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PROPAGATION OF NON-STATIONARY WAVES IN VISCOELASTIC STRIP COMPOSED OF ORTHOTROPIC LAYERS

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Abstract. *The analytical solution for transient waves caused by transverse impact on an infinite layered strip with free-free boundaries is presented in this work. The strip is composed of two horizontal layers of different heights. The materials of both layers are assumed to be linear viscoelastic and orthotropic. The case of special orthotropy is assumed for simplicity. The dissipative behaviour of each layer is modelled by the discrete model of standard linear viscoelastic solid in Zener configuration. The solving procedure used in this work follows the methods applied in previously published works dealing with the problems of a viscoelastic orthotropic strip and the symmetric case of a layered strip. The system of four linear partial integro-differential equations describing the non-stationary state of plane stress in the strip is solved by means of integral transform method. Concretely, the Laplace transform in time domain and the Fourier transform in spatial domain are applied. As a results of this procedure, the final formulas for displacement components in both layers are derived in Laplace domain. These transforms contain eight spectra of Fourier integrals which can be found as the solution of the system of eight complex equations arising from boundary conditions of the problem. When the spectra are known, the resulting formulas for the Laplace transforms of displacement components are obtained.*

The presented new formulation of non-symmetric problem enables to study the wave phenomena in general two-layered solids of strip-like geometry and it is fundamental for solving a problem with arbitrary number of layers.

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1 INTRODUCTION

Propagation of stationary and non-stationary waves in layered elastic and viscoelastic media is a subject of extensive investigation already for many decades. This interest results from various applications of this field of elastodynamics, e.g. in seismology, non-destructive testing, design of impact shields and armour systems, aircraft industry, etc. Many different methods exist and are used for the investigation of transient or steady-state responses of such structures. In [1] the authors used the combination of the Fourier transform and the method of characteristics to obtain the response of a multilayered viscoelastic medium to a dynamic out-of-plane shear traction. The same methods are used by the authors in [2] for the investigation of two-dimensional transient waves in cylindrical layered media composed of N isotropic homogeneous layers. The method of characteristics is also used in [3]. In this case, the method is applied to a problem of optimal design of an elastic two-layered strip under transient pressure load. The authors used this analytical approach in order to minimise stress amplitudes with potential application in design of multilayered armour. An analogous optimisation problem is solved by Luo et al. in [4]. The authors use the transfer matrix method to reach the optimal design of an elastic multilayered medium subjected to transverse pulse loading.

Problems of waves in layered media become much more complicated when anisotropic material properties of layers are introduced. This demand on problem generalisation (in the sense of material behaviour) was invoked by extensive application of composites and laminated structures in many fields of industry. In this case, the number of analytical studies dealing with transient and steady-state responses is limited. As an example, one can mention the work [5] in which the authors investigate the transition from transient to steady-state response in the case of anisotropic layered media. The authors of other works usually handle such problems by a numerical approach, e.g. by standard finite element method or by more effective strip element method [6].

In the present study, the classical method of integral transforms is used to obtain the transient response of an infinite two-layered strip subjected to dynamic loading of impact character. Each layer has different height and is modelled as linear viscoelastic with special orthotropic properties. This work is a continuation of authors' previous works [7] and [8] in which analogous wave problems for an orthotropic viscoelastic strip and for a strip with two symmetric orthotropic viscoelastic layers were solved, respectively. The structure of the present work is organise in such a way to emphasise the fundamental steps leading to the derivation of Laplace transforms of displacement components for each layer.

2 FORMULATION OF THE PROBLEM

We will assume an infinite thin strip composed of two horizontal layers. Let us denote the characteristic dimensions of the strip as h_1 and h_2 as shown in Fig. 1. So the total height of the strip will be h_2 and the height of the first lower layer is h_1. Further, we will assume that the material of each layer is linear viscoelastic with special orthotropic properties, i.e. the material and geometric axes coincide. The horizontal and vertical axes of the coordinate system with the origin lying on the lower free boundary of the strip will be denoted as x_1 and x_2, respectively (see Fig. 1). The upper edge of the strip will be loaded in x_1–x_2 plane by a pressure pulse of duration t_0 and described in time and space by the function $\sigma(x_1, t) = \sigma_0(x_1)\,(H(t) - H(t - t_0))$, where $H(t)$ denotes the Heaviside step function. The function $\sigma_0(x_1)$ defining the pressure amplitude distribution will be considered as even in x_1 for simplicity.

Based on the description given above, the problem will be solved as a symmetric plane stress

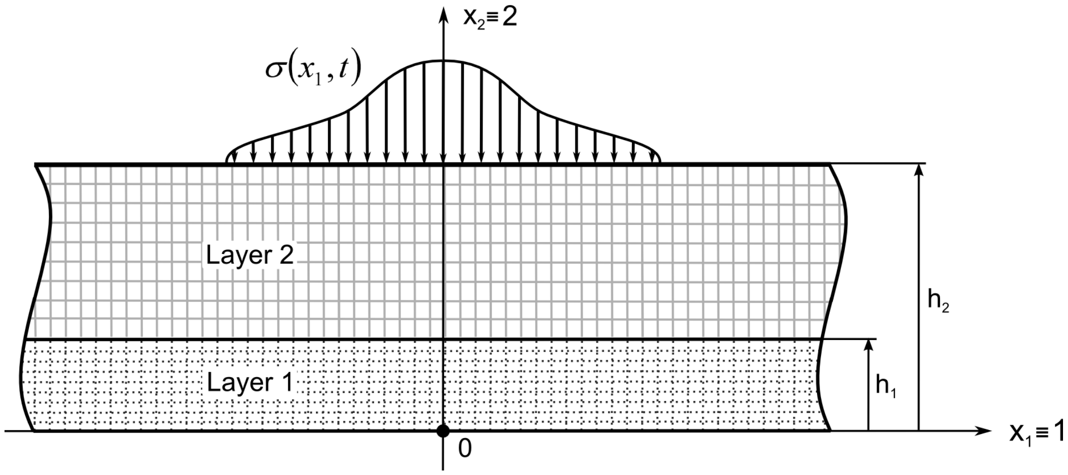

Figure 1: Scheme of the problem solved.

problem on a domain $\langle 0, +\infty\rangle \times \langle 0, h_2\rangle$. We will look for four functions of displacements $u_{1,L}(x_1, x_2, t)$ and $u_{2,L}(x_1, x_2, t)$ describing the transient response in the lower ($L = 1$) and upper ($L = 2$) layer of the strip.

3 SOLUTION OF THE PROBLEM

3.1 Kinematic and constitutive equations

The solution will be derived under the assumption of small displacements and small strains. Therefore, the kinematic relations for normal strains $\varepsilon_{11,L}(x_1, x_2, t)$, $\varepsilon_{22,L}(x_1, x_2, t)$ and shear strain $\varepsilon_{12,L}(x_1, x_2, t)$ can be written for each layer L in the form $\varepsilon_{ij,L} = \frac{1}{2}\left(\frac{\partial u_{i,L}}{\partial x_j} + \frac{\partial u_{j,L}}{\partial x_i}\right)$ for $i, j = 1, 2$. The last non-zero strain component $\varepsilon_{33,L}(x_1, x_2, t)$ can then be determined by using the first two normal strains and the appropriate Poisson's ratio.

As mentioned previously, the material of each layer will be assumed to be linear viscoelastic and specially orthotropic. Its viscoelastic behaviour, which results in the attenuation of propagated waves, will be modelled in both directions x_1 and x_2 by the Zener model of standard viscoelastic solid. It means that the material relaxation is described by one relaxation time in each direction. This model is composed of the Maxwell element and an elastic spring in parallel (see [9]). Due to the material linearity, the required constitutive equations can be derived using the superposition principle of stresses and strain rates occurring on elastic and viscous elements of the Zener model. When we denote the elastic and viscous Poisson's ratios by ν_{ij} and μ_{ij}, respectively, and take into account the equalities $\nu_{12}/E_1 = \nu_{21}/E_2$ and $\mu_{12}/\lambda_1 = \mu_{21}/\lambda_2$, see [10, 9], the constitutive equations for strictly elastic and viscous orthotropic material under the state of plane stress can be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{12} & b_{22} & 0 \\ 0 & 0 & 2b_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} \quad \text{and} \quad \frac{\partial}{\partial t}\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1} & -\frac{\mu_{12}}{\lambda_1} & 0 \\ -\frac{\mu_{12}}{\lambda_1} & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{2\eta_{12}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}. \quad (1)$$

The reduced stiffnesses b_{ij} appearing in (1) are defined as [10]

$$b_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad b_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \quad b_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} \quad \text{and} \quad b_{66} = G_{12}, \quad (2)$$

where E_1, E_2 and G_{12} represent Young's and shear moduli of elastic material in appropriate directions. The last parameters λ_i and η_{ij} present in (1) are the coefficients of normal and shear viscosities, respectively. Using these parameters one can define the ratios $\alpha_1 = E_1/\lambda_1$, $\alpha_2 = E_2/\lambda_2$ and $\beta = G_{12}/\eta_{12}$ which represent the reciprocal values of relaxation times. In the following text we will consider the equality between the elastic and viscous Poisson's ratios for simplicity, i.e. $\nu_{ij} = \mu_{ij}$. Taking this assumption into account one can write $\alpha_1 = \alpha_2 = \alpha$. Further, we will assume that $\beta = \alpha$.

The derivation of the final form of constitutive equations for Lth layer can be divided into two steps: (i) the superposition of strain rates of elastic and viscous element in the Maxwell model, (ii) the superposition of stresses of the Maxwell model and the elastic element in parallel. The first step leads to a system of 3 linear ODEs, the solution of which can be found by convolution integral. Doing so, performing the second mentioned step and introducing the kinematic equations yield to the final formulas for stresses in orthotropic viscoelastic medium:

$$\begin{aligned}
\sigma_{11,L} &= (b_{0,11,L} + b_{11,L})\,\frac{\partial u_{1,L}}{\partial x_1} + (\nu_{0,21,L} b_{0,11,L} + \nu_{21,L} b_{11,L})\,\frac{\partial u_{2,L}}{\partial x_2} \\
&\quad -\alpha_L b_{11,L} \int\limits_0^t \left(\frac{\partial u_{1,L}}{\partial x_1} + \nu_{21,L} \frac{\partial u_{2,L}}{\partial x_2} \right) e^{-\alpha_L (t-\tau)} d\tau \,, \\
\sigma_{22,L} &= (b_{0,22,L} + b_{22,L})\,\frac{\partial u_{2,L}}{\partial x_2} + (\nu_{0,12,L} b_{0,22,L} + \nu_{12,L} b_{22,L})\,\frac{\partial u_{1,L}}{\partial x_1} \\
&\quad -\alpha_L b_{22,L} \int\limits_0^t \left(\frac{\partial u_{2,L}}{\partial x_2} + \nu_{12,L} \frac{\partial u_{1,L}}{\partial x_1} \right) e^{-\alpha_L (t-\tau)} d\tau \,,
\end{aligned}$$

$$\sigma_{12,L} = (b_{0,66,L} + b_{66,L}) \left(\frac{\partial u_{1,L}}{\partial x_2} + \frac{\partial u_{2,L}}{\partial x_1} \right) - \alpha_L b_{66,L} \int\limits_0^t \left(\frac{\partial u_{1,L}}{\partial x_2} + \frac{\partial u_{2,L}}{\partial x_1} \right) e^{-\alpha_L (t-\tau)} d\tau \,. \quad (3)$$

The material parameters corresponding to the elastic element in parallel with the Maxwell model are distinguished in (3) by additional subscript 0.

3.2 Equations of motion

The system of PDEs describing the wave phenomena in the strip can be derived by using the basic equations of motion for the state of plane stress as follows [11]

$$\rho_L \frac{\partial^2 u_{1,L}}{\partial t^2} = \frac{\partial \sigma_{11,L}}{\partial x_1} + \frac{\partial \sigma_{12,L}}{\partial x_2} \,, \quad \rho_L \frac{\partial^2 u_{2,L}}{\partial t^2} = \frac{\partial \sigma_{22,L}}{\partial x_2} + \frac{\partial \sigma_{12,L}}{\partial x_1} \,, \quad (4)$$

where ρ_L represents the material density of the Lth layer. If we make further simplification that $\nu_{0,ij,L} = \nu_{ij,L}$, i.e. that the Poisson's ratios of elastic elements in the Zener model are equal, and substitute (3) into (4), we obtain the final system of four partial integro-differential equations to be solved:

$$\begin{aligned}
\frac{\partial^2 u_{1,L}}{\partial t^2} &= \left(c^2_{0,11,L} + c^2_{11,L} \right) \left(\frac{\partial^2 u_{1,L}}{\partial x_1^2} + \nu_{21,L} \frac{\partial^2 u_{2,L}}{\partial x_1 \partial x_2} \right) + \left(c^2_{0,12,L} + c^2_{12,L} \right) \left(\frac{\partial^2 u_{1,L}}{\partial x_2^2} + \frac{\partial^2 u_{2,L}}{\partial x_1 \partial x_2} \right) \\
&\quad -\alpha_L \int\limits_0^t \left[c^2_{11,L} \frac{\partial^2 u_{1,L}}{\partial x_1^2} + c^2_{12,L} \frac{\partial^2 u_{1,L}}{\partial x_2^2} + \left(\nu_{21,L} c^2_{11,L} + c^2_{12,L} \right) \frac{\partial^2 u_{2,L}}{\partial x_1 \partial x_2} \right] e^{-\alpha_L (t-\tau)} d\tau \,,
\end{aligned}$$

$$\frac{\partial^2 u_{2,L}}{\partial t^2} = \left(c_{0,22,L}^2 + c_{22,L}^2\right)\left(\frac{\partial^2 u_{2,L}}{\partial x_2^2} + \nu_{12,L}\,\frac{\partial^2 u_{1,L}}{\partial x_1 \partial x_2}\right) + \left(c_{0,12,L}^2 + c_{12,L}^2\right)\left(\frac{\partial^2 u_{2,L}}{\partial x_1^2} + \frac{\partial^2 u_{1,L}}{\partial x_1 \partial x_2}\right)$$

$$-\alpha_L \int\limits_0^t \left[c_{22,L}^2 \frac{\partial^2 u_{2,L}}{\partial x_2^2} + c_{12,L}^2 \frac{\partial^2 u_{2,L}}{\partial x_1^2} + \left(\nu_{12,L} c_{22,L}^2 + c_{12,L}^2\right) \frac{\partial^2 u_{1,L}}{\partial x_1 \partial x_2}\right] e^{-\alpha_L (t-\tau)} d\tau \qquad (5)$$

for $L = 1, 2$. The constants $c_{0,ij,L}$ and $c_{ij,L}$ are analogous to the phase velocities in an elastic orthotropic medium and they are defined in a similar way as in [7].

3.3 Solving procedure

As mentioned above, the method of integral transforms is used for solving the problem in this work. First, the Laplace transform in time domain is applied to the system (5). Introducing a new complex variable p and taking into account the zero initial conditions of the problem, the system (5) can be transformed to a system of four PDEs for the Laplace transforms $U_{1,L}(x_1, x_2, p)$ and $U_{2,L}(x_1, x_2, p)$ of the displacement components $u_{1,L}(x_1, x_2, t)$ and $u_{2,L}(x_1, x_2, t)$, respectively, which has the form

$$p^2 U_{1,L} = C_{11,L}^2 \left(\frac{\partial^2 U_{1,L}}{\partial x_1^2} + \nu_{21,L}\,\frac{\partial^2 U_{2,L}}{\partial x_1 \partial x_2}\right) + C_{12,L}^2 \left(\frac{\partial^2 U_{1,L}}{\partial x_2^2} + \frac{\partial^2 U_{2,L}}{\partial x_1 \partial x_2}\right),$$

$$p^2 U_{2,L} = C_{22,L}^2 \left(\frac{\partial^2 U_{2,L}}{\partial x_2^2} + \nu_{12,L}\,\frac{\partial^2 U_{1,L}}{\partial x_1 \partial x_2}\right) + C_{12,L}^2 \left(\frac{\partial^2 U_{2,L}}{\partial x_1^2} + \frac{\partial^2 U_{1,L}}{\partial x_1 \partial x_2}\right). \qquad (6)$$

The complex functions $C_{11,L}(p)$, $C_{22,L}(p)$ and $C_{12,L}(p)$ used in (6) are defined by relations

$$C_{11,L} = \left[c_{0,11,L}^2 + \left(1 - \frac{\alpha_L}{p + \alpha_L}\right) c_{11,L}^2\right]^{1/2}, \quad C_{22,L} = \left[c_{0,22,L}^2 + \left(1 - \frac{\alpha_L}{p + \alpha_L}\right) c_{22,L}^2\right]^{1/2},$$

$$C_{12,L} = \left[c_{0,12,L}^2 + \left(1 - \frac{\alpha_L}{p + \alpha_L}\right) c_{12,L}^2\right]^{1/2}. \qquad (7)$$

As the next step of the solving process, the Fourier transform in x_1 variable will be applied. Due to the even property of the function $\sigma(x_1, t)$, the unknown displacement transforms $U_{1,L}(x_1, x_2, p)$ and $U_{2,L}(x_1, x_2, p)$ are given by sine and cosine Fourier integrals, respectively, such that

$$U_{1,L} = \frac{1}{\pi} \int\limits_0^\infty S_{1,L} \sin(\omega x_1)\, d\omega\,, \quad U_{2,L} = \frac{1}{\pi} \int\limits_0^\infty S_{2,L} \cos(\omega x_1)\, d\omega\,, \qquad (8)$$

where the functions $S_{1,L}(\omega, x_2, p)$ and $S_{2,L}(\omega, x_2, p)$ are the Fourier spectra which need to be find. For this purpose we substitute (8) into (6) and after some algebra we obtain a system of four linear ODEs for the unknown spectra. This system is coupled, contrary to the isotropic case, and it can be proved that its solution can be assumed in the form

$$S_{1,L} = P_L \operatorname{sh}(\Lambda_{1,L} x_2) + Q_L \operatorname{ch}(\Lambda_{1,L} x_2) + R_L \operatorname{sh}(\Lambda_{2,L} x_2) + S_L \operatorname{ch}(\Lambda_{2,L} x_2),$$

$$S_{2,L} = \xi_{1,L}\left(P_L \operatorname{ch}(\Lambda_{1,L} x_2) + Q_L \operatorname{sh}(\Lambda_{1,L} x_2)\right) + \xi_{2,L}\left(R_L \operatorname{ch}(\Lambda_{2,L} x_2) + S_L \operatorname{sh}(\Lambda_{2,L} x_2)\right). \qquad (9)$$

The notations sh and ch used in (9) stand for hyperbolic functions $\sinh$ and $\cosh$, respectively, and the complex functions $\xi_{1,L}(\omega, p)$ and $\xi_{2,L}(\omega, p)$ are introduced by formulas

$$\xi_{1,L} = \frac{\Lambda_{1,L}^2 - a_{1,L}}{\Lambda_{1,L} a_{2,L}} \quad \text{and} \quad \xi_{2,L} = \frac{\Lambda_{2,L}^2 - a_{1,L}}{\Lambda_{2,L} a_{2,L}}\,, \qquad (10)$$

in which

$$\Lambda_{1,L} = \left[\frac{1}{2}\left(k_L + s_L^{1/2}\right)\right]^{1/2}, \quad \Lambda_{2,L} = \left[\frac{1}{2}\left(k_L - s_L^{1/2}\right)\right]^{1/2}, \quad k_L = a_{1,L} + b_{1,L} - a_{2,L}b_{2,L}\,,$$
$$s_L = a_{2,L}^2 b_{2,L}^2 - 2a_{2,L}b_{2,L}\left(a_{1,L} + b_{1,L}\right) + \left(a_{1,L} - b_{1,L}\right)^2. \tag{11}$$

The functions $a_{i,L}(\omega, p)$ and $b_{i,L}(\omega, p)$ present in (10-11) follow from the coefficients of bi-quadratic characteristic equation of the mentioned system of ODEs and they can be expressed for each layer L analogously as in [7].

In the next step, the unknown functions $P_L(\omega, p)$, $Q_L(\omega, p)$, $R_L(\omega, p)$ and $S_L(\omega, p)$ have to be found for $L = 1, 2$. When the problem of a single-layered strip is solved (see [7]), the number of these unknowns reduces to four and they can be quite simply derived from four boundary conditions valid at the strip edges. But in the case of a two-layered strip, we receive eight functions which can be determined based on the boundary conditions formulated at the strip edges and at the layer interface. With respect to the problem formulation in Section 2 and according to Fig. 1, the boundary conditions are as follows

$$\begin{gathered}
\sigma_{22,2}(x_1, h_2, t) = \sigma(x_1, t)\,, \quad \sigma_{12,2}(x_1, h_2, t) = 0\,, \\
u_{1,1}(x_1, h_1, t) = u_{1,2}(x_1, h_1, t)\,, \quad u_{2,1}(x_1, h_1, t) = u_{2,2}(x_1, h_1, t)\,, \\
\sigma_{22,1}(x_1, h_1, t) = \sigma_{22,2}(x_1, h_1, t)\,, \quad \sigma_{12,1}(x_1, h_1, t) = \sigma_{12,2}(x_1, h_1, t)\,, \\
\sigma_{22,1}(x_1, 0, t) = 0\,, \quad \sigma_{12,1}(x_1, 0, t) = 0\,.
\end{gathered} \tag{12}$$

Making the Laplace transforms of constitutive relations (3) and substituting them together with (8-9) into transformed conditions (12) one obtains the following system of equations

$$\begin{aligned}
k_{1,2}\left[P_2\,\mathrm{sh}(h_2\Lambda_{1,2}) + Q_2\,\mathrm{ch}(h_2\Lambda_{1,2})\right] + k_{2,2}\left[R_2\,\mathrm{sh}(h_2\Lambda_{2,2}) + S_2\,\mathrm{ch}(h_2\Lambda_{2,2})\right] &= f, \\
k_{3,2}\left[P_2\,\mathrm{ch}(h_2\Lambda_{1,2}) + Q_2\,\mathrm{sh}(h_2\Lambda_{1,2})\right] + k_{4,2}\left[R_2\,\mathrm{ch}(h_2\Lambda_{2,2}) + S_2\,\mathrm{sh}(h_2\Lambda_{2,2})\right] &= 0, \\
P_1\,\mathrm{sh}(h_1\Lambda_{1,1}) + Q_1\,\mathrm{ch}(h_1\Lambda_{1,1}) + R_1\,\mathrm{sh}(h_1\Lambda_{2,1}) + S_1\,\mathrm{ch}(h_1\Lambda_{2,1}) & \\
-P_2\,\mathrm{sh}(h_1\Lambda_{1,2}) - Q_2\,\mathrm{ch}(h_1\Lambda_{1,2}) - R_2\,\mathrm{sh}(h_1\Lambda_{2,2}) - S_2\,\mathrm{ch}(h_1\Lambda_{2,2}) &= 0, \\
\xi_{1,1}\left[P_1\,\mathrm{ch}(h_1\Lambda_{1,1}) + Q_1\,\mathrm{sh}(h_1\Lambda_{1,1})\right] + \xi_{2,1}\left[R_1\,\mathrm{ch}(h_1\Lambda_{2,1}) + S_1\,\mathrm{sh}(h_1\Lambda_{2,1})\right] & \\
-\xi_{1,2}\left[P_2\,\mathrm{ch}(h_1\Lambda_{1,2}) + Q_2\,\mathrm{sh}(h_1\Lambda_{1,2})\right] - \xi_{2,2}\left[R_2\,\mathrm{ch}(h_1\Lambda_{2,2}) + S_2\,\mathrm{sh}(h_1\Lambda_{2,2})\right] &= 0, \\
k_{1,1}\left[P_1\,\mathrm{sh}(h_1\Lambda_{1,1}) + Q_1\,\mathrm{ch}(h_1\Lambda_{1,1})\right] + k_{2,1}\left[R_1\,\mathrm{sh}(h_1\Lambda_{2,1}) + S_1\,\mathrm{ch}(h_1\Lambda_{2,1})\right] & \\
-k_{1,2}\left[P_2\,\mathrm{sh}(h_1\Lambda_{1,2}) + Q_2\,\mathrm{ch}(h_1\Lambda_{1,2})\right] - k_{2,2}\left[R_2\,\mathrm{sh}(h_1\Lambda_{2,2}) + S_2\,\mathrm{ch}(h_1\Lambda_{2,2})\right] &= 0, \\
k_{3,1}\left[P_1\,\mathrm{ch}(h_1\Lambda_{1,1}) + Q_1\,\mathrm{sh}(h_1\Lambda_{1,1})\right] + k_{4,1}\left[R_1\,\mathrm{ch}(h_1\Lambda_{2,1}) + S_1\,\mathrm{sh}(h_1\Lambda_{2,1})\right] & \\
-k_{3,2}\left[P_2\,\mathrm{ch}(h_1\Lambda_{1,2}) + Q_2\,\mathrm{sh}(h_1\Lambda_{1,2})\right] - k_{4,2}\left[R_2\,\mathrm{ch}(h_1\Lambda_{2,2}) + S_2\,\mathrm{sh}(h_1\Lambda_{2,2})\right] &= 0, \\
k_{1,1}\,Q_1 + k_{2,1}\,S_1 &= 0, \\
k_{3,1}\,P_1 + k_{4,1}\,R_1 &= 0,
\end{aligned} \tag{13}$$

where the function $f(\omega, p)$ corresponds to applied load and there hold

$$f(\omega, p) = a(\omega)\,\frac{1 - \exp\left(-pt_0\right)}{p} \quad \text{and} \quad \sigma_0(x_1) = \frac{1}{\pi}\int_0^\infty a(\omega)\cos(\omega\, x_1)d\omega\,. \tag{14}$$

The other functions appearing in the system (13) are defined as

$$\begin{gathered}
k_{1,L}(\omega, p) = h_{22,L}\left(\nu_{12,L}\,\omega + \xi_{1,L}\Lambda_{1,L}\right), \quad k_{2,L}(\omega, p) = h_{22,L}\left(\nu_{12,L}\,\omega + \xi_{2,L}\Lambda_{2,L}\right), \\
k_{3,L}(\omega, p) = h_{12,L}\left(\Lambda_{1,L} - \omega\,\xi_{1,L}\right), \quad k_{4,L}(\omega, p) = h_{12,L}\left(\Lambda_{2,L} - \omega\,\xi_{2,L}\right),
\end{gathered}$$

$$h_{22,L}(p) = b_{0,22,L} + \frac{p}{p+\alpha_L} b_{22,L}\,, \quad h_{12,L}(p) = b_{0,66,L} + \frac{p}{p+\alpha_L} b_{66,L}\,. \tag{15}$$

When the solution of the system (13) is found, the relations (8) and (9) can be used to obtain the final transforms of displacement components in Laplace domain. The formulas for other mechanical quantities can then be derived by using presented kinematic and constitutive equations. Basically, two approaches to the solution of (13) exist. First, with respect to the complexity of the problem, we tried to find the numerical solution for all the required values of ω and p using a Matlab code. The analysis of the system (13) showed that the spectral radius of the system matrix is too large to obtain correct numerical values of P_L, Q_L, R_L and S_L. Therefore the symbolic system Maple was used to derive the exact analytical solution of the system (13). Contrary to the problem with symmetric layers solved in [8], the system matrix has larger number of nonzero elements, so the derived solution is quite complicated and it is not possible to present it in this work. But the first computations showed that its evaluation process is very sensitive to numerical errors and very time consuming.

The Laplace inversion back to time domain is the last step of the evaluation process. The exact inversion (e.g. by means of the residue theorem) is theoretically possible but due to the complexity of the derived formulas the numerical approach is preferred. From the large number of existing algorithms we can recommend the FFT based algorithm combined with the Wynn's epsilon accelerator (see [12]). This algorithm was tested in [13] on different problems of elastodynamics and it was showed that it is very efficient and robust and that it can be used to obtain results of accuracy comparable to those from analytical approach (see e.g. [14]).

4 CONCLUSIONS

Transient response of an infinite two-layered strip to a dynamic loading is investigated using an analytical approach in this work. The material properties of each layer are modelled by the Zener viscoelastic model with special orthotropic properties. Using the method of integral transform the formulas for displacement components are derived in Laplace domain and the process of their evaluation is discussed. The presented solution can be used for studying wave phenomena in solids of strip-like geometry, strips with thin layers, beams on viscoelastic foundations, etc. Additionally, the non-symmetric formulation enables us to derive the solution for analogous problem of a multilayered strip with large field of potential applications.

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