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HARMONIC AND RANDOM VIBRATIONS OF ANISOTROPIC BEAM

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Abstract. *Vibrations of a thin elastic beam-strip are studied. A beam is made of an anisotropic material heterogeneous in the thickness direction. The 1D model of second-order accuracy is delivered by using asymptotic expansions in powers of the relative beam thickness. A special attention is paid to the slanted anisotropy with 6 elastic modules. A spectrum of bending vibrations in the case of simply supported ends of a beam is constructed. Forced vibrations under action of a harmonic and a random excitations are studied. In the last case, the root-mean-square of deflections are found in dependence of a type of excitation.*

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1 INTRODUCTION

In this work vibrations of a thin elastic beam (strip) made of an anisotropic material heterogeneous in the thickness direction are studied. The 1D model of second-order accuracy is put forwards using asymptotic expansions in powers of the relative beam thickness. A lot of investigations are devoted to obtaining approximate equations describing beams, plates and shells deformations. For homogeneous isotropic material the well-known Kirchhoff–Love and Timoshenko–Reissner models may be used. For anisotropic materials with a general anisotropy (with 21 elastic modules) the additional difficulties arise [1, 2, 3, 4, 5]. The more exact equations of second-order accuracy for beams and plates made of a transversely isotropic heterogeneous (or multi-layered) material were constructed in [6, 7, 8, 9]. In the case of the general anisotropy the construction of models of second-order accuracy is more difficult. In [10] such a model for an infinite long-waved beam vibrations and waves was constructed.

In the present paper the same problem is solved for a beam with finite length and simply supported edges. Forced harmonic and random vibrations are studied. The effects of general anisotropy are discussed.

2 THE MAIN EQUATIONS AND ASSUMPTIONS

A linear plane dynamic problem for a multi-layered anisotropic beam-strip of constant thickness h and length L is studied. A system of equations is as follows:

$$\frac{\partial\,\sigma_{11}}{\partial x}+\frac{\partial\,\sigma_{13}}{\partial z}-\rho\frac{\partial^2 u}{\partial t^2}+f_1(x,z,t)=0,\qquad \frac{\partial\,\sigma_{13}}{\partial x}+\frac{\partial\,\sigma_{33}}{\partial z}-\rho\frac{\partial^2 w}{\partial t^2}+f_3(x,z,t)=0, \tag{1}$$

where $x,\ z$ are the Cartesian co-ordinates ($0\le x\le L,\ 0\le z\le h$), t is the time, σ_{ij} are the stresses, $f_1,\ f_3$ are the projections of the external body load intensity, $u(x,z,t),\ w(x,z,t)$ are the deflection projections on the x- and z-directions, respectively, $\rho=\rho(z)$ is the material density (Fig. 1).

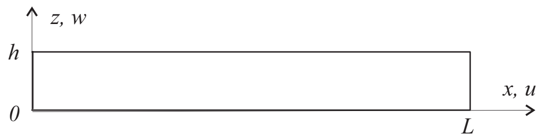

Figure 1: A beam.

The planes $z=0$ and $z=h$ are free, that leads to the boundary conditions

$$\sigma_{13}(x,z,t)=\sigma_{33}(x,z,t)=0,\quad z=0,h. \tag{2}$$

In the case of general anisotropy, the elasticity relations read as

$$\begin{aligned}\sigma_{11}&=E_{11}\varepsilon_{11}+H_1\varepsilon_{13}+E_{13}\varepsilon_{33},\\ \sigma_{13}&=H_1\varepsilon_{11}+G_{13}\varepsilon_{13}+H_3\varepsilon_{33},\\ \sigma_{33}&=E_{13}\varepsilon_{11}+H_3\varepsilon_{13}+E_{33}\varepsilon_{33},\end{aligned}\qquad \mathbf{E}=\begin{pmatrix}E_{11}&H_1&E_{13}\\ H_1&G_{13}&H_3\\ E_{13}&H_3&E_{33}\end{pmatrix}, \tag{3}$$

$$\varepsilon_{11}=\frac{\partial u}{\partial x},\quad \varepsilon_{13}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x},\quad \varepsilon_{33}=\frac{\partial w}{\partial z}, \tag{4}$$

where ε_{ij} are the strains.

Assume that the modules of elasticity in Eqs. (3) and the density ρ are independent of x, but they may depend on z. Therefore, the multilayered beams and the beams made of functionally graded material are not excluded from consideration. The matrix **E** is supposed to be positively definite.

From Eqs. (3) we have

$$\begin{aligned} \varepsilon_{33} &= c_3\sigma_{33} - c_\nu\varepsilon_{11} - c_1\varepsilon_{13}, \\ \varepsilon_{13} &= -c_h\varepsilon_{11} + c_g\sigma_{13} - c_1\sigma_{33}, \\ \sigma_{11} &= c_0\varepsilon_{11} + c_h\sigma_{13} + c_\nu\sigma_{33}, \end{aligned} \tag{5}$$

where

$$\begin{gathered} c_0 = E_{11} - \frac{E_{13}^2 G_{13} - 2H_1H_3E_{13} + H_1^2E_{33}}{\Delta_1} = \frac{\Delta}{\Delta_1} > 0, \quad \Delta_1 = G_{13}E_{33} - H_3^2 > 0, \\ c_3 = \frac{G_{13}}{\Delta_1}, \quad c_1 = \frac{H_3}{\Delta_1}, \quad c_h = \frac{E_{33}H_1 - E_{13}H_3}{\Delta_1}, \quad c_\nu = \frac{E_{13}G_{13} - H_1H_3}{\Delta_1}, \quad c_g = \frac{E_{33}}{\Delta_1}, \end{gathered} \tag{6}$$

and $\Delta > 0$ is the determinant of the matrix **E**.

The main unknowns in Eqs. (1), (3) are $w, u, \sigma_{13}, \sigma_{33}$. According to Eqs. (5) they satisfy the system:

$$\begin{aligned} \frac{\partial w}{\partial z} &= -c_\nu\frac{\partial u}{\partial x} - c_1\sigma_{13} + c_3\sigma_{33}, \\ \frac{\partial u}{\partial z} &= -\frac{\partial w}{\partial x} - c_h\frac{\partial u}{\partial x} + c_g\sigma_{13} - c_1\sigma_{33}, \\ \frac{\partial \sigma_{13}}{\partial z} &= -c_0\frac{\partial^2 u}{\partial x^2} - c_h\frac{\partial \sigma_{13}}{\partial x} - c_\nu\frac{\partial \sigma_{33}}{\partial x} + \rho\frac{\partial^2 u}{\partial t^2} - f_1(x,z,t), \\ \frac{\partial \sigma_{33}}{\partial z} &= -\frac{\partial \sigma_{13}}{\partial x} + \rho\frac{\partial^2 w}{\partial t^2} - f_3(x,z,t). \end{aligned} \tag{7}$$

We introduce the dimensionless variables (with the hat-sign):

$$\begin{gathered} \{u,w,z\} = h\{\hat{u},\hat{w},\hat{z}\}, \quad \{\sigma_{ij}, E_{ij}, G_{13}, H_i, c_0\} = E_0\{\hat{\sigma}_{ij}, \hat{E}_{ij}, \hat{G}_{13}, \hat{H}_i, \hat{c}_0\}, \\ x = L\hat{x}, \quad \{c_1, c_3, c_g\} = \frac{1}{E_0}\{\hat{c}_1, \hat{c}_3, \hat{c}_g\}, \quad \rho = \rho_0\hat{\rho}, \quad t = T\hat{t}, \quad \mu = \frac{h}{L}, \\ \{f_1, f_3\} = \frac{E_0}{h}\{\hat{f}_1, \hat{f}_3\}, \quad E_0 = \frac{1}{h}\int_0^h c_0(z)dz, \quad \rho_0 = \frac{1}{h}\int_0^h \rho(z)dz, \quad T = \sqrt{\frac{\rho_0 L^4}{E_0 h^2}}. \end{gathered} \tag{8}$$

Here, $E_0,\ \rho_0$ are the average values of the equivalent longitudinal stresses c_0 and of material density. The dimensionless time $\hat{t}$ is introduced so that the minimal natural frequency is of the order of unity, μ is the small thickness parameter. Further we omit the hat-sign.

In particular, for a beam made of an orthotropic material in the dimensionless notation we get:

$$H_1 = H_3 = 0,\ c_1 = c_h = 0,\ c_0 = E_{11} - \frac{E_{13}^2}{E_{33}},\ c_g = \frac{1}{G_{13}},\ c_\nu = \frac{E_{13}}{E_{33}},\ c_3 = \frac{1}{E_{33}} \tag{9}$$

and for an isotropic material:

$$c_0 = \frac{E}{E_0(1-\nu^2)},\ c_g = \frac{2(1+\nu)E_0}{E},\ c_\nu = \frac{\nu}{1-\nu},\ c_3 = \frac{E_0(1-2\nu)(1+\nu)}{E(1-\nu)}, \tag{10}$$

where $E(z), \nu(z)$ are, respectively, the Young modulus and the Poisson ratio.

If $H_1 \neq 0$ and/or $H_3 \neq 0$, we have the slanted anisotropy with $c_1 \neq 0$ and/or $c_h \neq 0$. In the opposite case the material is orthotropic or isotropic.

In the dimensionless variables the boundary-value problem (7), (2) is written as:

$$\begin{aligned}
&\frac{\partial w}{\partial z} = -\mu c_\nu \frac{\partial u}{\partial x} - c_1\sigma_{13} + c_3\sigma_{33}, \\
&\frac{\partial u}{\partial z} = -\mu \frac{\partial w}{\partial x} - \mu c_h \frac{\partial u}{\partial x} + c_g\sigma_{13} - c_1\sigma_{33}, \\
&\frac{\partial \sigma_{13}}{\partial z} = -\mu^2 c_0 \frac{\partial^2 u}{\partial x^2} - \mu c_h \frac{\partial \sigma_{13}}{\partial x} - \mu c_\nu \frac{\partial \sigma_{33}}{\partial x} + \mu^4 \rho \frac{\partial^2 u}{\partial t^2} - f_1 \equiv Z_1 \\
&\frac{\partial \sigma_{33}}{\partial z} = -\mu \frac{\partial \sigma_{13}}{\partial x} + \mu^4 \rho \frac{\partial^2 w}{\partial t^2} - f_3 \equiv Z_2, \\
&\sigma_{13}(x,z,t) = \sigma_{33}(x,z,t) = 0, \quad z = 0, 1.
\end{aligned} \tag{11}$$

The right-hand sides of Eqs. (11) are small — this is why the asymptotic expansions to construct the solution may be applied.

3 THE ASYMPTOTIC SOLUTION OF THE PROBLEM (11)

We assume that the plate is under action of the normal external pressure applied to the plane $z = 0$. Then in Eqs. (11)

$$f_1(x,z,t) = 0, \quad f_3(x,z,t) = F_3(t)\delta(z), \tag{12}$$

where $\delta(z)$ is the Dirac delta function, and $F_3(t)$ is independent of x.

We assume that for low-frequency transversal vibrations the deflection w is of the order of unity, and the differentiation in x and t does not change the orders of the unknown functions in Eqs. (11). Then the orders of the unknown functions are as follows:

$$w = O(1), \quad u = O(\mu), \quad \sigma_{13} = O(\mu^3), \quad \sigma_{33} = O(\mu^4), \tag{13}$$

with $F_3 = O(\mu^4)$.

We seek the solution of Eqs. (11) as a formal asymptotic series:

$$\begin{aligned}
&w(x,z,t,\mu) = w^{(0)}(x,z,t) + \mu w^{(1)}(x,z,t) + \mu^2 w^{(2)}(x,z,t), \\
&u(x,z,t,\mu) = \mu\left(u^{(0)}(x,z,t) + \mu u^{(1)}(x,z,t) + \mu^2 u^{(2)}(x,z,t)\right), \\
&\sigma_{13}(x,z,t,\mu) = \mu^3\left(\sigma_{13}^{(0)}(x,z,t) + \mu\sigma_{13}^{(1)}(x,z,t) + \mu^2\sigma_{13}^{(2)}(x,z,t)\right), \\
&\sigma_{33}(x,z,t,\mu) = \mu^4\left(\sigma_{33}^{(0)}(x,z,t) + \mu\sigma_{33}^{(1)}(x,z,t) + \mu^2\sigma_{33}^{(2)}(x,z,t)\right).
\end{aligned} \tag{14}$$

Further we shall content ourselves only with the first three terms of these series.

The integration in z of the two first equations (11) introduces arbitrary functions $w_n(x,t),\ u_n(x,t),\ n = 0,1,2$, and equations for these functions follow from the compatibility conditions of the rest two equations (11) and the boundary conditions $\sigma_{13} = \sigma_{13} = 0$ at $z = 0,1$:

$$\langle Z_k(z)\rangle = 0, \quad k = 1,2, \qquad \langle Y\rangle = \int_0^1 Y(z)\,dz, \tag{15}$$

where $\langle Y\rangle$ is the averaging operator in the thickness direction. In particular, the relations $\langle c_0\rangle = 1$ and $\langle \rho\rangle = 1$ are valid.

For shortness, we introduce the integral operator $\mathbf{I}(X) = \int_0^z X dz$. For any functions $X(z)$ and $Y(z)$, we have

$$\langle \mathbf{I}(X)\rangle = \langle X\rangle - \langle zX\rangle\,, \quad \langle X\mathbf{I}(Y)\rangle + \langle Y\mathbf{I}(X)\rangle = \langle X\rangle\,\langle Y\rangle\,. \tag{16}$$

The zero approximation. According to the first of equations (11) the function $w^{(0)}(x,z,t)$ does not depend on z, $w^{(0)}(x,z,t) = w_0(x,t)$. Then we find consequently

$$u^{(0)} = u_0(x,t) + z_*\frac{\partial w_0}{\partial x}, \quad z_* = a - z, \quad a = \langle c_0 z\rangle\,, \quad \sigma_{13}^{(0)} = -\mathbf{I}(c_0 z_*)\frac{\partial^3 w_0}{\partial x^3}, \tag{17}$$

$$D\frac{\partial^4 w_0}{\partial x^4} = -\frac{\partial^2 w_0}{\partial t^2} + F_3(t), \quad D = \langle \mathbf{I}(c_0 z_*)\rangle = \left\langle c_0 z_*^2\right\rangle, \qquad \frac{\partial^2 u_0}{\partial x^2} = 0, \tag{18}$$

$$\sigma_{33}^{(0)} = \mathbf{I}(\mathbf{I}(c_0 z_*))\frac{\partial^4 w_0}{\partial x^4} + \mathbf{I}(\rho)\frac{\partial^2 w_0}{\partial t^2} - F_3(t), \tag{19}$$

where a is the neutral lower co-ordinate, D is the dimensionless bending stiffness, u_0 is the horizontal deflection of the neutral layer. Equation (18) may be obtained in the frames of the Kirchhoff–Love Hypotheses with the corresponding value of the equivalent longitudinal stiffness c_0 (see (6)).

The zero and the second approximations coincide with those obtained in [7] - [9] for a transversely isotropic plate, because at $c_h = c_1 = 0$ the both problems are described by the same Eqs. (11).

For the slanted anisotropy in the ***first approximation*** again we have $w^{(1)} = w_1(x,t)$, where w_1 are independent of z, and

$$u^{(1)} = u_1(x,t) + z_*\frac{\partial w_1}{\partial x} - \mathbf{I}(c_h)\frac{\partial u_0}{\partial x} - \mathbf{I}(c_h z_*)\frac{\partial^2 w_0}{\partial x^2}. \tag{20}$$

The arbitrary functions u_1 and w_1 may be found from the compatibility conditions (15); they satisfy equations

$$\begin{aligned} &\frac{\partial^2 u_1}{\partial x^2} = U_1\frac{\partial^4 w_0}{\partial x^4} = \frac{U_1}{D}\left(F_3 - \frac{\partial^2 w_0}{\partial t^2}\right), \quad U_1 = \langle c_0\mathbf{I}(c_h z_*) + c_h\mathbf{I}(c_0 z_*)\rangle\,, \\ &D\frac{\partial^4 w_1}{\partial x^4} + \frac{\partial^2 w_1}{\partial t^2} = 0. \end{aligned} \tag{21}$$

Thus, we find $\sigma_{13}^{(1)}$ and $\sigma_{33}^{(1)}$. For the orthotropic beam $U_1 = 0$, the first approximation identically vanishing.

In *the second approximation*

$$
\begin{aligned}
w^{(2)} &= w_2 - \mathbf{I}(c_\nu)\frac{\partial u_0}{\partial x} - \mathbf{I}(c_\nu z_*)\frac{\partial^2 w_0}{\partial x^2}, \\
u^{(2)} &= u_2 + z_*\frac{\partial w_2}{\partial x} - \mathbf{I}(c_h)\frac{\partial u_1}{\partial x} - \mathbf{I}(c_h z_*)\frac{\partial^2 w_1}{\partial x^2} + \\
&\quad (\mathbf{II}(c_\nu z_*) + \mathbf{I}(c_h\mathbf{I}(c_h z_*)) - \mathbf{I}(c_g\mathbf{I}(c_0 z_*)))\frac{\partial^3 w_0}{\partial x^3}.
\end{aligned} \tag{22}
$$

As before, we find equations for the functions u_2 and w_2 from Eqs. (15):

$$
\begin{aligned}
&\frac{\partial^2 u_2}{\partial x^2} = \frac{\partial^2 u_0}{\partial t^2} - \frac{U_1}{D}\frac{\partial^2 w_1}{\partial t^2} - \frac{U_2}{D}\frac{\partial^3 w_0}{\partial x \partial t^2}, \\
&D\frac{\partial^4 w_2}{\partial x^4} = -\frac{\partial^2 w_2}{\partial t^2} - \frac{W_2}{D}\frac{\partial^4 w_0}{\partial x^2 \partial t^2} + (a_\rho - a)\frac{\partial^3 u_0}{\partial t^2 \partial x}, \quad a_\rho = \langle z\rho \rangle .
\end{aligned} \tag{23}
$$

According to the expansions (14) and Eqs. (18), (21) and (22) we get the equations of second-order accuracy for the functions $u(x,t) = \mu u_0 + \mu^2 u_1 + \mu^3 u_2$ and $w(x,t) = w_0 + \mu w_1 + \mu^2 w_2$:

$$
\frac{\partial^2 u}{\partial x^2} = \mu^2\frac{\partial^2 u}{\partial t^2} - \mu^2\frac{U_1}{D}\frac{\partial^2 w}{\partial t^2} - \mu^3\frac{U_2}{D}\frac{\partial^3 w}{\partial x \partial t^2} + \mu^2\frac{U_1}{D}F_3, \tag{24}
$$

$$
D\frac{\partial^4 w}{\partial x^4} = -\frac{\partial^2 w}{\partial t^2} + F_3 - \mu^2\frac{W_2}{D}\frac{\partial^4 w}{\partial x^2 \partial t^2} + \mu(a_\rho - a)\frac{\partial^3 u}{\partial t^2 \partial x}. \tag{25}
$$

Here $u(x,t)$ is the horizontal deflection of a point from the neutral line $z = a$, and $w(x,t)$ is the vertical deflection from the line $z = 0$.

In the zero approximation it is possible to study the longitudinal and bending vibrations separately. In the first approximation (for slanted anisotropy with $U_1 \neq 0$) the longitudinal vibrations depend on the bending ones. In the second approximation Eqs. (24) and (25) in the general case are connected with each other. But for $a_\rho = a$ Eq.(24) may be solved separately. Here $z = a_\rho$ is the co-ordinate of the beam center of gravity.

The coefficients U_2 and W_2 in Eqs. (24) and (25) are very complicated functions of beam parameters, and are presented in [10]. For a homogeneous material, we get

$$
D = \frac{1}{12}, \quad a = \frac{1}{2}, \quad U_1 = \frac{c_h}{6}, \quad U_2 = \frac{c_h^2 + c_g - c_\nu}{24}, \quad W_2 = -\frac{1}{144} - \frac{3c_g - c_\nu + c_h^2}{360}. \tag{26}
$$

Calculations [10] show that Eqs. (24), (25) give a more exact result compared with the zero and with the first approximations especially in the cases when the orders of elastic modules in (3) differ from each other.

4 THE BOUNDARY CONDITIONS AND THE NATURAL FREQUENCIES

Let the boundary conditions

$$
w(x,z) = 0, \quad \sigma_{11}(x,z) = 0 \quad \text{at all} \quad z \in [0,1] \quad \text{and} \quad x = 0, 1 \tag{27}
$$

be given. Replacing the condition $\sigma_{11} = 0$ by the two conditions $\langle \sigma_{11} \rangle = 0$ and $\langle z_*\sigma_{11} \rangle = 0$ we obtain the conditions

$$
\frac{\partial u}{\partial x} = 0, \quad w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0, 1. \tag{28}
$$

Dividing variables,

$$u(x,t) = u(x)e^{i\omega t}, \quad w(x,t) = w(x)e^{i\omega t}, \quad F_3(t) = F_3^0 e^{i\omega t}, \quad i = \sqrt{-1} \tag{29}$$

Equations (24), (25) read as:

$$\frac{d^2u}{dx^2} + \mu^2\omega^2 u = \mu^2\omega^2 b_1 w - \mu^3\omega^2 b_2 \frac{dw}{dx} + \mu^2 b_1 F_3^0, \tag{30}$$

$$D\frac{d^4w}{dx^4} - \omega^2 w = F_3^0 - \mu^2\omega^2 b_3 \frac{d^2w}{dx^2} - \mu\omega^2 b_4 \frac{du}{dx} \tag{31}$$

with

$$b_1 = \frac{U_1}{D}, \quad b_2 = -\frac{U_2}{D}, \quad b_3 = -\frac{W_2}{D} > 0, \quad b_4 = a_\rho - a.$$

Here the coefficients b_1; b_2, b_3, and b_4 describe the effects of slanted anisotropy, of non-classic transversal (mainly, shear) deformations, and of the difference between the neutral layer and the center of gravity position.

At $F_3^0 = 0$ equations (30), (31), and (28) describe the boundary-value problem for the lower part of the free vibrations spectrum. With $b_4 = 0$ Eq. (30) and (31) give the bending part of the spectrum

$$\begin{aligned} &w(x) = w_n(x) = \sin n\pi x, \quad \omega_n^2 = \frac{D(n\pi)^4}{1+\mu^2(n\pi)^2 b_3}, \\ &u_n(x) = \frac{\mu^2\omega_n^2 b_1(\sin n\pi x - v_n(x)) - \mu^3\omega_n^2 b_2 n\pi \cos n\pi x}{p_n^2 - (n\pi)^2}, \quad n = 1, 2, \ldots, \\ &v_n(x) = n\pi p_n^{-1}(\cos p_n x(\cot p_n - (-1)^n) + \sin p_n x), \quad p_n = \mu\omega_n, \end{aligned} \tag{32}$$

where $v_n(x)$ is the solution to the homogeneous equation (30). From Eq. (32) it follows that the bending vibrations are accompanied by small longitudinal deflections u, and in the case of slanted anisotropy ($b_1 \neq 0$) the order of u is larger. With $b_4 \neq 0$ the more complicated formulas close to (32) have place.

If $p_n \approx n\pi$ the special consideration is necessary. In this case the “internal resonance” appears at which the bending frequency is close to the longitudinal one. For small enough numbers n (such that $(\mu n\pi)^2(D - b_3) < 1$) the internal resonance does not appear.

5 THE FORCED HARMONIC AND RANDOM VIBRATIONS

The forced stable vibrations under tge action of the harmonic excitation $F_3(t) = F_3^0 e^{i\omega t}$ are described by Eqs. (30), (31), and (28). The solution may be written as:

$$u(x,t,\omega) = s_u(x,\omega)e^{i\omega t}, \quad w(x,t,\omega) = s_w(x,\omega)e^{i\omega t}. \tag{33}$$

The functions $s_u(x,\omega)$ and $s_w(x,\omega)$ go off to infinity at $\omega \to \omega_n$ where ω_n are the natural frequencies. To get finite amplitudes it is necessary to place constraints.

The simplest way consists in the replacing the elastic material by a visco-elastic one with the complex modules. If we assume that all modules E_{jk}, G_{13}, $H_j i$ in the matrix (3) are changed by $(1+i\gamma)E_{jk},\ (1+i\gamma)G_{13},\ (1+i\gamma)H_j$ with the same small dimensionless coefficient γ, then Eqs. (30), (31) read as:

$$\begin{aligned}(1+i\gamma)\frac{d^2u}{dx^2}+\mu^2\omega^2u&=\mu^2\omega^2b_1w-\mu^3\omega^2b_2\frac{dw}{dx}+\mu^2b_1F_3^0,\\(1+i\gamma)D\frac{d^4w}{dx^4}-\omega^2w&=F_3^0-\mu^2\omega^2b_3\frac{d^2w}{dx^2}-\mu\omega^2b_4\frac{du}{dx}.\end{aligned}\tag{34}$$

Instead of Eqs. (33) we get the bounded solution

$$u(x,t,\omega)=\mathbf{Re}(s_u(x,\omega)e^{i\omega t}),\quad w(x,t,\omega)=\mathbf{Re}(s_w(x,\omega)e^{i\omega t}),\tag{35}$$

where s_u and s_w are solutions of Eqs. (34).

Let now $F_3(t)$ be the random stationary process with spectral density $S_F(\omega)$. Then for a fixed x the processes $u(x,t),\ w(x,t)$ are also stationary with spectral densities

$$S_u(x,\omega)=|s_u(x,\omega)|^2S_F(\omega),\qquad S_w(x,\omega)=|s_w(x,\omega)|^2S_F(\omega)\tag{36}$$

with the same s_u and s_w. There are various characteristics of the processes $u(x,t),\ w(x,t)$. For example, the root-mean-squares

$$\sigma_u^2(x)=\int_{-\infty}^{\infty}S_u(x,\omega)d\omega,\qquad\sigma_w^2(x)=\int_{-\infty}^{\infty}S_w(x,\omega)d\omega.\tag{37}$$

6 NUMERICAL EXAMPLE

Consider a soft isotropic homogeneous material (matrix) uniformly reinforced by a system of hard straight fibres, at angle α with the x-axis. After averaging we get a material with slanted anisotropy (Fig. 2).

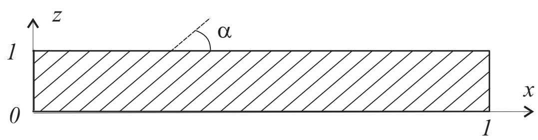

Figure 2: A reinforced beam.

For the elastic modules in (3) the following relations are valid [11]:

$$\begin{aligned}E_{11}&=\frac{E(1-\delta)}{1-\nu^2}+E_n\delta\cos^4\alpha,\qquad &E_{13}&=\frac{E\nu(1-\delta)}{1-\nu^2}+E_n\delta\sin^2\alpha\cos^2\alpha,\\E_{33}&=\frac{E(1-\delta)}{1-\nu^2}+E_n\delta\sin^4\alpha,\qquad &G_{13}&=\frac{E(1-\delta)}{2(1+\nu)}+E_n\delta\sin^2\alpha\cos^2\alpha,\\H_1&=E_n\delta\sin\alpha\cos^3\alpha,&H_3&=E_n\delta\sin^2\alpha\cos\alpha,\end{aligned}\tag{38}$$

where E and ν are, respectively, the Young modulus and the Poisson ratio of a matrix, E_n is the Young modulus of fibers, δ is the part of volume occupied by fibers.

We accept the following dimensionless parameters:

$$\nu=0.3,\quad E_n/E=1000,\quad\delta=0.1,\quad\rho=1,\quad\alpha=\pi/18,\quad\mu=0.05,\quad\gamma=0.1.\tag{39}$$

The material is homogeneous, therefore $b_4 = 0$, and from Eqs. (6), (8), (26) we find the coefficients in Eqs. (30), (31):

$$b_1 = 9.95, \quad b_2 = -13.82, \quad b_3 = 1.24. \tag{40}$$

Equation (32) gives the first bending frequencies

$$\omega_1 = 2.81, \quad \omega_2 = 10.76, \quad \omega_3 = 22.71, \quad \omega_4 = 32.35, \quad \omega_5 = 53.62, \ldots$$

The even modes are not exited in the studied case when F_3 does not depend on x.

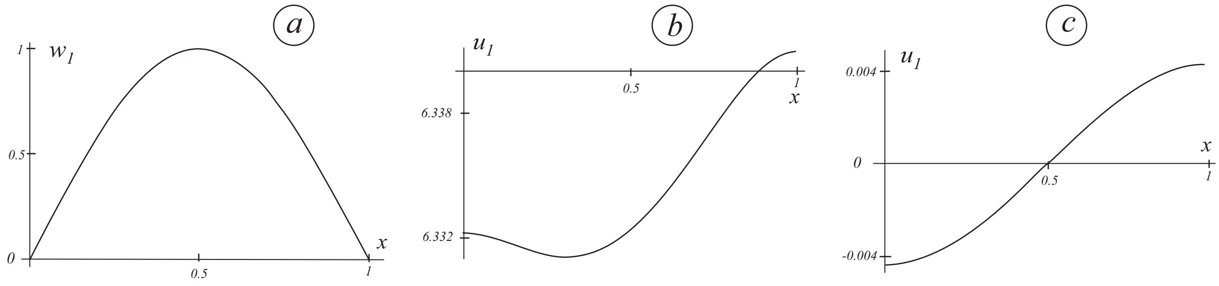

Figure 3: Eigen-functions $w_1(x),\ u_1(x)$, and $u_1(x)$ at $b_1 = 0$.

Let us discuss the first eigen-functions (32). The function $w_1(x)$ is an ordinary sine (see Fig. 3(*a*)). The function u_1 is a solution of Eq. (30) at $\omega = \omega_1$, $w = \sin \pi x$ and $F_3^0 = 0$. To satisfy boundary conditions

$$\frac{du}{dx} = 0 \quad \text{at} \quad x = 0, \ \ x = 1 \tag{41}$$

it is necessary to add a solution $v_1(x)$ of the homogeneous equation $\frac{d^2u}{dx^2} + \mu^2\omega_1^2 u = 0$. As a result we get the function (32) $u_1(x)$ which is large compared with $w_1(x)$ and almost constant (see Fig. 3(*b*)). If we put $b_1 = 0$ (an orthotropic material) then $u_1(x)$ becomes very small (see Fig. 3(*c*)). Also the function $u_1(x)$ is small if one or two conditions (41) be replaced by $u = 0$.

The functions $s_w(x, \omega)$ and $s_u(x, \omega)$ may be written in terms of the Fourier expansions:

$$\begin{aligned} s_w(x,\omega) &= \sum_{n=1,3,\ldots} \frac{4\sin n\pi x}{Dn^5\pi^4(\omega_n^2(1+i\gamma)-\omega^2)}, \\ s_u(x,\omega) &= \sum_{n=1,3,\ldots} \frac{4\left(\mu^2\omega_n^2 b_1(\sin n\pi x - v_n(x)) - \mu^3\omega_n^2 b_2 n\pi\cos n\pi x\right)}{Dn^5\pi^4(\omega_n^2(1+i\gamma)-\omega^2)((1+i\gamma)p_n^2-(n\pi)^2)}. \end{aligned} \tag{42}$$

Only the first terms of these series are essential because the second terms are 3^5 times smaller.

We consider a random stationary excitation $F(t)$ with unit dispersion and spectral density

$$S_F(\omega) = \frac{1}{\pi}\,\frac{2\alpha(\alpha^2+\beta^2)}{(\omega^2-\alpha^2-\beta^2)^2+4\alpha^2\omega^2}, \quad \alpha = 0.2\beta. \tag{43}$$

The function $S_F(\omega)$ has a maximum at $\omega \approx \beta$. At $\beta = 1$ the function $S_F(\omega)$ is shown in Fig. 4(*a*). In Fig. 4(*b*) and in Fig. 4(*c*) the root-mean-squares $\sigma_u(\beta)$ and $\sigma_w(\beta)$ of the deflections of the point $x = 1/2$ calculated by Eqs. (36) and (37) are shown.

The functions $\sigma_u(\beta)$ and $\sigma_w(\beta)$ have a maximum at β close to $\omega_1 = 2.81$ namely a resonance at random excitation also has place. As for the eigen-functions, the inequality $\sigma_u(\beta) > \sigma_w(\beta)$ is valid.

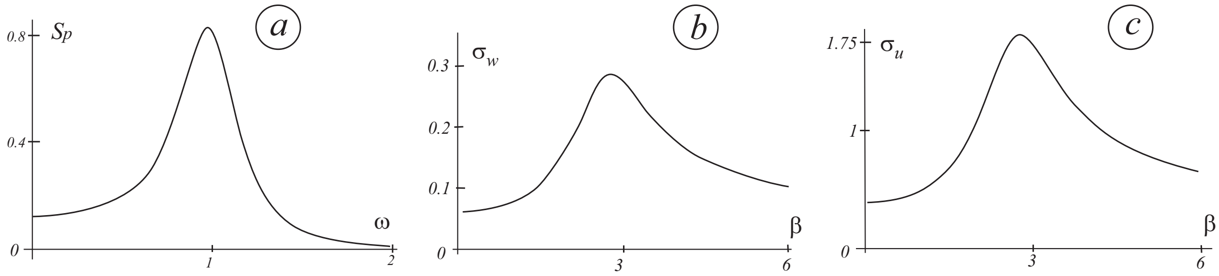

Figure 4: The spectral density of excitation $S_F(\omega)$ at $\beta = 1$, and the root-mean-squares $\sigma_u(\beta)$ and $\sigma_w(\beta)$ of the deflections of the point $x = 1/2$.

7 CONCLUSIONS

Vibrations of a thin elastic beam-strip made of an anisotropic material are studied. The 1D model of second-order accuracy is delivered. A special attention is paid to the slanted anisotropy with 6 elastic modules. The main peculiarity is that in the case of slanted anisotropy the beam with simply supported edges has very large horizontal deflections at a vertical excitation.

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