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ISOGEOMETRIC METHODS IN DYNAMIC ANALYSIS OF CURVED BEAMS INCLUDING WARPING AND DISTORTIONAL EFFECTS

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Abstract. *In this research effort, Isogeometric tools are employed either in the Analog Equation Method (AEM), a boundary element based method, or in the Finite Element Method for the analysis of curved homogenous beams of arbitrary cross section (thin- or thick- walled) taking into account nonuniform warping, shear deformation effects (shear lag due to both flexure and torsion) and distortion. A set of distortional (in-plane warping) parameters have been considered and multiplied by the corresponding distortional functions, which constitute together with the warping ones the Navier equations for 2-D elasticity problem. One-dimensional boundary value problems described by second-order differential equations are formulated with respect to the displacement and rotation components as well as to the independent in- and out-of-plane warping parameters and solved. The in- and out-of-plane warping functions as well as the geometric constants including the additional ones due to warping and distortion (stiffness matrices) are evaluated employing a pure BEM approach. In order to derive the differential equations of motion with respect to the kinematical components, the terms of inertia contributions have to be added in the previous ones and, thus, the spatial mass matrix can be derived. Numerical results are worked out to illustrate the method, designate its efficiency, accuracy and computational cost, as well as verify its integrity comparing with the results of 3-D FEM (Finite Element Method) models.*

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1 INTRODUCTION

Refined models either straight or curved with shell or solid elements are widely used in structures, such as for example the deck of a bridge with a thin-walled cross section, for stress or strain analysis. The analysis of such members employing the so-called "Higher-Order Beam Theories" [1-3] is of increased interest due to their important advantages over more elaborate approaches based on shell or solid finite elements [4], which are mainly incorporated in commercial software.

In-plane deformations, such as distortion, occurring when thin-walled sections undergo bending and torsional deformations can considerably weaken the flexural and torsional stiffness of thin-walled beams. Even though distortion is larger in magnitude near the beam's ends, it does not remain local (exponentially decays away from the support) and thus it should be considered over the entire domain of the beam to account for its stiffness-weakening effect. Schardt [5-6] developed an advanced formulation known as Generalized Beam Theory (GBT) which is a generalization of the classical Vlasov beam theory in order to incorporate flexural and torsional distortional effects. Further developments of GBT avoid some of its cumbersome procedures through eigenvalue cross sectional analysis [7-12]. Ferradi and Cespedes [2] presented the formulation of a 3D beam element solving an eigenvalue problem for the distortional behavior of the cross section (in-plane problem) and computing warping functions separately by using an iterative equilibrium scheme. Finally, Dikaros and Sapountzakis [13] presented a general boundary element formulation for the analysis of composite beams of arbitrary cross section taking into account the influence of generalized cross sectional warping and distortion due to both flexure and torsion. In this proposal, distortional and warping functions are evaluated by the same eigenvalue problem and in order of importance.

Regarding horizontally curved beams subjected to vertical or radial loads, they inherently exhibit a more complex behavior comparing to straight formulations due to the fact that the effects of primary and secondary torsion are always coupled to those of bending and cross section distortion either for centered or eccentric loads. Petrov and Geradin [14] employing the same concept with El Fatmi and Ghazouani [3] for straight beams formulated a theory for curved and pre-twisted beams of arbitrary homogeneous cross sections, covering geometrically nonlinear range as well. Kim and Kim [15] developed a theory for thin-walled curved beams of rectangular cross section by extending the theory developed earlier for straight beams taking into account warping and distortional deformations. Park et al. [16] expanded their previous work [17], which was limited to straight box girder bridges, to curved formulations. They developed a curved box beam element which was employed in order to develop design charts for adequate spacing of the intermediate diaphragms of curved bridges. Flexural and torsional displacement functions have been based on those proposed for doubly symmetric cross section by Kang and Yoo [18] while distortional functions have been derived for a mono-symmetric cross section [19]. Yoo et al. [20] applied the concept of the BEF analogy for the analysis of distortional stresses of horizontally curved box-girders. The proposed procedure is capable of handling simple or continuous single cell box girders (or separated multi-cell box girders) with rigid or deformable interior diaphragms or cross-frames. Towards establishing a more general theory, Arici and Granata [21] employed the Hamiltonian Structural Analysis Method for the analysis of straight and curved thin-walled structures on elastic foundation extending the so-called GBT. To the authors' knowledge, there are no research efforts that introduce a unified distortional and warping eigenvalue analysis of arbitrarily shaped cross sections to the analysis of curved beams.

As far as the free vibration and dynamic response is concerned, it has been noticed in studies of the last decades, mainly for straight beam formulations, that the thin-walled members' behavior can be highly affected by cross section's in-plane (distortion) deformations. There are a number of investigations with various approaches in order to determine the cross sectional deformations which are either restricted to quadrilateral cross sections [22-23] or to the evaluation of two distortional modes that are roughly approximated by cubic polynomials [24]. Thin-walled closed piecewise straight beams with angled joints were also studied by Jang and Kim [25], but arbitrarily shaped sections were not investigated. More recently, Petrolo et al. [26] as well as Carrera and Varello [27] developed a beam formulation which can be exploited for the analyses of compact, thin-walled structures and bridge-like cross-sections. However, this approach is only capable of handling problems that involve a limited range of deformation types due to the fact that the displacement field, which is based on, is not formulated in the most general way. In addition to these, Jang et al. [28] and Bebiano et al. [29] developed more refined beam models with open- or closed-shaped cross section for the vibration problem. However, their cross sectional analyses are based on beam-frame and plate models for the discretization of the cross section. However, these approaches depend on the section's shape, the nodal topology and the number of intermediate nodes employed which make the procedure cumbersome while deformation mode selection becomes important for the analysis. Regarding vibration analysis of curved beams including distortional effects, few research efforts have taken into account the complete coupling of torsion, warping, and distortion deformations together with the curvature effect. Zhu et al. [30] provided a dynamic theory for the spatial vibration analysis of horizontally curved thin-walled rectangular box-shaped beams based on the displacement fields proposed by Kim and Kim [22]. Thus, to the authors' knowledge, there is no study on the vibration problem of curved beams with arbitrary cross section including in- and out-of-plane deformations.

In modern regulations and design specifications, the importance of torsional and distortional effects in stress or strain analysis of structural members is recognized. Particularly, in sub-sections 6.2.7.1 and 6.2.7.2 of EN 1993-2, Eurocode 3: Design of steel structures - Part 2: Steel bridges, regarding torsion, the designer is obliged to keep the distortional stresses under a specific limiting value or follow some general design rules in case of neglecting distortion. It should also be noted that most of the provisions of Eurocode 3 regarding torsion are valid only when distortional deformation can be neglected. Special attention is paid for distortion by the following bridge design specifications. The Guide Specifications for Horizontally Curved Highway Bridges by the American Association of State Highway and Transportation Officials – AASHTO (1993) specify the maximum spacing of the intermediate diaphragms through an approximate formula. The Hanshin Expressway Public Corporation of Japan provides the Guidelines for the Design of Horizontally Curved Girder Bridges – HEPCJ (1988) specifying the maximum spacing of the intermediate diaphragms in curved box girder with respect to that in straight box girders multiplied by a reduction factor. It should be noted here that the boundary conditions and the cross section shape are not taken into account directly for both specifications. Finally, up to the 1980s the design of a bridge structure was based on static analysis, corrected by a dynamic amplification factor which is based on the first natural frequency. An extensive effort was made by Hamed and Frosting [31] to introduce the effects of warping and distortion of bridge cross-sections. An analytical model is developed in their works where the bridge is idealized as being made of panels which behave as plates in the transversal direction and as Euler–Bernoulli beams in the longitudinal direction [26].

In this study, the static and dynamic analysis of horizontally curved beams of arbitrary cross section, loading and boundary conditions including generalized cross sectional warping and distortional effects due to both flexure and torsion is presented. The aim of this Chapter is to propose a new formulation by enriching the beam's kinematics both with out-of- and in-plane deformation modes and, thus, take into account both cross section's warping and distortion in the final 1D analysis of curved members, towards developing GBT further for curved geometries while employing independent warping parameters, which are commonly used in Higher Order Beam Theories (HOBT). The approximating methods and schemes proposed by Dikaros and Sapountzakis [13, 32] are employed and extended in this study. Adopting the concept of end-effects and their exponential decay away from the support [3], appropriate residual strains are added to those corresponding to rigid body movements. Further, applying Hooke's stress-strain law and employing the equilibrium equations of 3D elasticity, a system of partial differential equations can be derived for each material over the 2D cross section's domain together with the corresponding boundary and initial conditions. Consequently, a coupled two-dimensional boundary value problem is formulated, with or without considering Poisson ratio. Applying a proper discretization scheme for the cross section, the above mentioned problem will lead to the formulation of an eigenvalue problem which the eigenvalues and the corresponding eigenvectors, for a desired number of modes, can be extracted from. The obtained set of modes contains axial, flexural and torsional modes in order of significance without distinction between them. To avoid the additional effort needed in order to recognize the most significant modes, the iterative local equilibrium scheme described in the work of Dikaros and Sapountzakis [32] is adopted until the error due to residual terms becomes minimal. The functions derived are evaluated employing 2D BEM [33]. The coefficient matrices containing the geometric and mass properties of the cross section can now be calculated. Thus, a set of boundary value problems are formulated with respect to the unknown kinematical components (displacements, rotations and independent parameters) for each time instant, the number of which is defined by the user depending on the accuracy of the results. This linear system is solved using Isogeometric tools, either integrated in the Finite Element Method (FEM) [34] or in the Analog Equation Method (AEM) [35], which is BEM based. Employing the principal of virtual work the new equilibrium equations are derived. Additionally, by employing a distributed mass model system accounting for longitudinal, transverse, rotatory, torsional, warping and distortional inertia, free vibration characteristics and responses of the stress resultants and displacements to static and moving loading can be evaluated. The results obtained from the beam element will be compared to those obtained from finite 3D solutions. Numerical examples are presented to illustrate the efficiency and the accuracy of this formulation. To the authors' knowledge, the numerical procedures previously mentioned have not been reported in the literature for the analysis of curved beams including distortional effects.

The essential features and novel aspects of the formulation that will be presented in the following compared with previous ones are summarized as follows.

i. The developed beam formulation is capable of the dynamic analysis of spatial curved beams of arbitrary composite cross section with one plane of constant curvature considering warping and distortional effects (in addition to the previous formulations) that are introduced in the same boundary value problem which describes the cross section's deformations.

ii. The cross sectional analysis is based on an iterative equilibrium scheme which results in a numerical procedure with less computational effort and complexity comparing to traditional eigenvalue analysis reported in the literature for similar problems. Particularly, modes attributed to different structural phenomena can be separated directly and make the supervision of the results easier. In addition to this, the data post-processing and the iterative procedure become faster due to the fact that warping and distortional functions are calculated separately.

iii. The accuracy level of the numerical method proposed can be decided by the user by setting the desirable number of the modes taken into account and, thus, increasing the number of higher modes added in the final solution.

iv. The numerical solution of the curved advanced beam is based on B-splines [36-37] and NURBS (Isogeometric Analysis) offering the advantage of integrating computer aided design (CAD) in the analysis.

2 S TATEMENT OF THE PROBLEM

Let us consider a straight or curved prismatic element of length L (Fig. 1) with an arbitrarily shaped composite cross section of m homogenous, isotropic and linearly elastic materials with modulus of elasticity E_m, shear modulus G_m and and Poisson ratio ν_m, occupying the region Ω_m of the yz plane with finite number of inclusions (Fig. 2). Let also the boundaries of the regions Ω_m be denoted by Γ_m. This boundary curve is piecewise smooth, i.e. it may have a finite number of corners. In Fig. 2 $CXYZ$ is the principal bending coordinate system through the cross section's centroid C, while y_C, z_C are its coordinates with respect to $Sxyz$ reference coordinate system through the cross section's shear center S. It holds that $y_C = y - Y$ and $z_C = z - Z$. The initial radius of curvature, denoted by R is considered constant and it is parallel to Y axis. The displacement vector $\bar{u}(x,y,z)$ of an arbitrary point of the cross section is obtained as the sum of SV solution vector corresponding to the rigid body motion combined with a residual (index R) displacement vector due to end-effects which are responsible for the generation of self-equilibrating stress distributions:

$$\begin{aligned}
\bar{u}(x,y,z) &= \bar{u}^{SV}(x,y,z) + \bar{u}^{R}(x,y,z) = \underbrace{u(x) + \theta_Y(x)Z - \theta_Z(x)Y}_{rigid\ body\ movement} + \underbrace{\sum_{i=1}^{m} \alpha_i(x) W_i(y,z)}_{\textbf{out-of-plane warping}} \\
\bar{v}(x,y,z) &= \bar{v}^{SV}(x,y,z) + \bar{v}^{R}(x,y,z) = \underbrace{v(x) - z\theta_x(x)}_{rigid\ body\ movement} + \underbrace{\sum_{i=1}^{m} \alpha_i(x)_{,x} DY_i(y,z)}_{\textbf{distortion in Y direction}} \\
\bar{w}(x,y,z) &= \bar{w}^{SV}(x,y,z) + \bar{w}^{R}(x,y,z) = \underbrace{w(x) + y\theta_x(x)}_{rigid\ body\ movement} + \underbrace{\sum_{i=1}^{m} \alpha_i(x)_{,x} DZ_i(y,z)}_{\textbf{distortion in Z direction}}
\end{aligned} \tag{1}$$

where $(\ )_{,j}$ is for differentiation with respect to j, i is the number of higher order cross sectional functions considered, $\bar{u}$, $\bar{v}$, $\bar{w}$ are the axial, transverse and radial beam displacement components with respect to the $Sxyz$ system of axes, respectively, $W(y,z)$ is the warping function, $DY(y,z)$ and $DZ(y,z)$ are the distortional functions of the in-plane deformation mode $D(y,z)$ while $\alpha(x)$ is a function describing the decay of deformation along beam length. Moreover, $v(x)$ and $w(x)$ describe the deflection of the centre of twist S, while $u(x)$ denotes the "average" axial displacement of the cross section. $\theta_x(x)$ is the angle of twist due to torsion, while $\theta_Y(x)$ and $\theta_Z(x)$ are the angles of rotation due to bending about the centroidal Y , Z axes, respectively.The derivation of rigid body motions is in more detail explained in the work of Kang and Yoo [18], while $\sin\theta_x \approx \theta_x, \cos\theta_x \approx 1$ assumption is adopted and higher order terms are neglected in this study. Considering the fact that end-effects decay exponentially away from the support, $\alpha(x) = e^{-cx}$ where c is a constant to be specified. However, different expressions of this parameter have also been adopted in other research efforts (i.e. polynomials of various degrees).

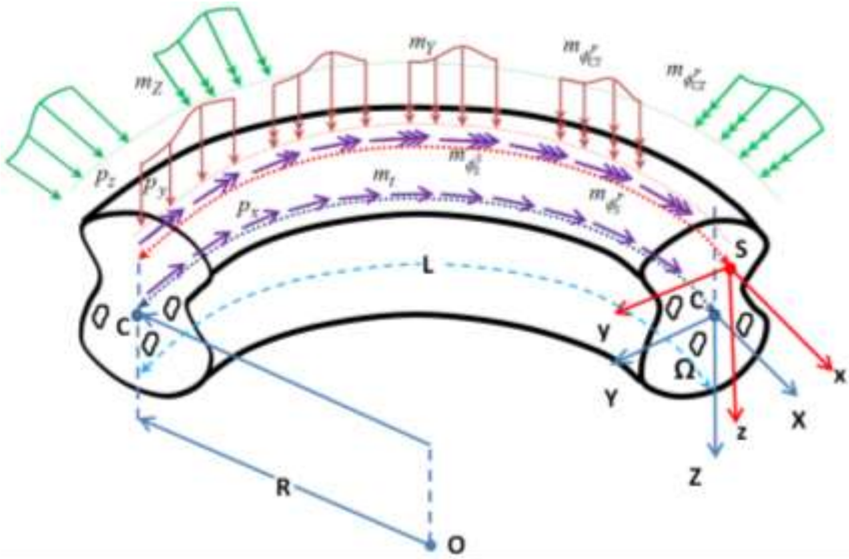

Figure 1: Prismatic curved beam under axial-flexural-torsional loading of an arbitrary homogenous cross section.

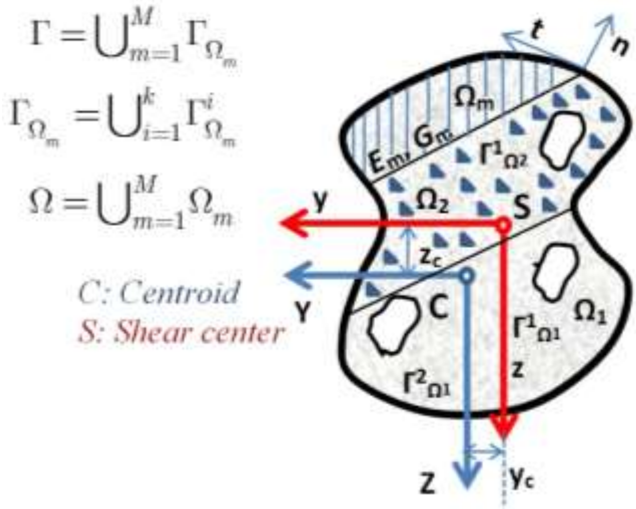

Figure 2: Arbitrary composite cross section of m homogenous materials occupying the two dimensional region Ω.

After establishing the displacement field, the strain components for *m*th material due to end-effects can be computed as

$$
\begin{aligned}
&(\varepsilon_{xx})_m = \left(\bar{u}^R(x,y,z)_{,x}\right)_m = \alpha_{,x}(W)_m \\
&(\varepsilon_{yy})_m = \left(\bar{v}^R(x,y,z)_{,y}\right)_m = \alpha_{,x}(DY_{,y})_m \\
&(\varepsilon_{zz})_m = \left(\bar{w}^R(x,y,z)_{,z}\right)_m = \alpha_{,x}(DZ_{,z})_m \\
&(\gamma_{xy})_m = (\gamma_{yx})_m = \left(\bar{v}^R(x,y,z)_{,x}\right)_m + \left(\bar{u}^R(x,y,z)_{,y}\right)_m = \alpha_{,xx}(DY)_m + \alpha(W_{,y})_m \\
&(\gamma_{xz})_m = (\gamma_{zx})_m = \left(\bar{w}^R(x,y,z)_{,x}\right)_m + \left(\bar{u}^R(x,y,z)_{,z}\right)_m = \alpha_{,xx}(DZ)_m + \alpha(W_{,z})_m \\
&(\gamma_{yz})_m = (\gamma_{zy})_m = \left(\bar{w}^R(x,y,z)_{,y}\right)_m + \left(\bar{v}^R(x,y,z)_{,z}\right)_m = \alpha_{,x}\left[(DZ_{,y})_m + (DY_{,z})_m\right]
\end{aligned}
\tag{2}
$$

Employing the well-known stress-strain constitutive relationship for elastic media and isotropic solid, the stress components of the *m*th material can be derived. Afterwards, employing local equilibrium equations of three-dimensional elasticity considering body forces to be absent, substituting stress components and the exponential function $\alpha(x)$, the system of partial differential equations for the *m*th material is stated together with the boundary conditions and, finally, employing the relation $\nu_m = \lambda_m / \left[2(\lambda_m + \mu_m)\right]$ and expanding the stresses in the boundary conditions, a boundary value problem together with the boundary conditions is formulated in eqs. (3a), (3b) and (3c) for the warping and two distortional functions, respectively.

$$
\begin{aligned}
&\left(\nabla^2 W\right)_m = c^2\left[-\frac{2}{1-\nu_m^e}(W)_m - \frac{1+\nu_m^e}{1-\nu_m^e}(\nabla D)_m\right] \\
&\begin{cases}
(W_{,n})_m = c^2\left[-(DY)_m n_y - (DZ)_m n_z\right] & \textit{on free surface} \\
g_m (W_{,n})_m + g_n (W_{,n})_n = c^2 (g_m - g_n)\left[-(DY)_m n_y - (DZ)_m n_z\right] & \textit{on Interfaces}
\end{cases}
\end{aligned}
\tag{3a}
$$

$$
\begin{aligned}
&\left(\nabla^2 DY\right)_m + \frac{1+\nu_m^e}{1-\nu_m^e}\left[(\nabla D)_{m,y} + (W_{,y})_m\right] = c^2\left[-(DY)_m\right] \\
&\begin{cases}
g_m\left\{(DY_{,n})_m + \left[(DY_{,y})_m n_y + (DZ_{,y})_m n_z\right]\right\} + g_m^*\left[(DY_{,y})_m + (DZ_{,z})_m\right]n_y \\
= -g_m^*(W)_m n_y \qquad \textit{on free surface} \\
g_m\left\{(DY_{,n})_m + \left[(DY_{,y})_m n_y + (DZ_{,y})_m n_z\right]\right\} + g_m^*\left[(DY_{,y})_m + (DZ_{,z})_m\right]n_y + \\
g_n\left\{(DY_{,n})_n + \left[(DY_{,y})_n n_y + (DZ_{,y})_n n_z\right]\right\} + g_n^*\left[(DY_{,y})_n + (DZ_{,z})_n\right]n_y \\
= -(g_m^* - g_n^*)(W)_m n_y \qquad \textit{on Interfaces}
\end{cases}
\end{aligned}
\tag{3b}
$$

Where $\nu_m^e = \nu_m / (1-\nu_m)$ is the effective Poisson ratio while $g_m = \mu_m / \mu_{ref}$, $g_m^* = \lambda_m / \mu_{ref}$ are weighted elastic constants with respect to μ_{ref} which is the shear modulus of reference

$$\left(\nabla^2 DZ\right)_m + \frac{1+\nu_m^e}{1-\nu_m^e}\left[(\nabla D)_{m,z} + \left(W_{,z}\right)_m\right] = c^2\left[-\left(DZ\right)_m\right]$$

$$\begin{cases} g_m\left\{\left(DZ_{,n}\right)_m + \left[\left(DY_{,z}\right)_m n_y + \left(DZ_{,z}\right)_m n_z\right]\right\} + g_m^*\left[\left(DY_{,y}\right)_m + \left(DZ_{,z}\right)_m\right]n_z \\ = -g_m^*\left(W\right)_m n_z \qquad \textit{on free surface} \\ g_m\left\{\left(DZ_{,n}\right)_m + \left[\left(DY_{,z}\right)_m n_y + \left(DZ_{,z}\right)_m n_z\right]\right\} + g_m^*\left[\left(DY_{,y}\right)_m + \left(DZ_{,z}\right)_m\right]n_z + \\ g_n\left\{\left(DZ_{,n}\right)_n + \left[\left(DY_{,z}\right)_n n_y + \left(DZ_{,z}\right)_n n_z\right]\right\} + g_n^*\left[\left(DY_{,y}\right)_n + \left(DZ_{,z}\right)_n\right]n_z \\ = -\left(g_m^* - g_n^*\right)\left(W\right)_m n_z \qquad \textit{on Interfaces} \end{cases} \tag{3c}$$

material. If a plane stress assumption is employed, ν_m^e is substituted by ν_m. When $\nu_m = 0$ it holds that $g_m = E_m / E_{ref}$, $g_m^* = 0$, with E_{ref} being the elastic modulus of reference material, and the aforementioned boundary value problem is simplified. Therefore, employing a proper discretization for the cross section, the above coupled boundary value problem (eqs. (3)) will lead to the formulation of a generalized eigenvalue problem of the form $AF = c^2 BF$ where A, B are known coefficient matrices, c is the eigenvalue and $F = \begin{bmatrix} W & DY & DZ \end{bmatrix}^T$ is the eigenvector of the problem. The solution of eigenvalue problem yields a set of eigenvalues together with the corresponding eigenvectors which constitute a basis of cross sectional deformation modes suitable for distortional analysis of beams.

As mentioned earlier, the iterative equilibrium scheme described by Ferradi, Cespedes and Arquier [1] as well as Dikaros and Sapountzakis [32] is employed here until a sufficient number of modes is obtained to represent accurately the non-uniform warping effects and the corresponding distortional ones. In order to initialize the above stated boundary value problem, the rigid body movements of the cross section are employed. These correspond to SV flexural and torsional warping modes. Afterwards, in order to restore equilibrium the secondary warping modes are determined together with their corresponding distortional ones. Following this concept, the iterative procedure is formulated converging to the exact shape of the warping in a section. Each functional vector F_{i+1} has to fulfil the orthogonality condition with respect to the functions F_i corresponding to the previous set of modes.

Within the context of the above considerations and considering up to secondary warping as well as distortional displacements, the enriched kinematics of an arbitrary point of the beam for *m*th material at any time instant is given as

$$\bar{u}(x,y,z,t) = \bar{u}^P(x,y,z,t) + \bar{u}^S(x,y,z,t) = \underbrace{u(x) + \theta_Y(x)Z - \theta_Z(x)Y + \eta_x(x)\phi_S^P(y,z)}_{\text{primary}} \underbrace{+\eta_Y(x)\phi_{CY}^P(y,z) + \eta_Z(x)\phi_{CZ}^P(y,z) + \xi_x(x)\phi_S^S(y,z)}_{\text{secondary}} \tag{4a}$$

$$\begin{aligned}\bar{v}(x,y,z,t) &= v(x) - z\theta_x(x) \\ &+ \underbrace{\zeta_x(x)v_S^P(y,z) + \zeta_Y(x)v_{CY}^P(y,z) + \zeta_Z(x)v_{CZ}^P(y,z)}_{\text{primary}} \\ &+ \underbrace{\chi_x(x)v_S^S(y,z) + \chi_Y(x)v_{CY}^S(y,z) + \chi_Z(x)v_{CZ}^S(y,z)}_{\text{secondary}}\end{aligned} \tag{4b}$$

$$\begin{aligned}\bar{w}(x,y,z,t) &= w(x) + y\theta_x(x) \\ &+ \underbrace{\zeta_x(x)w_S^P(y,z) + \zeta_Y(x)w_{CY}^P(y,z) + \zeta_Z(x)w_{CZ}^P(y,z)}_{\text{primary}} \\ &+ \underbrace{\chi_x(x)w_S^S(y,z) + \chi_Y(x)w_{CY}^S(y,z) + \chi_Z(x)w_{CZ}^S(y,z)}_{\text{secondary}}\end{aligned} \tag{4c}$$

where $\bar{u}^P$, $\bar{u}^S$, denote the primary and secondary longitudinal displacements, respectively. $\eta_x(x)$, $\xi_x(x)$ are the independent warping parameters introduced to describe the nonuniform distribution of primary and secondary torsional warping, while $\eta_Y(x)$, $\eta_Z(x)$ are the independent warping parameters introduced to describe the nonuniform distribution of primary warping due to shear. Similarly, $\zeta_x(x)$, $\chi_x(x)$ are the independent distortional parameters introduced to describe the nonuniform distribution of primary and secondary distortion due to torsion, while $\zeta_Y(x)$, $\zeta_Z(x)$, $\chi_Y(x)$, $\chi_Z(x)$ are the independent distortional parameters introduced to describe the nonuniform distribution of primary and secondary distortion due to flexure. All these parameters are multiplied by the corresponding warping and distortional functions which are components of the $W(y,z)$ and $D(y,z)$ vectors derived by the solution of the coupled boundary value problem stated in eqs. (3). In eqs. (4), 16 degrees of freedom have been employed in 3D space. These activate 12 cross sectional deformation modes, namely rigid (4), primary (4) and secondary motions (4), including extension.

If tertiary displacements have to be employed for accuracy reasons, the beam's kinematics is enriched further. In this case 22 degrees of freedom have been employed in order to describe the beam's behavior. The additional 6 degrees, namely $\xi_Y(x)$, $\xi_Z(x)$, $\omega_x(x)$, $\psi_x(x)$, $\psi_Y(x)$ and $\psi_Z(x)$, account for 3 tertiary warping and 3 tertiary distortional effects, respectively. These activate 4 additional cross sectional deformation modes including extension. The enrichment of the beam's kinematics can be done automatically by increasing the number of modes, which are an input value for the boundary value problem to be solved. This results in the evaluation of additional cross sectional operators which will be employed in the analysis of the beam model, after establishing the strain components as it will be described in the following.

After establishing the displacement field, the linear strain-displacement relations in the system (x,y,z) can be written. Employing the expressions of the displacement components (eqs. (4)), the strains and stresses can be computed. Applying the principle of virtual work or any other variational principle following standard arguments in the calculus of variations, the governing differential equations for the beam in terms of the kinematical components can be derived. Thus, the local stiffness matrix $\left[k_l\right]$ of the spatial curved beam can be evaluated after solving a system of linear equations. Finally, the matrix form of stiffness matrix is derived as follows

$$\left[Aux1\right]=\begin{bmatrix} 1 & 0 & 0 & 0 & Z & -Y & \phi_S^P & \phi_{CY}^P & \phi_{CZ}^P & \phi_S^S & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & v_S^P & v_{CY}^P & v_{CZ}^P & v_S^S & v_{CY}^S & v_{CZ}^S \\ 0 & 0 & 1 & y & 0 & 0 & 0 & 0 & 0 & 0 & w_S^P & w_{CY}^P & w_{CZ}^P & w_S^S & w_{CY}^S & w_{CZ}^S \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} e(R) \tag{5a}$$

$$\left[Aux2\right]=\left[\begin{matrix} 0 & -\frac{1}{R}e(R) & 0 & \frac{z}{R}e(R) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{R}e(R) & 0 & 0 & 0 & \frac{Z}{R}e(R) & -1-\frac{Y}{R}e(R) & \frac{\phi_S^P}{R}e(R)+\phi_{S,y}^P & \frac{\phi_{CY}^P}{R}e(R)+\phi_{CY,y}^P & \frac{\phi_{CZ}^P}{R}e(R)+\phi_{CZ,y}^P & \frac{\phi_S^S}{R}e(R)+\phi_{S,y}^S \\ 0 & 0 & 0 & 0 & 1 & 0 & \phi_{S,z}^P & \phi_{CY,z}^P & \phi_{CZ,z}^P & \phi_{S,z}^S \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right. \cdots$$

$$\cdots\left.\begin{matrix} \frac{v_S^P}{R}e(R) & \frac{v_{CY}^P}{R}e(R) & \frac{v_{CZ}^P}{R}e(R) & \frac{v_S^S}{R}e(R) & \frac{v_{CY}^S}{R}e(R) & \frac{v_{CZ}^S}{R}e(R) \\ v_{S,y}^P & v_{CY,y}^P & v_{CZ,y}^P & v_{S,y}^S & v_{CY,y}^S & v_{CZ,y}^S \\ w_{S,z}^P & w_{CY,z}^P & w_{CZ,z}^P & w_{S,z}^S & w_{CY,z}^S & w_{CZ,z}^S \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ w_{S,y}^P+v_{S,z}^P & w_{CY,y}^P+v_{CY,z}^P & w_{CZ,y}^P+v_{CZ,z}^P & w_{S,y}^S+v_{S,z}^S & w_{CY,y}^S+v_{CY,z}^S & w_{CZ,y}^S+v_{CZ,z}^S \end{matrix}\right] \tag{5b}$$

$$\varepsilon=\left[Aux1\right]u_{,x}+\left[Aux2\right]u \tag{5c}$$

$$\sigma=\left[C\right]\left[Aux1\right]u_{,x}+\left[C\right]\left[Aux2\right]u \tag{5d}$$

$$\delta U=\int_0^L\int_\Omega\left(\delta u_{,x}^T\left[Aux1\right]^T+\delta u^T\left[Aux2\right]^T\right)\left(\left[C\right]\left[Aux1\right]u_{,x}+\left[C\right]\left[Aux2\right]u\right)\frac{1}{e(R)}d\Omega dx\Rightarrow$$

$$\Rightarrow\delta U=\int_0^L\left(\delta u_{,x}^T k_{22}u_{,x}+\delta u^T k_{12}u_{,x}+\delta u_{,x}^T k_{21}u+\delta u^T k_{11}u\right)dx\Rightarrow \text{by parts integration} \tag{5e}$$

$$\Rightarrow\delta U=\int_0^L\left(\delta u^T\left\{-k_{22}u_{,xx}+\left[k_{12}-k_{21}\right]u_{,x}+k_{11}u\right\}\right)dx+\left[\delta u^T\left\{k_{22}u_{,x}+k_{21}u\right\}\right]_0^L$$

$$\left[k_l\right]=\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \tag{5f}$$

where [*Aux1*], [*Aux2*] are auxiliary matrices to express strains ε in matrix form, [*C*] is the elasticity matrix employed to derive stresses σ, $\frac{R}{R-Y}=e(R)$, $\frac{1}{e(R)}d\Omega dx=dV$ is the differential volume of the curved beam for constant radius of curvature, δU is the virtual strain energy and k_{11}, k_{12}, k_{21} and k_{22} are 16X16 coefficient matrices containing the geometric properties of the cross section. These are calculated as follows

$$\begin{aligned} k_{11} &= \int_{\Omega_m} [Aux1]^T [C][Aux1] \frac{1}{e(R)} d\Omega, \qquad k_{12} = \int_{\Omega_m} [Aux1]^T [C][Aux2] \frac{1}{e(R)} d\Omega \\ k_{21} &= \int_{\Omega_m} [Aux2]^T [C][Aux1] \frac{1}{e(R)} d\Omega, \qquad k_{22} = \int_{\Omega_m} [Aux2]^T [C][Aux2] \frac{1}{e(R)} d\Omega \end{aligned} \tag{6}$$

Moreover, the external work can be derived as follows

$$[Aux]=\begin{bmatrix} 1 & 0 & 0 & 0 & Z & -Y & \phi_S^P & \phi_{CY}^P & \phi_{CZ}^P & \phi_S^S & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & v_S^P & v_{CY}^P & v_{CZ}^P & v_S^S & v_{CY}^S & v_{CZ}^S \\ 0 & 0 & 1 & y & 0 & 0 & 0 & 0 & 0 & 0 & w_S^P & w_{CY}^P & w_{CZ}^P & w_S^S & w_{CY}^S & w_{CZ}^S \end{bmatrix} \tag{7}$$

$$\delta W = \underbrace{\int_0^L \left(\delta u^T [Aux]^T t\right) dx}_{\delta u^T p} + \left[\int_\Omega \left(\delta u^T [Aux]^T t\right) d\Omega \right]_0^L \tag{8}$$

where t is the traction vector applied on the lateral surface of the beam including the end cross sections and p is the external load vector of the beam.

Employing the expressions of the displacement components, the cross sectional operators (eqs. (6)) and the governing differential equations of the curved beam can be obtained in a similar way when tertiary or higher warping and distortional effects are considered.

In order to derive the differential equations of motion with respect to the kinematical components, the terms of inertia contributions $\delta W_{\text{mass}} = \int_V \rho\left(\bar{u}_{,tt}\delta\bar{u} + \bar{v}_{,tt}\delta\bar{v} + \bar{w}_{,tt}\delta\bar{w}\right)\mathrm{d}V$ (with $dV=\frac{1}{e(R)}d\Omega dx$) have to be added in the previous (eqns. (5e) and (8)) and constitutive equations should be employed. ρ is the density of the material and $\bar{u},\bar{v},\bar{w}$ are the generalized displacements as previously described. Thus, the local spatial mass coefficient matrix $[m_l]$ can finally be derived. This can be extracted in matrix form from the following expression

$$\delta W_{\text{mass}} = \int_0^L \int_\Omega \rho\left(\delta u^T [Aux]^T [Aux] u_{,tt}\right) \frac{1}{e(R)} d\Omega dx \Rightarrow \delta U = \int_0^L \left(\delta u^T [m_l] u_{,tt}\right) dx \tag{9}$$

with $[Aux]$ given in eqn. (7), u representing the total displacement and $[m_l]$ being a 16X16 coefficient matrix when displacements of eqns. (4) are employed.

Except for the boundary conditions there are also the initial conditions at $x \in (0, L)$ for each kinematical component. After establishing the stiffness and mass matrices of the spatial curved beam element the equation of motion in matrix form can be stated.

The natural frequencies and modes in which the beam vibrates for the different motions (including also distortional ones) can be obtained by separation of variables and $u_i(x,t)$ is assumed as $u_i(x,t) = u_i(x)e^{i\omega t}$. Finally, the typical generalized eigenvalue problem $\left([k_t] - \omega^2 [m_t]\right)u_i = \{0\}$ is formulated and solved.

3 NUMERICAL SOLUTIONS

The evaluation of the warping and distortional functions is accomplished by solving the problems described by eqns. (3). Warping functions W and their derivatives are at first computed by solving eqns. (3a). Afterwards, these values are inserted as generalized body forces in eqns. (3b,c) which are solved as a 2D elasticity problem in order to obtain distortional functions D. The solution of the problem is accomplished employing BEM within the context of the method of subdomains and BEM for Navier operator [33, 38]. Afterwards, the values are normalized and the procedure is repeated for the desired number of modes. Finally, the functions calculated are employed in order to obtain the cross sectional operation factors given in eqns. (6) as well as mass operation factors derived by eqn. (9). These are used as input values together with the elasticity and function matrices to solve the curved beam model with the methods described below.

According to the precedent analysis, the analysis of curved beams of arbitrary cross section including generalized warping and distortional effects reduces in establishing the components the kinematical components u_i having continuous derivatives up to the second order with respect to x at the interval $(0, L)$, up to the first order at $x = 0, L$, and for the dynamic problem up to the second order with respect to time *t*, satisfying the initial-boundary value problem described by the coupled governing differential equations of equilibrium along the beam and the boundary conditions at the beam ends, at $x = 0, L$ as well as the initial conditions [37]. The problem is solved using the Analog Equation Method in a similar way as the one described in detail [37]. However, some differences arise here due to the nature of the problem stated. The number of the kinematical components depends on the number of modes employed.

4 NUMERICAL EXAMPLES

4.1 Doubly Symmetric box-shaped cross section

A cantilever beam model is studied. This has a rectangular box-shaped cross section 5.0X3.5 *m* with plate thickness 0.30 *m* ($E = 3E7\,kN/m^2$, $G = 1.5E7\,kN/m^2$, $\nu = 0$, $t/d = 0.085$, $d/L = 0.087$). Its cross section is shown in Fig. 3. This beam is examined as curved with $R = 25.465m$ and an arc length L of 40 *m*. The displacement field considered is the one described in eqns. (4).

Finally, in Table 1 the first eight eigenfrequencies are compiled for the curved ($\rho = 2.5kN\,sec^2/m^4$) rectangular box-shaped cross section when employing FEM solid

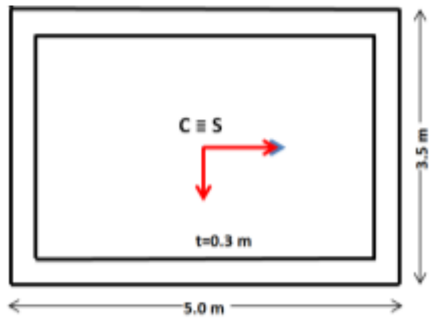

Figure 3: Rectangular box-shaped cross section (t=0.3m and d=3.5m).

Mode Number	FEMsolid 2880 NO Diaph.	FEMsolid 2880 8 Diaphs.	FEMsolid 2880 NO Diaph. (straight)	10 cubic B-splines in AEM	Type of mode (Only for NO Diaph.)
1	1.605	1.726	1.630	1.611	1^{st} mode of Vertical displacement (insignificant distortion)
2	2.221	2.261	2.168	2.155	1^{st} mode of Lateral displacement (insignificant distortion)
3	7.038	7.329	9.167	7.063	2^{nd} mode of Vertical displacement (significant distortion)
4	9.440	9.626	12.099	9.296	2^{nd} mode of Lateral displacement (significant distortion)
5	14.455	16.108	12.791	14.795	1^{st} mode of Torsion (significant distortion)
6	19.131	22.770	21.591	20.552	3^{rd} mode of Vertical displacement (excessive distortion)
7	23.306	32.479	29.194	22.961	3^{rd} mode of Lateral displacement (excessive distortion)
8	23.478	41.895	22.848	25.312	2^{nd} mode of Torsion (excessive distortion)

Table 1: Eigenfrequencies for the doubly symmetric box-shaped cross section curved or straight beam.

models with 8 or without diaphragms (NO Diaph.) and the proposed beam formulation with cubic B-splines in AEM as well as for the corresponding straight beam element (forth column of the Table 1) with FEM solid elements without the use of any diaphragms. It is obvious that the values obtained by the proposed beam formulation are in well coincidence with those of the FEM solid solution without any diaphragms. The placement of diaphragms results in a slight increase of the eigenfrequencies of the first four modes and a significant increase for the rest four modes due to the fact that distortion becomes more important as indicated from the description of the modes (last column of the Table 1). In addition to these, comparing to the straight beam formulation, the behavior of the beam is different. The first two modes exhibit similar eigenfrequencies for both the curved and straight model. Regarding the rest of the modes, significant discrepancies can be noticed either in the values of the eigenfrequencies or in the order of modes' significance. Particularly, it seems that torsional modes (5^{th} and 7^{th} for the straight beam) are more of importance in the straight beam model comparing to bending modes due to the lower value of the 5^{th} eigenfrequency and the altered order of significance between the 7^{th} and 8^{th} modes.

4.2 Monosymmetric open-shaped cross sections

A cantilever beam of a monosymmetric C-shaped cross section ($E = 73000\,kN/m^2$, $G = 28000\,kN/m^2$, $\nu = 0.3$, $t/d = 0.049$, $d/L = 0.055$), as this is shown in Fig. 4, is examined. The beam is analysed as curved with $R = 0.636m$ and an arc length L of 1 m.

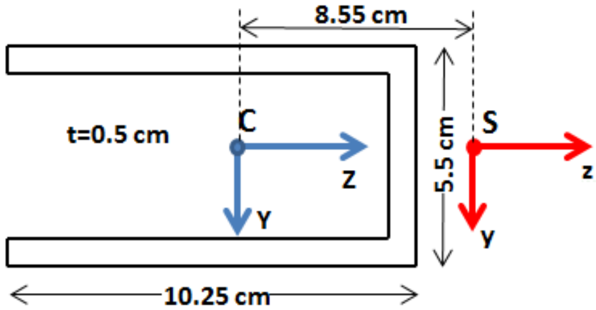

Figure 4: C-shaped cross section (t=0.5cm and d=5.5cm).

In Table 2 the first five eigenfrequencies are compiled for the curved ($\rho = 0.785 N\, sec^2 / m^4$) C-shaped cross section when employing a FEM solid model without

Mode Number	FEMsolid NO Diaph.	FEMsolid NO Diaph. (straight)	10 cubic NURBS	Type of mode
1	0.341	0.630	0.355	1^{st} mode of Vertical displacement
2	1.843	1.749	1.849	1^{st} mode of Lateral displacement
3	2.506	2.712	2.581	2^{nd} mode of Vertical displacement (excessive Torsion)
4	7.244	3.288	6.626	1^{st} mode of Torsion
5	7.908	5.990	8.376	2^{nd} mode of Torsion

Table 2: Eigenfrequencies for the C-shaped cross section curved or straight beam.

diaphragms (NO Diaph.) and the proposed beam formulation with cubic NURBS as well as for the corresponding straight beam element (second column of the Table 2) with FEM solid elements without the use of any diaphragms. It is obvious that the values obtained by the proposed beam formulation are in well coincidence with those of the FEM solid solution without any diaphragms. Comparing to the straight beam formulation, the behavior of the beam is different. The first mode exhibits a much lower eigenfrequency for the curved beam model. In general, modes of vertical displacement in curved arrangement are coupled with torsional modes. Particularly, the 4th mode of the straight beam is pure torsional, while all of the vertical modes of the curved beam exhibit torsional displacements, too. It should also be noted that after the 3rd mode, the eigenfrequencies of the straight beam are lower than those of the curved model, which seems to be stiffer.

4.3 Monosymmetric box-shaped cross sections

The last case studied is a box-shaped cross section curved beam ($E = 3.25E7\, kN/m^2$, $G = 1.39E7\, kN/m^2$, $R = 100\, m$, $\nu = 0.1667$, $t/d = 0.1$, $d/L = 0.065$) with an arc length L of 33 *m*. Its cross section is shown in Fig. 5.

In Table 3 the first eight eigenfrequencies are compiled for the curved ($\rho = 2.5 kN\, sec^2/m^4$) monosymmetric box-shaped cross section when employing FEM solid models with 1 or without diaphragms (NO Diaph.) and the proposed beam formulation with cubic NURBS. Both clamped and cantilever beam models have been studied for the FEM solid model without diaphragms (NO Diaph.). It is obvious that the values obtained by the proposed beam formulation are in well coincidence with those of the FEM solid solution without any diaphragms and the accuracy is improved comparing to the corresponding values compiled in Table 4.19 (where distortional effects had not been considered). The placement of the diaphragm at the midpoint of the curved length (forth column of the Table 5.13) results in a slight increase of the eigenfrequencies of the first four modes and a significant increase for the 5th and 6th modes. In addition to these, comparing to the cantilevered model, the behavior of the beam is much different. All of the eigenfrequencies are decreased while the order of significance is altered for the 6th and7th modes. In general distortional effects are of more importance for most of the modes (3rd to 8th) and torsional modes become more significant comparing to the clamped beam model. It should also be noted here that the procedure of finding the modes of the cantilever beam corresponding to the same ones of the clamped beam is quite cumbersome due to the fact that many local vibrational modes arise in the FEM solid model.

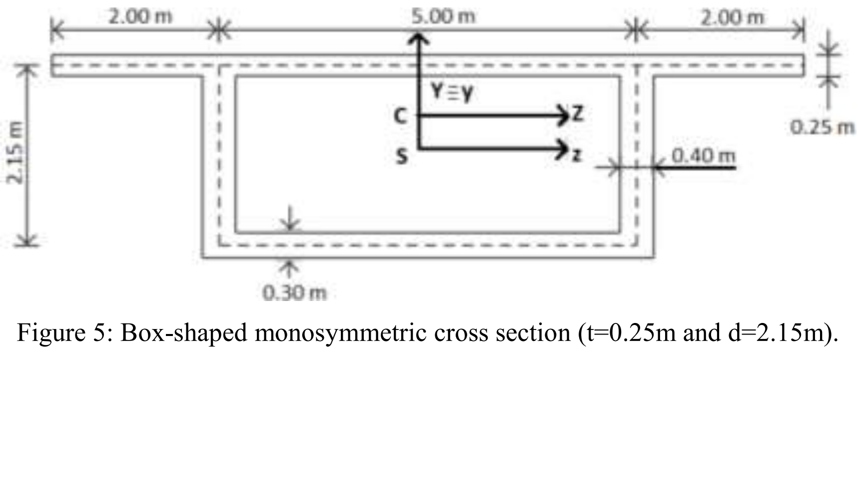

Figure 5: Box-shaped monosymmetric cross section (t=0.25m and d=2.15m).

Mode Number	FEMsolid 6600 NO Diaph. (cantilever)	FEMsolid 6600 NO Diaph. (clamped)	FEMsolid 6600 1 Diaph. (clamped)	10 cubic NURBS (clamped)	Type of mode
1	1.725	9.328	9.414	9.470	1st mode of Vertical displacement
2	4.065	17.099	19.230	16.887	1st mode of Lateral displacement
3	9.084	20.495	21.160	21.154	1st mode of Torsion
4	10.183	21.174	22.126	21.949	2nd mode of Vertical displacement
5	19.191	27.898	35.428	26.003	2nd mode of Lateral displacement
6	22.321	31.948	36.230	32.789	3rd mode of Vertical displacement
7	21.649	43.247	42.768	44.500	2nd mode of Torsion
8	29.165	47.490	47.013	49.602	3rd mode of Torsion

Table 3: Eigenfrequencies for the monosymmetric box-shaped cross section cantilever or clamped beam.

5 CONCLUSIONS

The importance of the proposed formulation is highlighted when considering the advantages of beam models compared to solid ones for the free vibrational case, as it is mentioned in the introduction. Therefore, the main purpose of this beam formulation is to remain simple and with the least number of degrees of freedom needed to describe its behavior accurately. In addition to this, creation of coarse curved models with quadrilateral solid elements and diaphragms is very time-consuming. The main conclusions that can be drawn from this investigation are:

- Highly accurate results can be obtained using B-splines in the AEM technique as well as NURBS in FE beam formulations for the analysis of the proposed beam elements. Computational cost and post-processing of the results is significantly reduced by the use of NURBS comparing to FEM solid models.
- The distortion of thin-walled box-shaped beams contributes significantly to lowering the natural frequency of torsional and bending vibration modes. Therefore, distortional effects must be considered in order to predict the dynamic behavior of beams accurately.
- Curved geometry alters the dynamic behavior of beam models with open or closed-shaped cross section and not necessarily in the same way.

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