -1.04928 & -1.1063 \end{pmatrix} \leftrightarrow \begin{pmatrix} -1.01613 \\ -1.02838 \\ -1.04928 \\ -1.1063 \end{pmatrix} = \frac{1+2\eta_j\sqrt{1-\eta_j^2}}{2\eta_j^2-1}, \quad (j = 1, \dots 4) \qquad (5.16)$$

The corresponding damping ratios η_j are computed in accordance with Eq. (2.4b).

The (108x8) real transformation matrix **Y**, computed in regard with Eq. (3.11), (3.12), has now the form (only the first twenty rows are printed):

$$\mathbf{Y} = \begin{pmatrix} 0 & 0 & 0.000022 & 0 & 0 & 0 & -0.000053 & 0 \\ 0 & 0 & 0.000028 & 0 & 0 & 0 & -0.000068 & 0 \\ 0.000009 & 0 & 0.000001 & 0 & -0.000028 & 0 & 0.00005 & 0 \\ 0.000002 & 0 & 0 & 0 & -0.000011 & 0 & 0.000038 & 0 \\ -0.000035 & 0 & -0.000003 & 0 & 0.00011 & 0 & -0.000195 & 0 \\ -0.000001 & 0 & 0.000111 & 0 & 0 & 0 & -0.000265 & 0 \\ -0.000017 & 0 & 0.000297 & 0 & 0.000039 & 0 & -0.000642 & 0 \\ -0.000053 & 0 & 0.002549 & 0 & 0.000082 & 0 & -0.005423 & 0 \\ 0.001935 & 0 & 0.000051 & 0 & -0.005447 & 0 & 0.008771 & 0 \\ 0.000069 & 0 & 0.00001 & 0 & -0.000499 & 0 & 0.001632 & 0 \\ -0.002448 & 0 & -0.000063 & 0 & 0.006585 & 0 & -0.010006 & 0 \\ -0.000067 & 0 & 0.00314 & 0 & 0.000099 & 0 & -0.006234 & 0 \\ -0.00004 & 0 & 0.000568 & 0 & 0.000073 & 0 & -0.00092 & 0 \\ -0.000223 & 0 & 0.009905 & 0 & 0.000305 & 0 & -0.017838 & 0 \\ 0.007945 & 0 & 0.000182 & 0 & -0.019879 & 0 & 0.027365 & 0 \\ 0.000184 & 0 & 0.000025 & 0 & -0.001282 & 0 & 0.003913 & 0 \\ -0.005433 & 0 & -0.00011 & 0 & 0.011877 & 0 & -0.012914 & 0 \\ -0.000156 & 0 & 0.006512 & 0 & 0.00018 & 0 & -0.009482 & 0 \\ -0.000067 & 0 & 0.000826 & 0 & 0.000081 & 0 & -0.000752 & 0 \\ -0.000542 & 0 & 0.022556 & 0 & 0.00058 & 0 & -0.031471 & 0 \end{pmatrix} \qquad (5.17)$$

The time-dependent "load" vector in the general modal transformed equations (2.23) is now calculated according to Eq. (3.13):

$$\begin{bmatrix} g_1(t) \\ h_1(t) \\ \cdots \\ g_4(t) \\ h_4(t) \end{bmatrix} = \begin{pmatrix} 5.691\,u^2 \\ 0 \\ 0.1578\,u^2 \\ 0 \\ 1.864\,u^2 \\ 0 \\ 0.8032\,u^2 \\ 0 \end{pmatrix} \qquad (5.18)$$

The time dependence is expressed through the time series for $u(t)$ in fig. 3. Note that in the special case of proportionally damped system $x_j = \dot{y}_j$, i.e. $h_j(t) = 0$.

In the resultant four uncoupled SDOFS block equations from type (3.10), the free frequencies and the modal damping ratios are:

$[\omega_i] = (16.604 \;\; 29.0381 \;\; 49.8988 \;\; 104.616)$

$[\eta_i] = (0.008 \;\; 0.0139909 \;\; 0.0240418 \;\; 0.0504049)$ (5.19a,b)

After step-by-step integration of the four modal equations, the time series of the modal coordinates $x_j(t)$, $y_j(t)$, $(j = 1,\ldots 4)$, are obtained – Fig. 8:

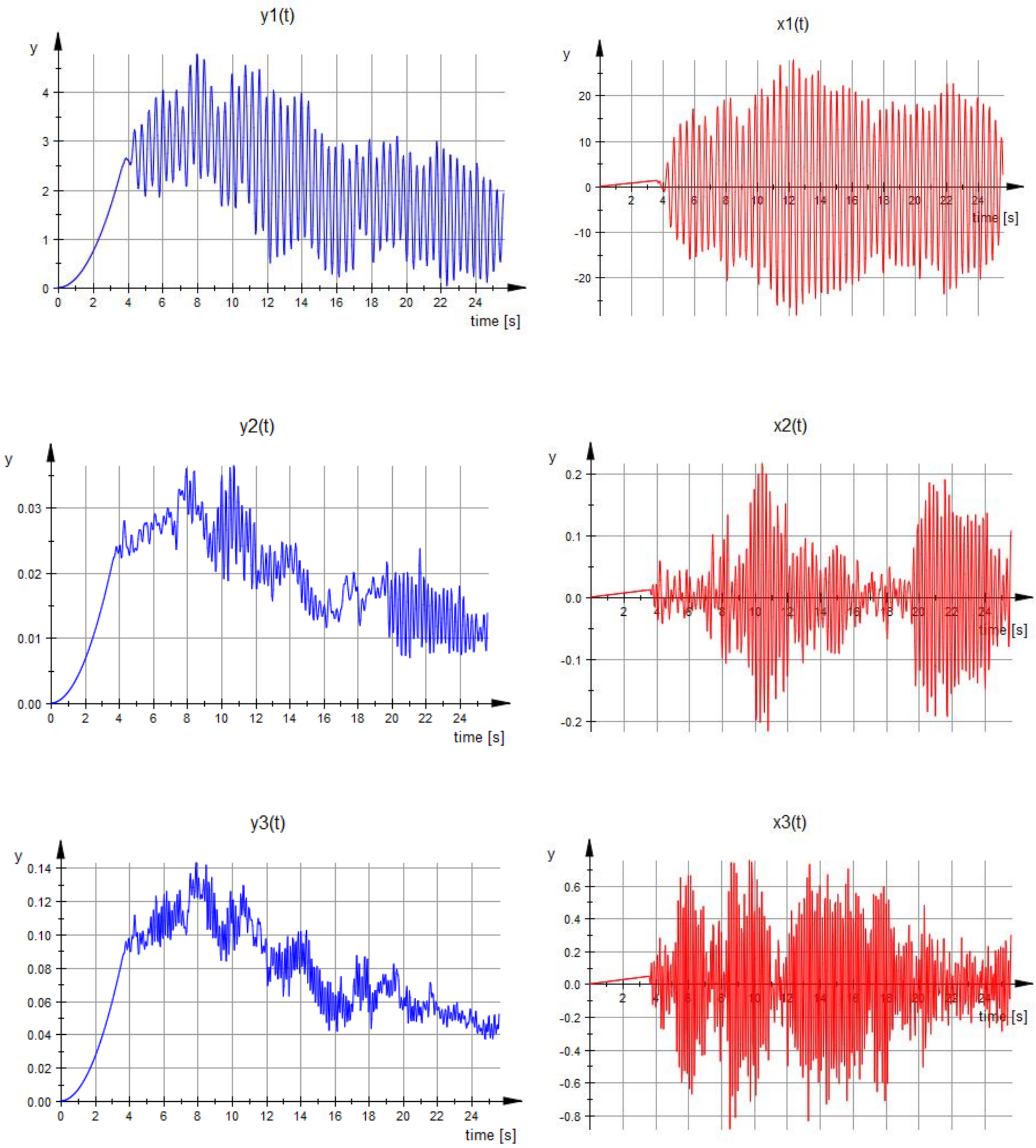


Fig. 8   Time response of the modal coordinates $y_j(t), x_j(t)$ $(j = 1,2,3)$ for the case "proportional damping"

The total responses $\mathbf{V}(t)$ are computed by a back transformation according to Eq. (3.11) – see Figs. 9a-d (here in the time range of 0 – 10 sec):

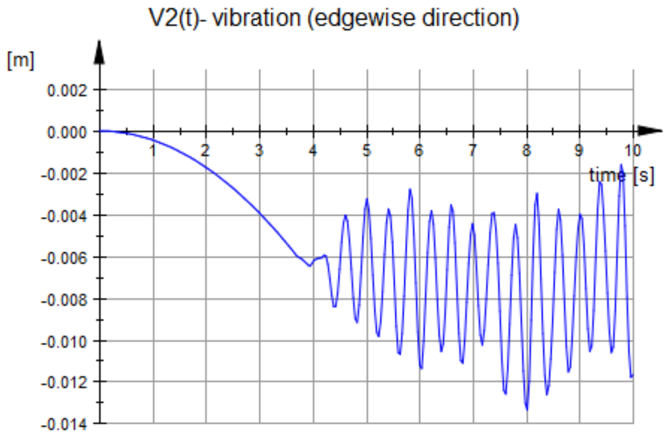


Fig 9a - Total vibration $u_2(t)$ $[m]$ at the rotor blade tip (y-direction at node #10) for the case “proportional damping”

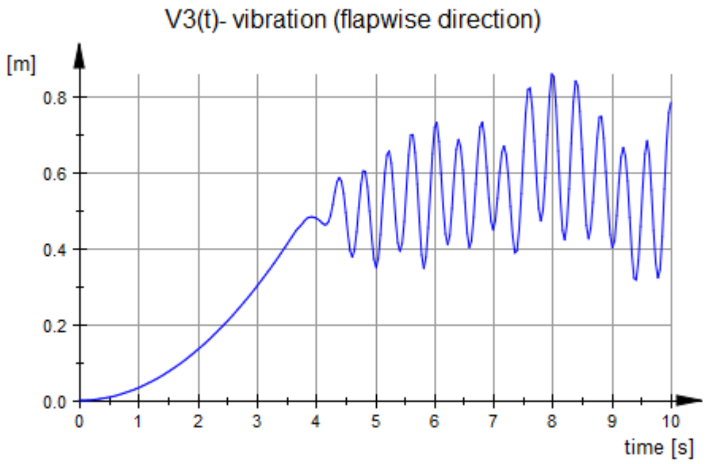


Fig 9b - Total vibration $u_3(t)$ $[m]$ at the rotor blade tip (z-direction at node #10) for the case “proportional damping”

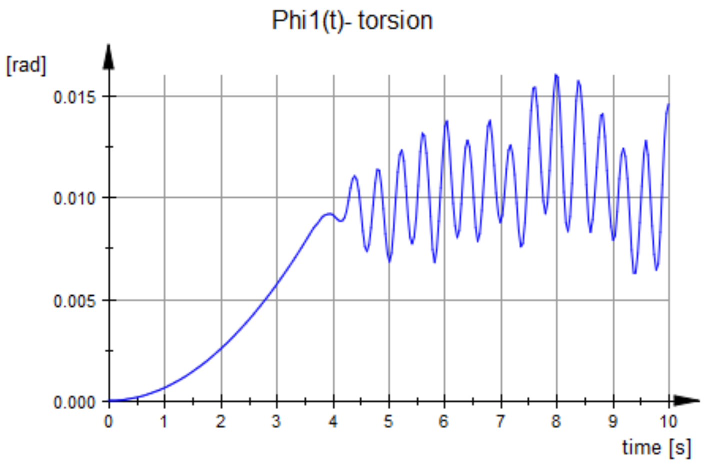


Fig 9c - Total torsion $\varphi_1(t)$ $[rad]$ about x-direction at node #10 for the case “proportional damping”

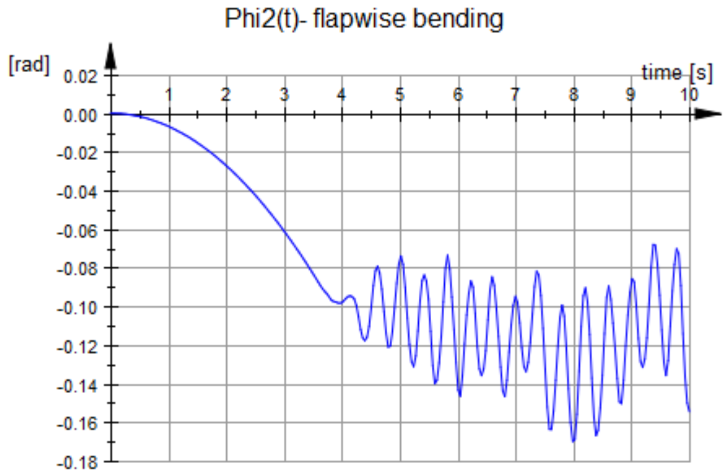


Fig 9d - Total rotation $\varphi_2(t)\ [rad]$ about y-axis at node #10 for the case “proportional damping”

The time series for the DOF calculated by direct step-by-step integration of the equations (5.3) are practically identical to the vibrations shown in Fig. 9a-d. The only difference occurs in the torsional vibration, see fig.10. The deviations may be explained by the absence of a torsional eigenmode in the four employed eigenvectors in the modal matrix.

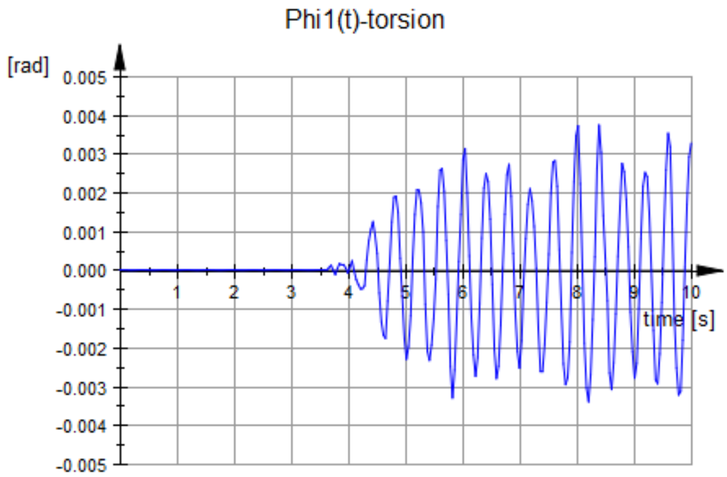


Fig 10 - Total torsion $\varphi_1(t)\ [rad]$ at node #10, calculated by direct step-by-step integration (“proportional damping”)

6 CONCLUSIONS

- A general modal decomposition method for MDOFS with non-proportional damping is presented in three variants. In general, all of them are based on the complex eigenvalue solution of a structural model with symmetric non-proportional damping matrix. The complex conjugate eigenpairs – eigenvalues and the corresponding eigenvectors – are to be computed first for the “state space” form of the equations of motion. By combining of two complex transformations, connected to the eigenvalue problems of the SDOFS and the MDOFS, three different kinds of a modal transformation matrix **Y** in real space are developed analytically to perform a modal decomposition of the equations of motion in real arithmetic.

- The first variant of the suggested procedure is described in Sec. 2. The real transformation matrix $\mathbf{Y}$ is assembled employing the right complex conjugate eigenvector pairs, normalized in both the SDOFS and MDOFS cases with respect to the corresponding mass matrix. The difference in the second variant, described in Sec. 3, is the normalization of the both modal matrices with respect to the corresponding stiffness matrices. In Sec. 4 has been presented the third variant, based on both the right and the left complex eigenvector pairs. In this version the complex conjugated eigenvectors for the MDOFS are normalized with respect to the stiffness matrix, the "left" modal matrix for the SDOFS don't need normalization.
- All variants of the presented modal procedure retain the common advantages of the classic modal decomposition of the equations of motion. Usually an uncomplete modal transformation should be performed by use of a few eigenmodes. Employing only the lowest few *(k)* eigenvector pairs in the $\mathbf{Y}$ -basis *(k<<n)* is leading with sufficient numerical accuracy to the total time response of all n DOF. The equations of motion are transformed into *k* uncoupled SDOFS block equations.
- The k uncoupled modal equations are easily numerically integrated like a SDOFS-equation – the result is the time response of the modal coordinates. Finally a back transformation to the original DOF has to be performed using the suggested new real $\mathbf{Y}$ –basis.
- The applications of the first and the second variants of the suggested method to the special case of proportionally damped system (employing a Rayleigh damping matrix) is leading to a simple analytical expression for the constant ratio $\frac{\Phi_{i(k)}}{\Phi_{r(k)}}$ for all k^{th} DOF of each considered eigenmode $(\mathbf{\Phi_r} \pm i\mathbf{\Phi_i})$. This is an indirect proof of the statement for synchronous free vibrations in the case of proportional damping. For more detailed investigations on this topic see [5].
- In Sec. 5 a numerical example – vibration of a rotor blade with 54 DOF - demonstrates the performance of the presented modal procedure in the variant of Sec. 3 for two cases – non-proportional and proportional (Rayleigh) damping. In the first variant the damping matrix of the system contains a stiffness-proportional part and a simple approximated aerodynamic damping part. In the second variant the formula for the constant phase of the resonance modes is verified numerically.
- Real life applications of the proposed modal analysis method and possible numerical complications are discussed more widely in [4], [5]. The present paper is a briefly overview of some possible variants of the suggested new modal transformation procedure in real space for viscous non-proportionally damped structural models.

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