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CYLINDRICAL WAVES IN STRUCTURES PERIODIC IN POLAR COORDINATES

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Abstract. *The paper is concerned with application of the Floquet theory for analysis of wave propagation in periodic in polar coordinates elastic structures. Although analysis of the periodicity effects in structures periodic in Cartesian coordinates is a well-established research subject, much less attention has been paid so far to vibroisolation in a membrane or a plate loaded by a point force or by a force distributed on the circle. The cancellation of the energy transfer in such a case may be achieved by employing periodicity of circular strips co-centric with the circle, where the driving force is applied. The paper addresses the similarities and differences in performance of infinite and finite structures periodic in Cartesian and polar coordinates.*

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1 INTRODUCTION

The periodicity effects are well-known and understood for a waveguide in Cartesian coordinates regardless its dimension. The underlying theory customarily referred to as the Floquet theory (or the Floquet theorem) relies on the translational invariance of the problem formulation for an infinite periodic structure.

In practice it is also of interest to consider periodic structures in other coordinate systems. Specifically, a compact source of vibrations (e.g., an operating pump) may be installed at a relatively large and flexible plate (e.g., a ship's deck). To avoid transmission of the structure-borne sound, a sequence of co-centric circular strips with alternating properties may be deployed with the point, where a pump is mounted, as the centre. It is natural to use polar coordinates to describe the wave propagation in such a structure. It is also plausible to expect that the destructive interference of transmitted and reflected waves produces the stop band effect as is known for a 'chessboard'-type periodic structure. Obviously, in the far-field zone, this effect must asymptotically match the predictions obtained for Cartesian coordinates. However, the applicability of 'straightforward' periodicity conditions in the near-field, i.e., in the relative vicinity of the origin of the polar coordinates needs verification. This verification constitutes the main research goal of this article. The case-study is concerned with circularly periodic elastic membrane. The location of pass and stop bands is compared with the power flow analysis for an infinite membrane with a periodic insert and with the distribution of eigenfrequencies of finite structures composed of several periodicity cells.

2 ANALYSIS OF THE AXISYMMETRIC WAVE IN A CIRCULARLY PERIODIC MEMBRANE

In a one-dimensional case, the Helmholtz equation, which describes time-harmonic plane wave propagation in an acoustic duct, or in an elastic string, or in an elastic rod, is the simplest model to formulate the Floquet theory for periodic structure composed of continuous constituents. This equation converted to polar coordinates and specialized for the axisymmetric case describes time harmonic propagation of the cylindrical wave of dilatation in an elastic layer under plane strain conditions, or in a membrane under constant uniform tension, and is perfectly suited for analysis of periodicity effects in this system of coordinates. It should also be noted that the Helmholtz equation in the axisymmetric case describes propagation of the cylindrical wave of anti-plane shear deformation in an elastic layer. All results obtained in this Section for the axisymmetric wave in a circularly periodic membrane, therefore, are fully applicable for the other problems listed above.

2.1 The power flow in an infinite periodic membrane with finite number of periodicity cells

We consider a circularly periodic membrane shown on Figure 1:

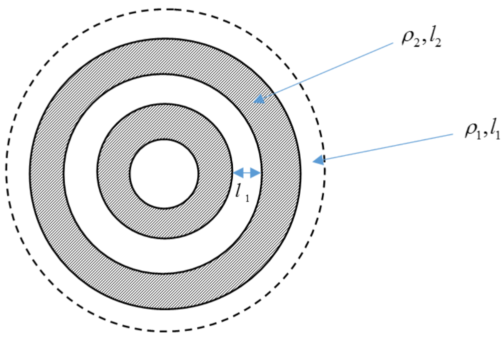

Figure 1: A circularly periodic elastic layer

Periodic membrane is considered in a way that allows to compare results with Cartesian coordinates case and as periodicity parameter equal radial length of pieces with respect to coordinate r is taken as shown on Fig.2 The “black” and “white” segments have different densities and lengths, but exposed to the same uniform tension. The following dimensionless parameters are used in this work:

$$\gamma = \frac{l_2}{l_1}; \sigma = \frac{c_2}{c_1}; k_1 l_1 = \Omega \tag{1}$$

In any application, a finite number of periodicity cells may be deployed to reduce power flow. In order to obtain basic understanding of waveguide properties of circularly periodic membrane, it is expedient to consider a forcing problem for an infinite structure with a finite number of circular inserts. Schematically, it can be illustrated as shown in Figure 2:

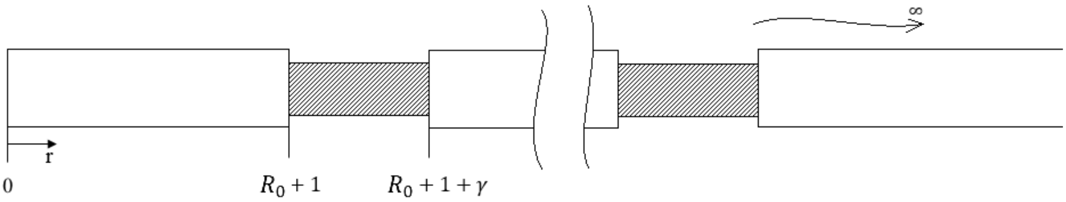

Figure 2: Scheme for energy flow

The convenient measure to characterize the effect of vibro-isolation is the insertion losses (IL), defined as $Il(n) = 10\log_{10}(\frac{E_n}{E_0})$, where E_0 is the power flow in an infinite uniform membrane, and E_n is the power flow under same excitation conditions in the membrane with n consequent equally spaced inserts, or periodicity cells.

We begin with solving the forcing problem for a circularly periodic membrane. In Cartesian coordinates, the position of the excitation force may be chosen arbitrarily. In polar coordinates, the distance from the origin of coordinates to the circle, where the excitation force is applied, is the independent parameter. In Figure 2, it is designated as R_0 and it is scaled with the length of the segment 1.

The energy flow (in order to obtain non-trivial power flow, the unit force is applied axisymmetrically at $r = R_0 + 1$) throughout an arbitrary circular contour, co-centric with the loaded circle of the membrane, is found as:

$$E(r) = -\frac{\Omega}{2} 2\pi r \mathrm{Re}((-iu(r))^* \cdot u'(r)) \tag{2}$$

where u^* is complex conjugated value of u.

As in the case of the axial vibration, it is expedient to plot the frequency-dependence of the energy flow. By these means, the zones, where the energy propagation is strongly attenuated can be clearly seen. Also, as shown in [1] certain number of symmetrical cell are required in order to see stop-bands, where the attenuation level is sensitive to this number. For illustrative matters, here the 1/10 part of insertion loss is plotted (the Floquet zones for Cartesian coordinates are taken from [2], Fig.2):

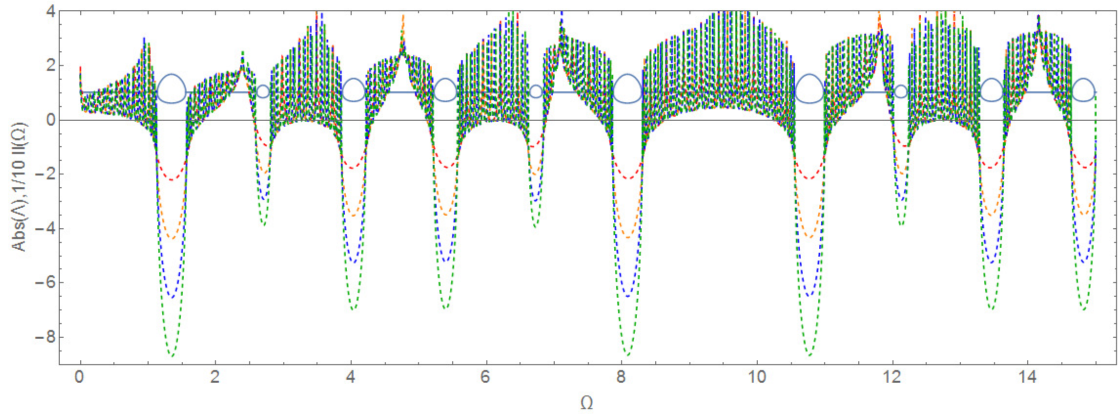

Figure 3: Insertion losses for five(red), ten(orange), fifteen(blue) twenty(green) periodicity cells compared to an axial Floquet zones.

These results suggest that the periodicity-induced attenuation exists for the cylindrical waves and that the positions of pass and stop bands determined for a structure periodic in Cartesian coordinates match very accurately positions of the high IL frequency bands. It is also seen that the dependence of attenuation levels upon the amount of periodicity cells is virtually the same as for a periodic structure in Cartesian coordinates. These results are hardly surprising. The increase in amount of periodicity cells means that the distance from the origin of coordinates to any added cell grows, and so the geometry of the polar system becomes very close to the Cartesian one. The same holds true then the amount of periodicity cells is kept constant, but the distance from the origin of coordinate system to the periodic insert grows.

However, the ‘far-field’ analysis of wave propagation in periodic structures in polar coordinates is not of much value both from the practical and from the theoretical points of view. Theoretically, the only point of interest is the convergence rate, while in any application the practical purpose is to reduce transmission of the vibro-acoustic energy as close to the source as possible.

2.2 Floquet theory approximations

The analysis of propagation of free waves in the structure shown in Fig.2 is relatively simple, because each of its components supports only one cylindrical wave, which propagates at any frequency. Therefore, various means, including the transfer matrix method, may be successfully used.

In order to derive the characteristic equation, we consider three adjacent components of a periodic layer located as shown in Fig.1 Since Bessel $Y(r)$ function contains singularity at point $r=0$, we consider any periodicity cell except for the one containing the origin of coordinates. Therefore, the left boundary of the first component of a periodicity cell is shifted on a distance R_0 from the origin of coordinates, $r=0$.

First, the interfacial conditions between adjacent components are formulated. They represent the continuity of displacements and balance of forces at the interface:

$$\begin{aligned} u_1(1+R_0)=u_2(1+R_0) \quad u_2((1+\gamma)+R_0)=u_3((1+\gamma)+R_0) \\ u_1'(1+R_0)=u_2'(1+R_0) \quad u_2'((1+\gamma)+R_0)=u_3'((1+\gamma)+R_0) \end{aligned} \tag{3}$$

In Cartesian coordinates, the Floquet theorem (with $\Lambda = const$) is used for displacements, and this condition may certainly be formulated in polar coordinates:

$$u_1(R_0)=\Lambda u_3((1+\gamma)+R_0) \tag{4}$$

However, the parameter Λ must in this case feature the dependence on the radial coordinate $\Lambda = \Lambda(R_0)$. Then differentiation of Eq.4 with respect to R_0 gives:

$$u_1'(R_0) = \Lambda u_3'((1+\gamma)+R_0) + \Lambda' u_3((1+\gamma)+R_0) \tag{5}$$

In Cartesian coordinates, the condition $\Lambda' = 0$ holds true and equations Eq.3- Eq.5 are invariant to the magnitude of R_0 because the general solutions have the exponential form. Then the canonical formulation of periodicity conditions is recovered.

In what follows, we explore several approximate formulations of the Floquet theory in polar coordinates Equations Eq.3- Eq.5 represent system of six homogenous linear algebraic equations. In order to find non-trivial solution, the condition $D(\Lambda,\Omega) = 0$ must be fulfilled, where $D(\Lambda,\Omega)$ is the determinant of the system Eq.3-Eq.5. In this case Floquet polynomial has a structure:

$$D(\Lambda,\Omega) = \Lambda^2 + a_1(R_0,\Omega)\Lambda + a_0(R_0,\Omega) \tag{6}$$

Coefficients $a_i(R_0,\Omega)$ are cumbersome combinations of Bessel functions and, therefore, not presented here. It should be emphasized, that, even if Λ is assumed to be constant, the end result still has dependence on R_0.

Obviously, transformation from Cartesian to polar coordinates means that the cylindrical waves are considered instead of the plane ones. Therefore, the 'natural' decay of the cylindrical wave in a homogeneous membrane needs to be accounted for. In Cartesian coordinates for a homogeneous membrane (i.e., when $\sigma = 1$) we obtain the periodicity parameter (system analogous to Eq.3- Eq.5 is used) in the simple form:

$$\Lambda_{cartesian} = \exp(\pm i\Omega(1+\gamma)) \tag{7}$$

This formula gives the limit for the values of Λ for a uniform membrane considered in polar coordinates when $R_0 \to \infty$.

To assess Λ for a uniform membrane in the far field, the single term approximation of a Bessel function of large argument may be used:

$$\begin{aligned} Y_m(z) &\approx \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{z}\right) \\ J_m(z) &\approx \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{z}\right) \end{aligned} \tag{8}$$

With approximation Eq.8 substituted into Eq.3- Eq.5 periodicity parameter for a homogenous membrane in polar coordinates has a form:

$$\Lambda_{polar}^{approx} = \sqrt{\frac{2+\gamma+R_0}{1+R_0}}\exp(\pm i\Omega(1+\gamma)) = \sqrt{\frac{2+\gamma+R_0}{1+R_0}}\Lambda_{cartesian} \tag{9}$$

Since Eq.9 is valid for both "all white" and "all black" homogenous membranes, it can be used in periodic membrane. This formula now features the dependence of Λ upon R_0, which is, however, approximate and tends to the correct limit when $R_0 \to \infty$.

Periodicity conditions for such a membrane with Eq.9 can be rewritten now following the Cartesian pattern:

$$u_1(R_0)=\sqrt{\frac{1+\gamma+R_0}{R_0}}\Lambda_{cart}u_3((1+\gamma)+R_0) \quad (10)$$

Thus, after differentiation with respect to R_0, we obtain:

$$u_1'(R_0)=\sqrt{\frac{1+\gamma+R_0}{R_0}}\Lambda_{cart}u_3'((1+\gamma)+R_0)+(\sqrt{\frac{1+\gamma+R_0}{R_0}})'\Lambda_{cart}u_3((1+\gamma)+R_0) \quad (11)$$

Interfacial Eq.3 and periodicity Eq.10-Eq.11 conditions form a new system with the determinant $D_{R_0}^{(1)}(\Lambda,\Omega)$, which is the second-order polynomial in Λ_{cart}. Solutions of the equation $D_{R_0}^{(1)}(\Lambda,\Omega)=0$ can now be plotted as shown in Figure 4:

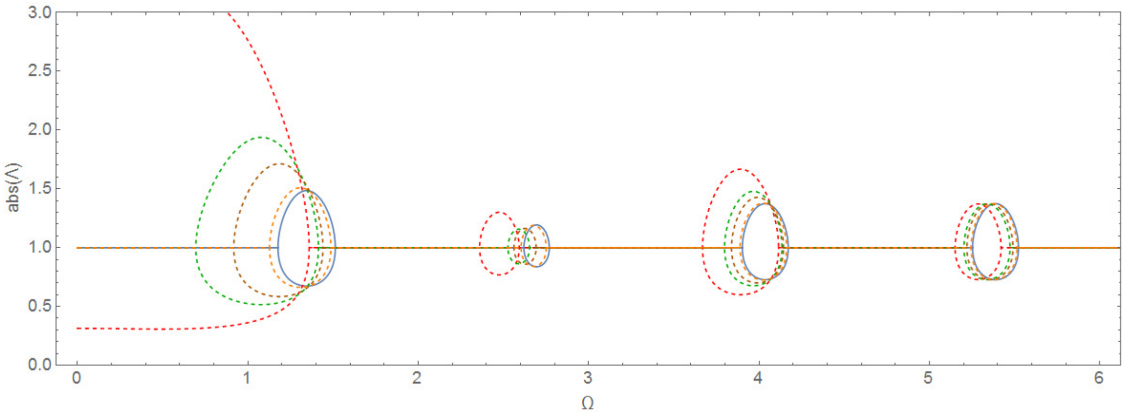

Figure 4: Approximation $D_{R_0}^{(1)}$ of Floquet zones of a membrane for $R_0=0.1,0.15,0.2,1$ (red,green,brown,orange dashed)

The approximation $D^{(1)}$ predicts the spurious stop band at zero frequency, because the series Eq.8 are not applicable for Bessel functions of a small argument. This spurious stop band vanishes at $R_0>0.14349$. The value $R_0\approx 0.14349$ is the first zero of discriminant of polynomial $D_{R_0}^{(1)}(\Lambda,\Omega)$ in Λ, and it may be considered as the limit of applicability of the approximation $D_{R_0}^{(1)}$.

Refined approximation can be written as:

$$\begin{aligned}Y_m(z)&\approx\sqrt{\frac{2}{\pi z}}(\cos(z-\frac{m\pi}{2}-\frac{\pi}{4})-\frac{1}{8z}\sin(z-\frac{m\pi}{2}-\frac{\pi}{4}))+O(\frac{1}{z^2})\\ J_m(z)&\approx\sqrt{\frac{2}{\pi z}}(\sin(z-\frac{m\pi}{2}-\frac{\pi}{4})-\frac{\pi}{4})-\frac{1}{8z}\cos(z-\frac{m\pi}{2}-\frac{\pi}{4}))+O(\frac{1}{z^2})\end{aligned} \quad (12)$$

Procedure of obtaining the periodicity coefficient for a homogenous membrane in this case is the same as before, and this coefficient acquires the form:

$$\Lambda_{polar}^{(2)}=c_0(R_0,\Omega)\exp(i\Omega(1+\gamma))+c_1(R_0,\Omega)(1+\gamma)\Omega\exp(i\Omega(1+\gamma)) \quad (13)$$

where:

$$c_0(R_0,\Omega)=\sqrt{\frac{1+\gamma+R_0}{R_0}}-\frac{3(1+\gamma)\sqrt{1+\gamma+R_0}}{R_0^{3/2}(64\Omega^2(1+\gamma+R_0)^2-3)}$$
$$c_1(R_0,\Omega)=(\frac{1+\gamma+R_0}{R_0})^{3/2}\frac{8}{64\Omega^2(1+\gamma+R_0)^2-3} \tag{14}$$

The term $c_0(R_0,\Omega)\exp(i\Omega(1+\gamma))$ should be treated as in previous case. For brevity, the term $c_1(R_0)(1+\gamma)\Omega\exp(i\Omega(1+\gamma))$ may be omitted as a higher-order one in the series expansion with respect to Ω. Then the periodicity conditions with Eq.4-Eq.5 are rewritten following the pattern, described above as:

$$u_1(R_0)=c_0(R_0,\Omega)\Lambda_{cart}u_3((1+\gamma)+R_0)$$
$$u_1'(R_0)=c_0(R_0,\Omega)\Lambda_{cart}u_3'((1+\gamma)+R_0)+\frac{\partial c_0(R_0,\Omega)}{\partial R_0}\Lambda_{cart}u_3((1+\gamma)+R_0) \tag{15}$$

Interfacial Eq.3 and periodicity Eq.15 conditions give the determinant $D_{R_0}^{(2)}(\Lambda,\Omega)$, which is again the second-order polynomial in Λ_{cart}. Solutions of the equation $D_{R_0}^{(2)}(\Lambda,\Omega)=0$ can now be plotted as shown in Figure 5:

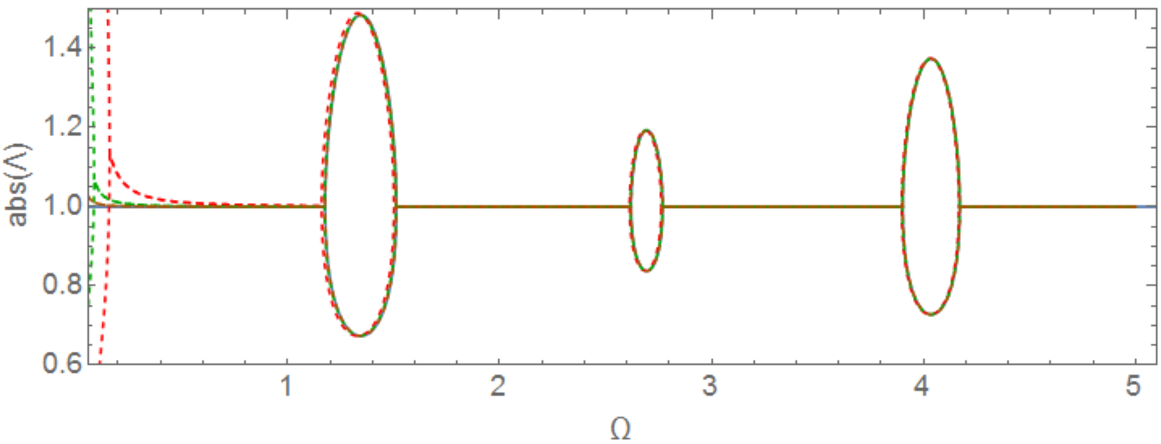

Figure 5: Approximation $D_{R_0}^{(2)}$ of Floquet zones of a membrane for $R_0=2,5,10$ (red, green, brown, dashed)

Although the authentic stop bands are found more accurately, introduction of an additional term in Bessel function approximation introduces singular points with respect to frequency Ω in the very low frequency range. These singularities do not have any physical meaning and, as the spurious stop band in the previous case, are present in the frequency range, where the series Eq.12 are invalid.

That limits the range of applicability of approximations Eq.8 and Eq.12 in the low frequency region. Nevertheless, both approximations give same results as the frequency grows. Therefore, it is concluded, that further work on formulation of periodicity conditions in the near field should be done.

2.3 Finite structures

Every finite periodic structure can be considered as series of 'unit' symmetrical periodicity cells. Such a periodicity cell is shown in Figure 6:

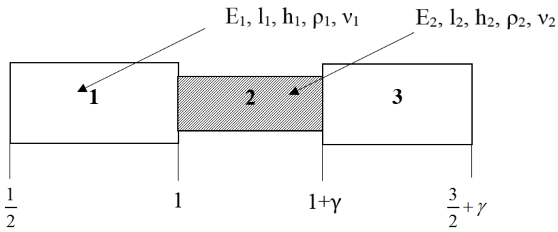

Figure 6: Symmetrical periodicity cell

In order to find eigenfrequencies interfacial conditions should be stated:

$$\begin{aligned} u_1(1+R_0) &= u_2(1+R_0) \\ u_1'(1+R_0) &= u_2'(1+R_0) \\ u_2(1+\gamma+R_0) &= u_3(1+\gamma+R_0) \\ u_2'(1+\gamma+R_0) &= u_3'(1+\gamma+R_0) \end{aligned} \tag{16}$$

The two types of boundary conditions are considered (see [1-2] for details):

$$\begin{aligned} u_1(1/2+R_0) &= 0 \\ u_3(3/2+\gamma+R_0) &= 0 \end{aligned} \tag{17}$$

$$\begin{aligned} u_1'(1/2+R_0) &= 0 \\ u_3'(3/2+\gamma+R_0) &= 0 \end{aligned} \tag{18}$$

Equations Eq.16-Eq.17 defines eigenfrequency problem with “fixed” ends or A-type boundary conditions, equations Eq.16 and Eq.18 defines problem with “free” ends or B-type boundary conditions (the terminology for these conditions follows Mead [2]). Roots, obviously, can be found only numerically, since no inverse of the Bessel function exists.

As proven in [1] for Cartesian coordinates, eigenfrequencies of symmetrical periodicity cell are placed on stop-band boundaries. With this formulation in polar coordinates, eigenfrequencies at the low-frequency range are slightly dependent on parameter R_0, therefore we can talk about this property only approximately:

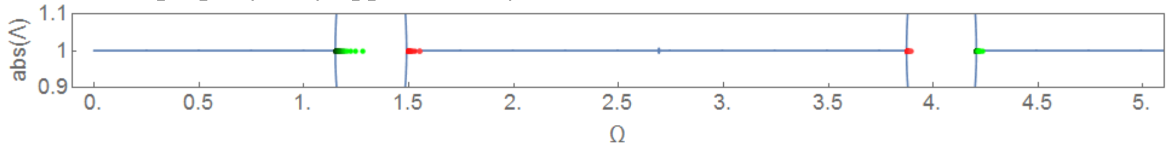

Figure 7: First eigenfrequency of single symmetric periodicity cells (red – eigenfrequencies of "fixed" problem, green – "free") and stop-band boundaries (blue)

As in the axial case, all eigenfrequencies of a structure, which consists of more than one periodicity cell appears in pass bands. Similarly, all eigenfrequencies of a structure consisting of a given amount of periodicity cells are contained in the spectrum of a structure with larger amount of these cells.

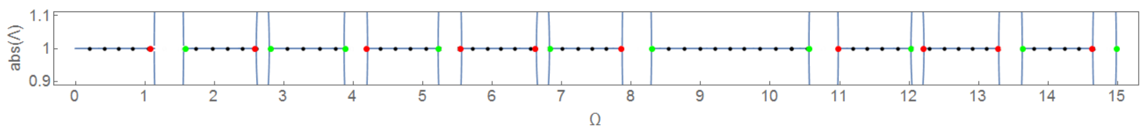

Figure 8**:** Eigenfrequency of structure with 5 symmetrical periodicity cells ($R_0 = 10$)

It should be emphasized that these property are found for any periodic structure regardless a value of the parameter R_0.

As seen, solutions of the membrane equation for structures, periodic in polar coordinates, have the properties, similar to the solutions in Cartesian coordinates. Furthermore, the Floquet theory with some adjustments can be used in polar coordinates. This constituted the on-going research work of the authors.

3 CONCLUSIONS

- The extension of the classical Floquet theory to the structures periodic in polar coordinates is done for a representative example of propagation of axisymmetric cylindrical waves in a periodic membrane. The frequency ranges, where power flow is suppressed, are identified and compared with their counterparts for the structure in Cartesian coordinates.
- The alternative formulations of periodicity conditions for polar coordinates are used to assess location of pass- and stop-bands and the results are discussed.
- The relation between eigenfrequencies of finite periodic structures and frequencies separating pass- and stop-bands known from classical Floquet theory is shown to be asymptotically valid in polar coordinates.

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