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ESTIMATION OF STABILITY LIMIT BASED ON GERSHGORIN'S THEOREM FOR EXPLICIT CONTACT-IMPACT ANALYSIS SIGNORINI PROBLEM USING BIPENALTY APPROACH

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Abstract. *The stability properties of the bipenalty method presented in Reference [4] is studied in application to one-dimensional bipenalized Signorini problem. The attention has been paid on the critical Courant numbers estimation based on Gershgorin's theorem. It is shown that Gershgorin's formula overestimates maximum eigenfrequency for all penalty ratios with exception of the critical penalty ratio. Thus, smaller safer values of critical Courant numbers are obtained in comparison with exact ones calculated from the solution of eigenvalue problem.*

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1 INTRODUCTION

In dynamic transient analysis, recent comprehensive studies have shown that using mass penalty together with standard stiffness penalty, the so-called bipenalty technique, preserves the critical time step in conditionally stable time integration schemes. In Reference [1] the bipenalty technique was introduced, where both penalty formulations were used simultaneously. The goal is to find the optimum of the so-called critical penalty ratio (CPR) defined as the ratio of stiffness and mass penalty parameters so that the maximum eigenfrequency and the critical time step are preserved. The calculation of CPR requires an analysis of the full bipenalised problem. Owing to mathematical difficulty, it limits the classes of elements that can be taken into account. In order to overcome this problem, a simple relationship between the CPR of an element and its maximum unpenalised eigenfrequency was derived in [2]. Thus, the multiple constraints and more complex element formulations can be directly accounted for [3].

In this work, previous investigation presented in Reference [4] was followed, where the stability of explicit contact-impact algorithm [5] using the bipenalty approach was studied. The upper bound of the stable Courant number on the stiffness penalty and mass penalty was derived based on the simple dynamic system with two degrees-of-freedom. It was shown that the critical Courant number tend towards zero for the stiffness penalty approaching infinity whereas the mass penalty was considered to be zero. On the other hand, when the penalty value was set to the CPR, which corresponded to the maximum eigenvalue of the unpenalised system, the critical Courant number was equal to one for the arbitrary value of the stiffness penalty. The derived upper bound of the stability was verified by means of the simple 1D dynamic problem. It was demonstrated decreasing the critical time step for the standard penalty method and its preserving for the bipenalty method.

In this paper, the attention is focused on the stability properties of one-dimensional bipenalized Signorini problem including application of Gershgorin's theorem for estimation of critical Courant numbers for explicit integration. In Section 2 the problem description adopted from Reference [4] including governing equations for bipenalized system is briefly outlined. The variational formulation of the contact problem for the bipenalty method is presented in Section 2.1, followed by the finite element discretization in Section 2.2. Next, a representative of explicit schemes—the central difference method—is discussed in Section 2.3 including the numerical stability of this method. In Section 2.4 Gershgorin's theorem for the estimation of maximum eigenfrequency of the bipenalized system is derived. In Section 3, the stability properties of one-dimensional bipenalized Signorini problem are investigated including application of Gershgorin's theorem, folowed by concluding remarks in Section 4.

2 PROBLEM DESCRIPTION AND GOVERNING EQUATIONS

Consider two deformable bodies coming into contact with each other. Its configuration is decribed by open domains Ω_1, Ω_2 with boundaries Γ_1 and Γ_2. A contact occurs if $\Gamma_1 \bigcap \Gamma_2 \neq \emptyset$ in the deformed state. The shared part of deformed boundaries $\Gamma_c = \Gamma_1 \bigcap \Gamma_2$ is referred to as the *contact boundary*. In order to identify the contact interface Γ_c, it is usual to introduce the so-called normal gap function g_{N}, which returns the closest distance between a given point $\mathbf{x}_1 \in \Omega_1$ and Ω_2. The normal gap function is negative if the point $\mathbf{x}_1$ penetrates into the body Ω_2. Otherwise it is non-negative.

2.1 Variational formulation

It is convenient to formulate the problem in a weak sense aiming at the finite element formulation of the contact-impact problem. For this purpose well-known Hamilton's variational principle is utilized. First, let us introduce the Lagrangian functional as

$$\mathcal{L}\left(\mathbf{u}, \dot{\mathbf{u}}\right) = \mathcal{T}\left(\dot{\mathbf{u}}\right) - \left(\mathcal{U}\left(\mathbf{u}\right) - \mathcal{W}\left(\mathbf{u}\right)\right) \tag{1}$$

where $\mathbf{u}$ and $\dot{\mathbf{u}}$ are the displacement and velocity vectors, respectively, and

$$\mathcal{T}\left(\dot{\mathbf{u}}\right) = \int_{\Omega} \frac{1}{2}\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,\mathrm{d}V \tag{2}$$

$$\mathcal{U}\left(\mathbf{u}\right) = \int_{\Omega} \psi(\mathbf{u})\,\mathrm{d}V \tag{3}$$

$$\mathcal{W}\left(\mathbf{u}\right) = \int_{\Omega} \mathbf{b}\cdot\mathbf{u}\,\mathrm{d}V + \int_{\Gamma_S} \mathbf{t}\cdot\mathbf{u}\,\mathrm{d}S \tag{4}$$

are the kinetic energy, the strain energy, and the work done by external forces. Note that for brevity the unified domains $\Omega = \Omega_1 \cup \Omega_2$ and $\Gamma = \Gamma_1 \cup \Gamma_2$ have been introduced. Further in these integrals, $\mathbf{b}$ denotes the body force vector and $\mathbf{t}$ is the traction vector prescribed on $\Gamma_S \subset \Gamma$; ψ denotes the strain energy function and ρ is the density.

Let us assume that the contact boundary Γ_c is known. Then the non-penetration condition $g_\mathrm{N} \geq 0$ can be enforced by the bipenalty method. To this end the Lagrangian functional is amended by additional terms based on the simultaneous use of stiffness penalty ϵ_s and inertia (mass) penalty ϵ_m. The standard stiffness penalty adds an extra term to the strain energy (3) to enforce the zero gap on the contact boundary

$$\mathcal{U}_\mathrm{p}\left(\mathbf{u}\right) = \int_{\Omega} \psi(\mathbf{u})\,\mathrm{d}V + \int_{\Gamma_\mathrm{c}} \frac{1}{2}\epsilon_\mathrm{s} g_\mathrm{N}^2\,\mathrm{d}S \tag{5}$$

The inertia penalty term can also be added to the kinetic energy (4) to enforce a zero gap rate on the contact interface

$$\mathcal{T}_\mathrm{p}\left(\dot{\mathbf{u}}\right) = \int_{\Omega} \frac{1}{2}\rho\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,\mathrm{d}V + \int_{\Gamma_\mathrm{c}} \frac{1}{2}\epsilon_\mathrm{m} \dot{g}_\mathrm{N}^2\,\mathrm{d}S \tag{6}$$

The penalized Lagrangian functional is now defined as

$$\mathcal{L}_\mathrm{p}(\mathbf{u}, \dot{\mathbf{u}}) = \mathcal{T}_\mathrm{p}(\dot{\mathbf{u}}) - (\mathcal{U}_\mathrm{p}(\mathbf{u}) - \mathcal{W}(\mathbf{u})) \tag{7}$$

According to Hamilton's variational principle, the unknown displacement field can be found as the one which renders the action functional stationary

$$\delta \int_0^T \mathcal{L}_\mathrm{p}\left(\mathbf{u}, \dot{\mathbf{u}}\right)\,\mathrm{d}t = 0 \tag{8}$$

where δ denotes the first variation or the directional derivative in the direction of the virtual displacement $\delta\mathbf{u}$. Using the standard procedures one arrives at the principle of virtual displacements

$$\int_{\Omega} \rho\ddot{\mathbf{u}}\cdot\delta\mathbf{u}\,\mathrm{d}V + \int_{\Omega} \delta\boldsymbol{\psi}(\mathbf{u})\,\mathrm{d}V + \int_{\Gamma_\mathrm{c}} (\epsilon_\mathrm{m}\ddot{g}_\mathrm{N} + \epsilon_\mathrm{s} g_\mathrm{N})\delta g_\mathrm{N}\,\mathrm{d}S = \int_{\Omega} \mathbf{b}\cdot\delta\mathbf{u}\,\mathrm{d}V + \int_{\Gamma_S} \mathbf{t}\cdot\delta\mathbf{u}\,\mathrm{d}S \tag{9}$$

which provides the base for the finite element discretization. The integrals in Equation (9) represent the virtual work of the inertia forces, internal forces, contact forces, body forces, and traction forces, respectively. It is worth noting that the integral of the virtual contact work are expressed with the aid of the inertia and the stiffness penalty.

2.2 Finite element discretization

Applying the finite element method to the variational formulation (9), the discrete problem in the form of nonlinear ordinary differential equations is obtained as

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} + \mathbf{R}_\mathrm{c}(\mathbf{u}, \ddot{\mathbf{u}}) = \mathbf{R} \tag{10}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{K}$ is the stiffness matrix and $\mathbf{R}_\mathrm{c}$ is the contact residual vector, which is the source of the nonlinearity. Further, $\mathbf{R}$ is the time-dependent load vector, and $\mathbf{u}$ and $\ddot{\mathbf{u}}$ contain nodal displacements and accelerations, respectively. The element mass and stiffness matrices are given by

$$\mathbf{M}_e = \int_{\Omega_e} \rho \mathbf{H}^\mathrm{T}\mathbf{H} \,\mathrm{d}V \qquad \mathbf{K}_e = \int_{\Omega_e} \mathbf{B}^\mathrm{T}\mathbf{C}\mathbf{B} \,\mathrm{d}V \tag{11}$$

where $\mathbf{C}$ is the elasticity matrix, $\mathbf{B}$ is the strain-displacement matrix, and the matrix $\mathbf{H}$ stores the shape functions. Note that the integration is carried out over the element domain Ω_e. Global matrices are assembled in the usual fashion.

In the case of geometrically linear kinematics, the contact residual vector can be written as

$$\mathbf{R}_\mathrm{c}(\mathbf{u}, \ddot{\mathbf{u}}) = \mathbf{M}_\mathrm{p}\ddot{\mathbf{u}} + \mathbf{K}_\mathrm{p}\mathbf{u} + \mathbf{f}_\mathrm{p} \tag{12}$$

where

$$\mathbf{M}_\mathrm{p} = \int_{\Gamma_\mathrm{c}} \epsilon_\mathrm{m}\mathbf{N}\mathbf{N}^\mathrm{T} \,\mathrm{d}S \qquad \mathbf{K}_\mathrm{p} = \int_{\Gamma_\mathrm{c}} \epsilon_\mathrm{s}\mathbf{N}\mathbf{N}^\mathrm{T} \,\mathrm{d}S \qquad \mathbf{f}_\mathrm{p} = \int_{\Gamma_\mathrm{c}} \epsilon_\mathrm{s}\mathbf{N}g_0 \,\mathrm{d}S \tag{13}$$

Here, $\mathbf{M}_\mathrm{p}$ is the additional mass matrix due to inertia penalty, $\mathbf{K}_\mathrm{p}$ is the additional stiffness matrix due to stiffness penalty, and $\mathbf{f}_\mathrm{p}$ is the part of the contact force due to the initial gap g_0. The matrix $\mathbf{N}$ represents an operator from the displacement field $\mathbf{u}$ to the gap function g_N

$$g_\mathrm{N} = \mathbf{N}^\mathrm{T}\mathbf{u} + g_0 \tag{14}$$

The particular form of the matrix $\mathbf{N}$ follows from the used contact discretization. For 1D case $\mathbf{N}^\mathrm{T} = [1, -1]$. Note that we also used the diagonalized version of $\mathbf{M}_\mathrm{p}$ as a row-sum mass matrix with absolute value of terms. The diagonalized matrix is necessary for the explicit time integration, see next section.

2.3 Explicit time integration method and numerical stability

Consider the numerical time integration of the semi-discretized system (10) by the central difference method (CDM) [7]

$$(\mathbf{M}^t + \mathbf{M}_\mathrm{p}^t)\frac{\mathbf{u}^{t+\Delta t} - 2\mathbf{u}^t + \mathbf{u}^{t-\Delta t}}{\Delta t^2} + (\mathbf{K}^t + \mathbf{K}_\mathrm{p}^t)\mathbf{u}^t + \mathbf{f}_\mathrm{p}^t - \mathbf{R}^t = \mathbf{0} \tag{15}$$

Assuming that displacements are known at time $t - \Delta t$ and t, one can resolve unknown displacements at time $t + \Delta t$, where Δt marks the time step size. Note, that the matrices $\mathbf{M}_\mathrm{p}^t$ and $\mathbf{K}_\mathrm{p}^t$ are time-dependent because they are associated with active contact constraints. This fact causes the system to be nonlinear.

In this paper, the following form of the central difference scheme is used for solving elastodynamic problems with contact constrains

- Given $\mathbf{u}^t$, $\dot{\mathbf{u}}^{t-\Delta t/2}$, $\mathbf{R}^t$
- For given $\mathbf{u}^t$ analyze contact, compute gap vector $\mathbf{g}$ and contact forces $\mathbf{f}_\mathrm{p}^t = -\mathbf{K}_\mathrm{p}^t\mathbf{u}^t + \mathbf{f}_\mathrm{p}^0$
- Compute accelerations $\ddot{\mathbf{u}}^t = (\mathbf{M}^t + \mathbf{M}_\mathrm{p}^t)^{-1}(\mathbf{R}^t - \mathbf{K}^t\mathbf{u}^t + \mathbf{f}_\mathrm{p}^t)$
- Compute mid-point velocities $\dot{\mathbf{u}}^{t+\Delta t/2} = \dot{\mathbf{u}}^{t-\Delta t/2} + \Delta t\ddot{\mathbf{u}}^t$
- Compute new displacements $\mathbf{u}^{t+\Delta t} = \mathbf{u}^t + \Delta t\dot{\mathbf{u}}^{t+\Delta t/2}$
- Set $t \to t + \Delta t$

It is well known that the CDM for a linear system is conditionally stable. The linear stability theory establishes the upper bound of the time step size as

$$\Delta t \leq \frac{2}{\omega_\mathrm{max}} \tag{16}$$

where ω_max is the maximum eigenfrequency of the finite element mesh. It is convenient to introduce the Courant dimensionless number defined as

$$C_\mathrm{r} = \frac{c_1\Delta t}{h} \tag{17}$$

where c_1 is the speed of the fastest wave propagating in a continuum, typically the longitudinal wave and h is the element size. For 1D case it corresponds the speed of longitudinal waves along a thin rod c_0. Using the latter definition, the stability condition (16) can be rephrased as

$$C_\mathrm{r} \leq \frac{2}{\bar{\omega}_\mathrm{max}} \tag{18}$$

where $\bar{\omega}_\mathrm{max}$ is the dimensionless frequency

$$\bar{\omega}_\mathrm{max} = \frac{\omega_\mathrm{max}h}{c_1} \tag{19}$$

Indeed, the computation of even a single eigenvalue of a large systems may be expensive. Therefore, it would be advantageous to have an estimate of the maximum eigenvalue that is easy to compute. Such an estimate is provided by the element eigenvalue inequality [6]

$$\omega_\mathrm{max} < \max_e \omega_\mathrm{max}^e \tag{20}$$

Note that the element eigenvalue inequality (20) is not limited to element level submatrices. The submatrices may be also an assembly of elements.

It should be mentioned that there are no stability theorems for contact-impact problems [7]. In this case the stability theory for linear problems can be applied carefully. In practise, for example, the stability may be preserved by checking the energy balance during a nonlinear computation.

2.4 Gershgorin's theorem

Another method for the estimation of maximum eigenfrequency is based on Gershgorin's theorem [8, 9]. This method can be used on local level separately for each element (20) or on global level for global mass and stiffness matrices. Using Gershgorin's theorem for the estimation of maximum eigenfrequency of a general $n \times n$ matrix $\mathbf{A}$ gives

$$\omega_{\max} \approx \omega_{\max}^{\mathrm{Ger}} = \max_i \left(\sqrt{A_{ii} + \sum_{j=1, i\neq j}^{n} |A_{ij}|} \right) \tag{21}$$

In order to apply Gershgorin's theorem to generalized eigenvalue problem with bipenalty terms

$$(\mathbf{K} + \mathbf{K}_{\mathrm{p}})\mathbf{u} = \omega^2 (\mathbf{M} + \mathbf{M}_{\mathrm{p}})\mathbf{u} \tag{22}$$

it should be reformulated as the standard eigenvalue one. To this end we multiply equation (22) by $(\mathbf{M} + \mathbf{M}_{\mathrm{p}})^{-1}$ obtaining

$$\left[(\mathbf{M} + \mathbf{M}_{\mathrm{p}})^{-1} (\mathbf{K} + \mathbf{K}_{\mathrm{p}}) - \omega^2 \mathbf{I} \right] \mathbf{u} = \mathbf{0} \tag{23}$$

where the matrix product $(\mathbf{M} + \mathbf{M}_{\mathrm{p}})^{-1}(\mathbf{K} + \mathbf{K}_{\mathrm{p}})$ corresponds to the matrix $\mathbf{A}$. Since the matrices $\mathbf{M}$ and $\mathbf{M}_{\mathrm{p}}$ are assumed to be diagonal, the inverse is simply the inverse of their diagonal elements. Thus, Gershgorin's estimate for the maximum eigenfrequency takes the form

$$\omega_{\max}^{\mathrm{Ger}} = \max_i \left(\sqrt{\frac{K_{ii} + K_{\mathrm{p}ii}}{M_{ii} + M_{\mathrm{p}ii}} + \sum_{j=1, i\neq j}^{n} \left| \frac{K_{ij} + K_{\mathrm{p}ij}}{M_{ii} + M_{\mathrm{p}ii}} \right|} \right) \tag{24}$$

where n is number of degrees of freedom of the problem. Finally, if triangle inequality theorem $|K_{ij} + K_{\mathrm{p}ij}| \leq |K_{ij}| + |K_{\mathrm{p}ij}|$ is further considered, mass matrix elements are positive and positive definiteness of the stiffness matrix is naturally ensured leading to a simple formula for estimation of maximum eigenfrequency for bipenalized system

$$\omega_{\max}^{\mathrm{Ger}} = \max_i \sqrt{\frac{\sum_{j=1}^{n} |K_{ij}| + |K_{\mathrm{p}\,ij}|}{M_{ii} + M_{\mathrm{p}\,ii}}} \tag{25}$$

Note that matrices with "p" subscript correspond to the additional mass matrix $\mathbf{M}_{\mathrm{p}}$ due to inertia penalty and the additional stiffness matrix $\mathbf{K}_{\mathrm{p}}$ due to stiffness penalty defined in Equation (13).

3 STABILITY ANALYSIS OF BIPENALIZED SIGNORINI PROBLEM

In Reference [4], the stability of simple one-dimensional bipenalized Signorini problem with two degrees of freedom depicted in Figure 1 was studied. The attention was paid the to the effect of the bipenalty ratio on the stability behaviour. In this work, the critical Courant numbers based on Gershgorin's theorem (25) are computed and compared with exact formula.

The system consists of one 1D constant strain truss element with lumped mass matrix. The active contact constraint is set in node 1. The aim was to determine the maximum eigenfrequency of this system to estimate the critical Courant number in the form (18). To this end, the eigenvalue problem (22) can be formulated as

$$\frac{EA}{h} \begin{bmatrix} 1 + \beta_{\mathrm{s}} & -1 \\ -1 & 1 \end{bmatrix} \mathbf{u} = \omega^2 \frac{\rho A h}{2} \begin{bmatrix} 1 + \beta_{\mathrm{m}} & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u} \tag{26}$$

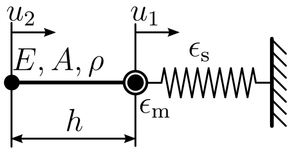

Figure 1: A scheme of the bipenalized Signorini problem.

where the displacement vector has the form $\mathbf{u} = [u_1, u_2]^T$. The dimensionless mass and stiffness penalty have been introduced as

$$\beta_m = \frac{2}{\rho A h}\epsilon_m \quad \beta_s = \frac{h}{EA}\epsilon_s \tag{27}$$

where ρ is the mass density, E is the Young's modulus, A marks the cross-sectional area of the bar and h corresponds to the finite element length. Next, a dimensionless penalty ratio r is defined as

$$r = \frac{1}{2}\frac{\beta_s}{\beta_m} \tag{28}$$

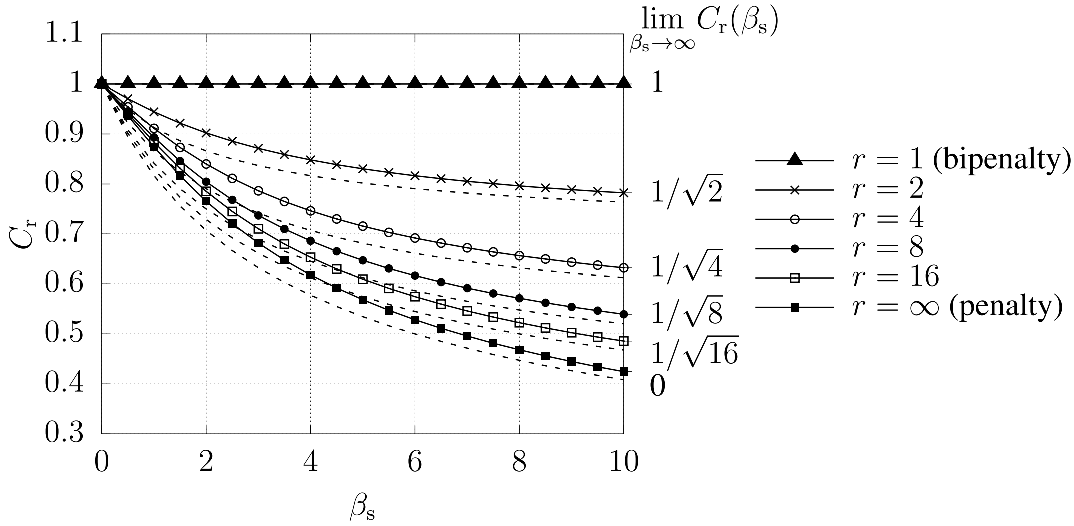

Figure 2: Bipenalized Signorini problem: Dependence of the critical Courant number C_r on the dimensionless stiffness penalty β_s for selected dimensionless penalty ratios r (dashed lines correspond to Gershgorin's theorem).

Finally, the critical Courant number C_r for the bipenalized Signorini problem was obtained as

$$C_r = \frac{2}{\sqrt{1 + \frac{(1+\beta_s)}{(1+\beta_m)} + \sqrt{1 + \frac{2(1-\beta_s)}{(1+\beta_m)} + \frac{(1+\beta_s)^2}{(1+\beta_m)^2}}}} \tag{29}$$

with the property that the critical Courant number for the bipenalty ratio $r = 1$ reaches the value $C_r = 1$ and it is independent on the chosen stiffness penalty β_s. Thus, the stable time step size remains unchanged for an arbitrary value of the dimensionless stiffness penalty β_s. As it was already stated in Reference [4], it is the main advantage of the bipenalty method. Using Gershgorin's theorem (25) expressed in the form (19), the estimation of dimensionless maximum eigenfrequency of the bipenalized Signorini problem gives

$$\bar{\omega}_{\max}^{\mathrm{Ger}} = \sqrt{2}\max\left(\sqrt{\frac{1+\beta_\mathrm{s}}{1+\beta_\mathrm{m}} + \left|\frac{-1}{1+\beta_\mathrm{m}}\right|}, \sqrt{1+|-1|}\right) \tag{30}$$

Alternatively, it is possible to apply Gershgorin's theorem directly for this simple one-dimensional problem. Since the mass matrix of bipenalized system (26) is diagonal, its inverse is trivial

$$(\mathbf{M}+\mathbf{M}_\mathrm{p})^{-1} = \frac{2}{\rho A h}\left[\begin{array}{cc} \frac{1}{1+\beta_\mathrm{m}} & 0 \\ 0 & 1 \end{array}\right] \tag{31}$$

The inverse mass matrix multiplied by the stiffness matrix of bipenalized system (26) gives

$$(\mathbf{M}+\mathbf{M}_\mathrm{p})^{-1}(\mathbf{K}+\mathbf{K}_\mathrm{p}) \;=\; \frac{2EA}{\rho A h^2}\left[\begin{array}{cc} \frac{1+\beta_\mathrm{s}}{1+\beta_\mathrm{m}} & \frac{-1}{1+\beta_\mathrm{m}} \\ -1 & 1 \end{array}\right] \tag{32}$$

which corresponds to the matrix $\mathbf{A}$. Note that the matrix $\mathbf{A}$ is not symmetric. Using original Gershgorin's estimation (21) expressed in normalized form (19) gives the same formula 30 for the estimation of dimensionless maximum eigenfrequency of the bipenalized Signorini problem.

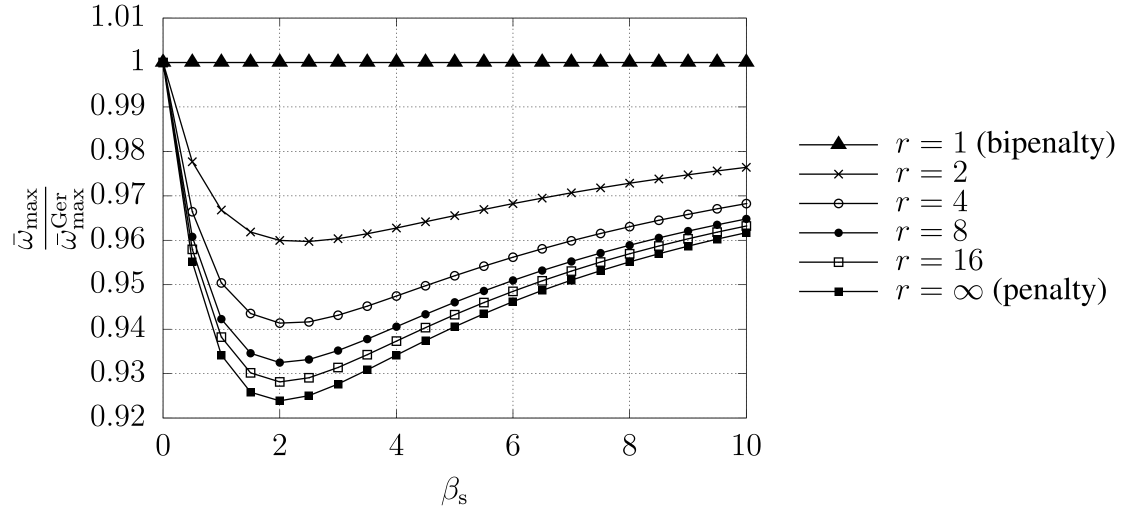

Figure 3: Bipenalized Signorini problem: Dependence of the ratio of exact dimensionless maximum eigenfrequency $\bar{\omega}_{\max}$ per dimensionless maximum eigenfrequency given by Gershgorin's formula $\bar{\omega}_{\max}^{\mathrm{Ger}}$ on the dimensionless stiffness penalty β_s for selected dimensionless penalty ratios r.

Substituting (30) into (16) using (17) the critical Courant number $C_\mathrm{r}^{\mathrm{Ger}}$ based on Gershgorin's theorem for the bipenalty Signorini model is obtained as

$$C_\mathrm{r}^{\mathrm{Ger}} = \frac{\sqrt{2}}{\max\left(\sqrt{\frac{1+\beta_\mathrm{s}}{1+\beta_\mathrm{m}} + \left|\frac{-1}{1+\beta_\mathrm{m}}\right|}, \sqrt{1+|-1|}\right)} \tag{33}$$

The dependence of the critical Courant number $C_{\rm r}$ is plotted in Figure 2 for selected dimensionless penalty ratios $r = 1, 2, 4, 8, 16$ and ∞. The thin dashed lines in Figure 2 correspond to the critical Courant number $C_{\rm r}^{\rm Ger}$ based on Gershgorin's formula. It should be stated that Gershgorin's theorem (25) overestimates maximum eigenfrequency for all penalty ratios r, with exception of the critical penalty ratio $r = 1$. Thus, smaller safer value of critical Courant numbers (33) are obtained in comparison with exact ones (29). For the critical penalty ratio $r = 1$ Gershgorin's theorem gives exact upper bound, which corresponds to row equally distributed elements. The quantitative difference between exact solution of the eigenvalue problem (26) of the bipenalized system and Gershgorin's estimation is shown in Figure 3, where the dependence of the ratio of exact dimensionless maximum eigenfrequency $\bar{\omega}_{\max}$ per dimensionless maximum eigenfrequency given by Gershgorin's formula $\bar{\omega}_{\max}^{\rm Ger}$ (30) on the dimensionless stiffness penalty $\beta_{\rm s}$ is plotted for selected dimensionless penalty ratios r. Apart from the critical penalty ratio $r = 1$ these values are smaller than one confirming that Gershgorin's theorem gives conservative results. It is interesting to note that the maximum difference occurs near dimensionless stiffness penalty $\beta_{\rm s} = 2$ regardless penalty ratio r.

4 CONCLUSIONS

In this paper, previous investigation presented in Reference [4] was followed, where the stability of explicit contact-impact algorithm [5] using the bipenalty approach was studied. The stability analysis was carried out for one-dimensional bipenalized discreted finite element systems, namely the Signorini problem. The main attention has been paid on an upper bound estimation of the stable Courant number for the bipenalty method with respect to stiffness penalty and mass penalty parameters. It was shown that the critical Courant number tend towards zero for the stiffness penalty parameter approaching infinity whereas the mass penalty parameter was considered to be zero, i.e. when only a pure penalty formulation was considered. On the other hand, setting the penalty ratio between mass and stiffness penalty parameters to *the critical penalty ratio* introduced by Askes in [1] preserved the stable Courant number at the level of the unpenalized system for an arbitrary value of the stiffness penalty parameter.

Next, the estimation of the critical Courant number of bipenalized Signorini problem based on Gershgorin's theorem was derived. It was shown that Gershgorin's formula overestimated maximum eigenfrequency for all penalty ratios r with exception of the critical penalty ratio $r = 1$. Thus, smaller safer values of critical Courant numbers are obtained in comparison with exact ones calculated from solution of eigenvalue problem. Further details of this work including extension to other one-dimensional contact models can be found in prepared paper [10].

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