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PARTITIONED FORMULATION OF CONTACT-IMPACT PROBLEMS WITH STABILIZED CONTACT CONSTRAINTS AND RECIPROCAL MASS MATRICES

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Abstract. *This work proposes an extremely efficient and accurate time explicit solution algorithm for the simulation of contact-impact problems with finite elements. The methodology combines three existent techniques. First, for the treatment of the contact problem, a bipenalty contact-impact algorithm for explicit dynamics with stabilization of the contact constraints through the addition of stiffness and mass penalties [1]. Second, for the reduction of the contact problem to the interface, the method of localized Lagrange multipliers [2], that is able to formulate the contact problem in partitioned form, uncoupling the contact terms from the free body matrices. And finally, for an efficient construction of the free-free inverse mass matrices, the method based on localized Lagrange multipliers to generate inverse mass matrices [3] with specific projectors for the application of boundary conditions. It is also demonstrated that the resultant methodology is very well suited for parallelization in multicore machines. Different numerical experiments will be used to prove the effectiveness of the proposed formulation.*

Keywords: Bipenalty contact, Inverse mass matrix, Explicit time integration, Localized Lagrange multipliers, Partitioned analysis.

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1 INTRODUCTION

The accurate and robust solution of contact-impact problems is still an open problem. Although many solution techniques exist, the penalty contact method is very popular particularly in explicit dynamics, mainly due to its simplicity. However, the stability limit of this method can be seriously affected by the large stiffness introduced to enforce the fulfillment of the kinematic contact constraints.

In this work, we present an explicit formulation for the partitioned solution of finite element contact-impact problems with localized Lagrange multipliers [2] and stabilization of the contact constraints using combined mass and stiffness penalization [1]. The key idea of the formulation is to use the method proposed by Tkachuk and Bischoff [4] for constructing a direct inverse of the mass matrix. The prevailing practice for computing the mass matrix $\mathbf{M}$ in the finite element method is to use a bilinear form of the system kinetic energy T, viz.,

$$T = \int_\Omega \tfrac{1}{2}\rho\dot{\boldsymbol{u}}(\mathbf{x},t)\cdot\dot{\boldsymbol{u}}(\mathbf{x},t)\,d\Omega \approx \tfrac{1}{2}\dot{\mathbf{u}}(t)^\mathsf{T}\mathbf{M}\,\dot{\mathbf{u}}(t) \tag{1}$$

where $\dot{\boldsymbol{u}}(\mathbf{x},t)$ and $\dot{\mathbf{u}}(t)$ are the continuum velocity and discrete approximate velocity, respectively. Since our objective is to generate the inverse mass matrix directly, instead of employing matrix inversion, we employ a dual form of the kinetic energy:

$$T = \int_\Omega \frac{1}{2\rho}\boldsymbol{p}(\mathbf{x},t)\cdot\boldsymbol{p}(\mathbf{x},t)\,d\Omega \approx \tfrac{1}{2}\mathbf{p}(t)^\mathsf{T}\mathbf{C}\mathbf{p}(t) \tag{2}$$

where $\boldsymbol{p}(\mathbf{x},t)$ and $\mathbf{p}(t)$ are respectively the continuum and discrete approximate momentum vectors, and $\mathbf{C}$ is a reciprocal mass matrix. It is demonstrated that an inverse of the mass matrix can be obtained via:

$$\mathbf{M}^{-1} = \mathbf{A}^{-\mathsf{T}}\mathbf{C}\mathbf{A}^{-1} \tag{3}$$

where $\mathbf{A}$ is the dual-base projection matrix, that can be transformed into a diagonal matrix employing $\mathbf{u}, \mathbf{p}$ biorthogonal shape functions [3]. Hence, the computational simplicity of obtaining an inverse of the mass-matrix via this new route.

However, the construction process of this inverse mass matrix gets complicated in bi-penalty contact schemes when a non-diagonal mass penalization is introduced. Our approach proposes to treat the stabilized contact problem using localized Lagrange multipliers. The benefit of this approach is twofold. First, it alleviates the special treatments needed for the construction of the reciprocal mass matrix and second, it completely decouples the interface contact problem from the body problem. This way, the constructed inverse mass matrix (3) appears unaltered in the equilibrium equations of the contacting bodies and it is not affected by the mass penalties, that remain exclusively in the partitioned equations of motion obtained for the contact interface.

The described formulation produces an extremely efficient explicit contact-impact algorithm that is able to maintain stability and produce accurate results even near the critical time-step, under conditions totally forbidden for the standard stiffness penalty method. All these properties are demonstrated analyzing the classical benchmark problem of an elastic cylinder impacting on a rigid wall.

2 BIPENALTY CONTACT WITH RECIPROCAL MASS MATRIX AND LOCALIZED LAGRANGE MULTIPLIERS

To this end, we invoke Hamilton's principle for partitioned constrained elastodynamics [3] including penalized contact constraints [1]. The variational form of the Hamiltonian can be expressed using the following four-field variational form:

$$\delta H(\boldsymbol{u},\boldsymbol{p},\boldsymbol{u}_b,\boldsymbol{\ell}) = \int_{t_1}^{t_2} \delta\left\{T(\dot{\boldsymbol{u}},\boldsymbol{p}) - U(\boldsymbol{u},\boldsymbol{u}_b,\boldsymbol{\ell}) + W(\boldsymbol{u})\right\}\, dt = 0 \tag{4}$$

were δT is the virtual kinetic energy, δU the virtual elastic energy and δW the virtual work done by the external loads, magnitudes that can be expressed:

$$\delta T = \int_\Omega \delta\left(\tfrac{1}{2}\boldsymbol{p}\cdot\dot{\boldsymbol{u}}\right)\, d\Omega + \int_{\Gamma_c} \delta\left(\tfrac{1}{2}\epsilon_m \dot{g}_N^2\right)\, d\Gamma \tag{5}$$

$$\delta U = \int_\Omega \delta\boldsymbol{\varepsilon} : \boldsymbol{\sigma}\, d\Omega + \int_\Gamma \delta\left\{\boldsymbol{\ell}\cdot(\boldsymbol{u}-\boldsymbol{u}_b)\right\}\, d\Gamma + \int_{\Gamma_c} \delta\left(\tfrac{1}{2}\epsilon_s g_N^2\right)\, d\Gamma \tag{6}$$

$$\delta W = \int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{f}\, d\Omega \tag{7}$$

with boundary displacements $\boldsymbol{u}_b$ at the contact zone Γ_c enforced by using a field of localized Lagrange multipliers $\boldsymbol{\ell}$ and external body forces per unit volume $\boldsymbol{f}$ acting on Ω. Note that the penalized contact condition appears not only in the variation of internal energy but also in the variation of kinetic energy, enforcing a zero gap rate in the contact region. This way, the stiffness and mass penalty parameters are defined respectively as ϵ_s and ϵ_m.

By making use of the momentum-velocity relation ($\boldsymbol{p} = \rho\dot{\boldsymbol{u}}$) in the following identity:

$$\tfrac{1}{2}\boldsymbol{p}\cdot\dot{\boldsymbol{u}} = \boldsymbol{p}\cdot\dot{\boldsymbol{u}} - \frac{1}{2\rho}\boldsymbol{p}\cdot\boldsymbol{p} \tag{8}$$

we can obtain an equivalent expression for the virtual kinetic energy per unit volume:

$$\delta\left(\tfrac{1}{2}\boldsymbol{p}\cdot\dot{\boldsymbol{u}}\right) = \delta\dot{\boldsymbol{u}}\cdot\boldsymbol{p} + \delta\boldsymbol{p}\cdot\left(\dot{\boldsymbol{u}} - \frac{1}{\rho}\boldsymbol{p}\right) \tag{9}$$

and finally performing an integration by parts of the second term of the last equation yields:

$$\int_{t_1}^{t_2} \delta\dot{\mathbf{u}}\cdot\boldsymbol{p}\, dt = -\int_{t_1}^{t_2} \delta\boldsymbol{u}\cdot\dot{\boldsymbol{p}}\, dt \tag{10}$$

relation that can be used to eliminate the virtual velocity field from the formulation.

Introducing previous identities, (9) and (10), into the principle of stationary action (4), we obtain the final four-field variational form of the Hamilton's principle for constrained elastodynamics with contact penalties:

$$\begin{aligned}\delta H(\boldsymbol{u},\boldsymbol{p},\boldsymbol{u}_b,\boldsymbol{\ell}) = \int_{t_1}^{t_2} \Bigg\{ &\int_\Omega \delta\boldsymbol{p}\cdot\left(\dot{\boldsymbol{u}} - \tfrac{1}{\rho}\boldsymbol{p}\right)\, d\Omega - \int_\Omega \left(\delta\boldsymbol{u}\cdot\dot{\boldsymbol{p}} + \delta\boldsymbol{\varepsilon} \colon \boldsymbol{\sigma}\right)\, d\Omega + \int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{f}\, d\Omega \\ + \int_{\Gamma_c} \delta\boldsymbol{u}\cdot\boldsymbol{\ell}\, d\Gamma - \int_{\Gamma_c} \delta\boldsymbol{\ell}\cdot(\boldsymbol{u}-\boldsymbol{u}_b)\, d\Gamma &- \int_{\Gamma_c} \delta\boldsymbol{u}_b\cdot\boldsymbol{\ell}\, d\Gamma + \int_{\Gamma_c} \delta g_N\left(\epsilon_m \ddot{g}_N + \epsilon_s g_N\right)\, d\Gamma \Bigg\}\, dt = 0\end{aligned} \tag{11}$$

expression that will be used to derive the equations of motion.

Discretization in space of this mixed form is performed by using independent shape functions for displacements, momenta and Lagrangian multipliers. We carry out then a standard mixed FEM discretization with independent shape functions for the different fields:

$$\boldsymbol{u} = \mathbf{N}_u\mathbf{u}, \qquad \boldsymbol{p} = \mathbf{N}_p\mathbf{p}, \qquad \boldsymbol{u}_b = \mathbf{N}_u\mathbf{u}_b, \qquad \boldsymbol{\ell} = \mathbf{N}_\lambda\boldsymbol{\lambda} \tag{12}$$

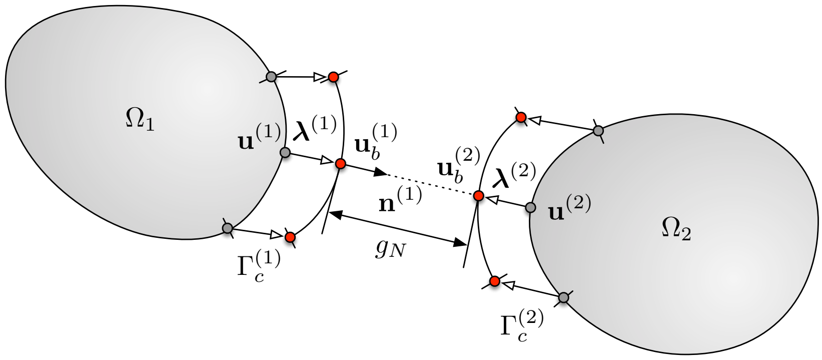

Figure 1: Two bodies in potential contact definition of the normal gap function g_N. Localized Lagrange multipliers $\boldsymbol{\lambda}^{(i)}$ with $i = 1, 2$ are introduced on both sides of the contact interface to decouple solid equilibrium equations from the contact problem.

where we use the same shape functions for the body surface and boundary displacements. Note also, see Figure 1, that the gap can be related to the normal projection of the boundary displacement and expressed:

$$g_N = \mathbf{H}\mathbf{u}_b + g_0 \tag{13}$$

where matrix $\mathbf{H}$ represents a geometrical operator from the boundary displacement field to the gap function. The particular form of this operator follows from the assumed contact discretization.

Introducing these approximations in (11) one obtains the following set of semi-discrete equations:

$$\mathbf{A}^{\mathsf{T}}\dot{\mathbf{u}} - \mathbf{C}\mathbf{p} = \mathbf{0} \qquad \textit{Momentum equation} \tag{14}$$
$$\mathbf{A}\dot{\mathbf{p}} + \mathbf{B}\boldsymbol{\lambda} = \mathbf{r} \qquad \textit{Equilibrium equation} \tag{15}$$
$$\mathbf{B}^{\mathsf{T}}\mathbf{u} - \mathbf{L}_b\mathbf{u}_b = \mathbf{0} \qquad \textit{Boundary and interface constraints} \tag{16}$$
$$\mathbf{L}_b^{\mathsf{T}}\boldsymbol{\lambda} = \mathbf{r}_c \qquad \textit{Equilibrium on the contact boundaries} \tag{17}$$

where vector $\mathbf{r} = \mathbf{f} - \mathbf{f}^{int}$ is the external-internal forces residual, the internal forces for linear problems are simply given by $\mathbf{f}^{int} = \mathbf{K}\mathbf{u}$ and the matrix components are expressed:

$$\mathbf{A} = \int_\Omega \mathbf{N}_u^{\mathsf{T}}\,\mathbf{N}_p\,d\Omega, \qquad \mathbf{C} = \int_\Omega \frac{1}{\rho}\mathbf{N}_p^{\mathsf{T}}\,\mathbf{N}_p\,d\Omega \tag{18}$$
$$\mathbf{B} = \int_\Gamma \mathbf{N}_u^{\mathsf{T}}\,\mathbf{N}_\lambda\,d\Gamma, \qquad \mathbf{L}_b = \int_\Gamma \mathbf{N}_\lambda^{\mathsf{T}}\,\mathbf{N}_u\,d\Gamma \tag{19}$$

where $\mathbf{A}$ is the global projection matrix, $\mathbf{C}$ the global reciprocal mass matrix, $\mathbf{B}$ the boundary assembly operator and $\mathbf{L}_b$ is the Localized multipliers assembly matrix.

In the case of geometrically linear kinematics, the contact residual vector can be written as

$$\mathbf{r}_c\left(\mathbf{u}_b, \ddot{\mathbf{u}}_b\right) = \mathbf{M}_p\ddot{\mathbf{u}}_b + \mathbf{K}_p\mathbf{u}_b + \mathbf{f}_p \tag{20}$$

where

$$\mathbf{M}_p = \int_{\Gamma_c} \epsilon_m\mathbf{H}^{\mathsf{T}}\mathbf{H}\,d\Gamma \qquad \mathbf{K}_p = \int_{\Gamma_c} \epsilon_s\mathbf{H}^{\mathsf{T}}\mathbf{H}\,d\Gamma \qquad \mathbf{f}_p = \int_{\Gamma_c} \epsilon_s\mathbf{H}\,g_0\,d\Gamma \tag{21}$$

are the additional mass matrix due to inertia penalty $\mathbf{M}_p$, the additional stiffness matrix due to stiffness penalty $\mathbf{K}_p$ and $\mathbf{f}_p$ is the part of the contact force due to the initial gap g_0.

Eliminating symbolically the momentum variable ($\mathbf{p}$) from (14) and (15), one obtains the classical equation of motion expressed in terms of displacements:

$$\left(\mathbf{A}\mathbf{C}^{-1}\mathbf{A}^{\mathsf{T}}\right)\ddot{\mathbf{u}} + \mathbf{B}\boldsymbol{\lambda} = \mathbf{r} \tag{22}$$

with the mass matrix approximated as:

$$\mathbf{M} = \mathbf{A}\mathbf{C}^{-1}\mathbf{A}^{\mathsf{T}} \tag{23}$$

and observe that there must exist an inverse mass matrix (denoted as $\mathbf{M}^{-1}$) given by:

$$\mathbf{M}^{-1} = \mathbf{A}^{-\mathsf{T}}\mathbf{C}\mathbf{A}^{-1} \tag{24}$$

assuming that the global projection matrix is invertible. Since the objective of the present work is to obtain inverse mass-matrices in efficient and accurate ways, one must insist on easily to invert *diagonal* or narrowly banded projection matrices. This can be accomplished by diagonalizing the projection matrix $\mathbf{A}$ defined in (18) using $\mathbf{N}_u - \mathbf{N}_p$ biorthogonal shape functions [3, 4].

3 PARTITIONED FORMULATION OF THE CONTACT PROBLEM

A standard bi-penalty explicit solution algorithm without partitioning, requires the consecutive inversion of the mass matrix $\mathbf{M}$ plus mass penalties $\mathbf{M}_p$ changing in time due to the different contact conditions. This solution procedure involving the total number of DOFs is extremely inefficient, so we propose a partitioning strategy to uncouple the interface contact problem from the body problem.

As we mentioned, in presence of contact conditions a direct computation of the mass-inverse requires complicated modifications of the biorthogonal shape functions. We eliminate the necessity of these modifications by considering the structures as free-floating and applying the boundary conditions through localized Lagrangian multipliers [2]. To explain the enforcement of boundary conditions, we reorganize the semi-discrete equations of motion (14)-(17) to obtain the following partitioned equation set:

$$\begin{bmatrix} \mathbf{M} & \mathbf{B} & \mathbf{0} \\ \mathbf{B}^{\mathsf{T}} & \mathbf{0} & -\mathbf{L}_b \\ \mathbf{0} & -\mathbf{L}_b^{\mathsf{T}} & \mathbf{M}_p \end{bmatrix} \left\{ \begin{array}{c} \ddot{\mathbf{u}} \\ \boldsymbol{\lambda} \\ \ddot{\mathbf{u}}_b \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{r} \\ \mathbf{0} \\ -\mathbf{r}_p \end{array} \right\} \tag{25}$$

where the second equation is simply twice time-differentiated expression of (16). These are the partitioned equations of motion including mass penalization, where vector $\mathbf{r}_p$ represents the classical stiffness penalty forces due to the contact conditions.

With the intention of reducing the problem to the contact variables, let us obtain first the partition accelerations ($\ddot{\mathbf{u}}$) from the first row of (25) as:

$$\ddot{\mathbf{u}} = \mathbf{M}^{-1}\left(\mathbf{r} - \mathbf{B}\boldsymbol{\lambda}\right) \tag{26}$$

and the interface accelerations from the third equation as:

$$\ddot{\mathbf{u}}_b = \mathbf{M}_p^{-1}\left(\mathbf{L}_b^{\mathsf{T}}\boldsymbol{\lambda} - \mathbf{r}_p\right) \tag{27}$$

Algorithm 1 Explicit predictor-corrector algorithm for contact-impact.

$\mathbf{u}^0$, $\dot{\mathbf{u}}^{-\frac{1}{2}}$ ▷ Initial conditions
for $n = 1 \dots n_{steps}$ **do time loop** ▷ Time integration loop
  $\ddot{\mathbf{u}}^n_{pred} = \mathbf{M}^{-1}\mathbf{r}^n$ ▷ Predicted accelerations
  $\dot{\mathbf{u}}^{n+\frac{1}{2}}_{pred} = \dot{\mathbf{u}}^{n-\frac{1}{2}} + \Delta t \ddot{\mathbf{u}}^n_{pred}$ ▷ Predicted mid-point velocities
  $\mathbf{u}^{n+1}_{pred} = \mathbf{u}^n + \Delta t \dot{\mathbf{u}}^{n+\frac{1}{2}}_{pred}$ ▷ Predicted displacements
  $\mathbf{u}^{n+1}_b = \left[\mathbf{L}_b^\mathsf{T}\mathbf{L}_b\right]^{-1}\mathbf{B}^\mathsf{T}\mathbf{u}^{n+1}_{pred}$ ▷ Predicted boundary displacements
  $\mathbf{r}^{n+1}_p = \mathbf{K}_p\mathbf{u}^{n+1}_b + \mathbf{f}_p$ ▷ Evaluate stiffness penalty forces
  $\boldsymbol{\lambda}^{n+1}_p = \left[\mathbf{B}^\mathsf{T}\mathbf{M}^{-1}\mathbf{B} + \mathbf{L}_b\mathbf{M}_p^{-1}\mathbf{L}_b^\mathsf{T}\right]^{-1}\mathbf{L}_b\mathbf{M}_p^{-1}\mathbf{r}^{n+1}_p$ ▷ Solve for bi-penalty multipliers
  $\ddot{\mathbf{u}}^n_{corr} = -\mathbf{M}^{-1}\mathbf{B}\boldsymbol{\lambda}^{n+1}_p$ ▷ Correction of accelerations
  $\ddot{\mathbf{u}}^n = \ddot{\mathbf{u}}^n_{pred} + \ddot{\mathbf{u}}^n_{corr}$ ▷ Obtain total accelerations
  $\dot{\mathbf{u}}^{n+\frac{1}{2}} = \dot{\mathbf{u}}^{n+\frac{1}{2}}_{pred} + \Delta t \ddot{\mathbf{u}}^n_{corr}$ ▷ New mid-point velocities
  $\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \dot{\mathbf{u}}^{n+\frac{1}{2}}$ ▷ New displacements
end for

in which once again we emphasize that the computation of $\mathbf{M}^{-1} = \mathbf{A}^{-\mathsf{T}}\mathbf{C}\mathbf{A}^{-1}$ does not requiere additional computational effort because the global projection matrix $\mathbf{A}$ is diagonal.

Then, substitute the accelerations obtained above into the second equation of (25):

$$\mathbf{B}^\mathsf{T}\mathbf{M}^{-1}\left(\mathbf{r} - \mathbf{B}\boldsymbol{\lambda}\right) + \mathbf{L}_b\mathbf{M}_p^{-1}\left(\mathbf{r}_p - \mathbf{L}_b^\mathsf{T}\boldsymbol{\lambda}\right) = \mathbf{0} \tag{28}$$

to compute the localized Lagrange multipliers:

$$\boldsymbol{\lambda} = \left[\mathbf{B}^\mathsf{T}\mathbf{M}^{-1}\mathbf{B} + \mathbf{L}_b\mathbf{M}_p^{-1}\mathbf{L}_b^\mathsf{T}\right]^{-1}\left(\mathbf{B}^\mathsf{T}\mathbf{M}^{-1}\mathbf{r} + \mathbf{L}_b\mathbf{M}_p^{-1}\mathbf{r}_p\right) \tag{29}$$

and substitute them back in (26) to obtain the partition accelerations:

$$\ddot{\mathbf{u}} = \boldsymbol{\mathcal{P}}\mathbf{M}^{-1}\mathbf{r} + (\mathbf{I} - \boldsymbol{\mathcal{P}})\mathbf{B}\mathbf{L}_b\mathbf{M}_p^{-1}\mathbf{r}_p \tag{30}$$

using a projection operator defined as:

$$\boldsymbol{\mathcal{P}} = \mathbf{I} - \mathbf{M}^{-1}\mathbf{B}\left[\mathbf{B}^\mathsf{T}\mathbf{M}^{-1}\mathbf{B} + \mathbf{L}_b\mathbf{M}_p^{-1}\mathbf{L}_b^\mathsf{T}\right]^{-1}\mathbf{B}^\mathsf{T} \tag{31}$$

that in the case of a complete separation, where mass penalties disappear ($\mathbf{M}_p^{-1} = \mathbf{0}$), projects the acceleration vector on a subspace fulfilling the zero contact conditions ($\mathbf{g}_N > \mathbf{0}$). In a general case, with partial contact, since the size of $\left[\mathbf{B}^T\mathbf{M}^{-1}\mathbf{B} + \mathbf{L}_b\mathbf{M}_p^{-1}\mathbf{L}_b^\mathsf{T}\right]$ is small, pertaining only to the contact degrees of freedom. Therefore, its inversion is trivial.

Finally note that to obtain the normal gap $\mathbf{g}_N$ and evaluate the contact conditions, we need the boundary displacements. This is done after computing the substructural displacements, by extracting the interface displacements using the expression:

$$\mathbf{u}_b = \left[\mathbf{L}_b^\mathsf{T}\mathbf{L}_b\right]^{-1}\mathbf{B}^\mathsf{T}\mathbf{u} \tag{32}$$

and we are ready to evaluate the normal gap through equation (13).

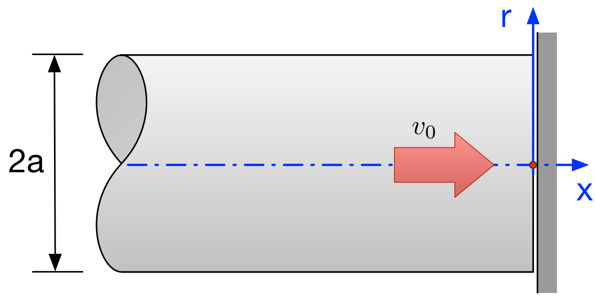

Figure 2: Scheme of an elastic cylinder of radius a and uniform velocity v_0 impacting on a rigid wall. The cylinder is meshed with an axisymmetric mesh of 8×40 quadrilateral elements.

3.1 Explicit predictor-corrector time integration scheme for contact-impact

Time integration of the semi-discrete equations of motion (25) is performed using the predictor-corrector scheme proposed by Wu [5] with splitting of bulk and contact accelerations, reproduced in Algorithm 1. It is assumed that the time integration has progressed until time t^n, with known displacements $\mathbf{u}^n$ and velocities at time $t^{n-\frac{1}{2}} = t^n - \frac{\Delta t}{2}$ defined as $\dot{\mathbf{u}}^{n-\frac{1}{2}}$, to advance the solution one step further with time step size $\Delta t = t^{n+1} - t^n$.

In the proposed splitting scheme, bulk accelerations in the predictor phase $\dot{\mathbf{u}}^n_{pred}$ and the correction accelerations $\dot{\mathbf{u}}^n_{corr}$ are computed using the direct inverse mass matrix (3) without interference of the mass penalties in $\mathbf{M}_p$, that are included through the localized Lagrange multipliers in a small problem reduced to the contact unknowns.

4 NUMERICAL EXAMPLE (TAYLOR TEST)

As a benchmark problem we adopt the Taylor test of a perfectly elastic cylinder, a problem with semi-analytical solution [6]. We consider the particular case of an cylinder of radius $a = 2$ m and length $L = 10$ m with an initial velocity $v_0 = 0.1$ m/s impacting on a rigid wall, represented in Figure 2. The problem is treated as axisymmetric, using a regular mesh of 8×40 quadrilateral finite elements of size $h_e = 0.25\,\mathrm{m}$. The material is considered elastic, with Young modulus $E = 1$ MPa, Poisson coefficient $\nu = 0.3$ and mass density $\rho = 1000\,\mathrm{kg/m^3}$.

The stiffness penalty parameter is fixed to $\epsilon_s = 20\rho c_L^2$. An eigenvalue analysis of the free cylinder without penalties, reveals that the critical time step is $\Delta t_{cr} = 3.43 \times 10^{-3}$ s and including the stiffness penalty ϵ_s the critical time step is reduced to $\Delta t^p_{cr} = 2.56 \times 10^{-3}$ s. The time integration of the semi-discrete equations of motion (25) is performed using the scheme described in Algorithm 1.

4.1 Study of the stability limit

It is well known that the classical penalty contact method [1], increasing the penalty stiffness ϵ_s with $\epsilon_m = 0$, significantly affects the stability limit of the time integration algorithm used. In general, the ratio between the Courant number with penalization $C^p_{cr} = 2c_L \Delta t^p_{cr}/h_e$ and the critical Courant number without penalization $C_{cr} = 2c_L \Delta t_{cr}/h_e$ is lower than one and decreases with the penalty stiffness ϵ_s. This effect is observed in Figure 3 for the curve $r = \infty$, corresponding to $\epsilon_m = 0$.

In the bi-penalty case, the ratio between stiffness and mass penalty is controlled by the non-

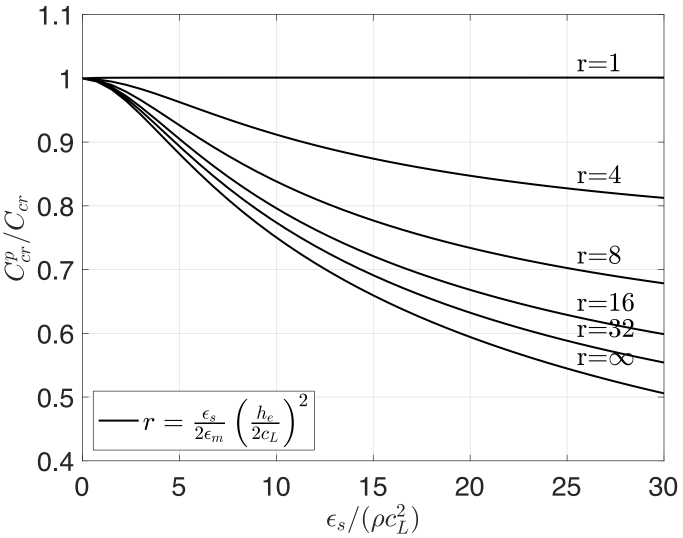

Figure 3: Representation of the stability limit of the cylinder impact problem as a function of the stiffness penalty ϵ_s and the mass-stiffness penalty ratio r.

dimensional parameter:

$$r = \frac{\epsilon_s}{2\epsilon_m}\left(\frac{h_e}{2c_L}\right)^2 \tag{33}$$

where c_L is the velocity of elastic longitudinal waves and h_e is the finite element size. This definition produces bi-penalty with similar mass and stiffness penalty coefficients for $r \to 1$.

In Figure 3, it is represented the reduction of the critical Courant number C_{cr}^p/C_{cr} for different values of the stiffness penalties ratio obtained for our benchmark problem. Note that, for a fixed stiffness penalty ϵ_s, it is possible to find the mass-stiffness penalty ratio r needed to achieve the critical Courant number necessary for stability. This value of the ratio r defines our required mass penalty ϵ_m for stability.

4.2 Transient analysis

We investigate in this Section the effects of the mass-penalty and time-step used for time integration. First, we solve the problem using a stable time step $\Delta t = 1.8\times10^{-3}$s corresponding to a Courant number $C = 0.7\, C_{cr}^p$. We observe in Figure 4 the time histories of displacements (left) and energies (right) obtained for bi-penalty with $r = 1$. Adding mass penalty when time-step is stable, has the benefit of stabilizing the oscillations occurring at the contact zone.

Next, we select a new time-step $\Delta t = 2.5 \times 10^{-3}$ s close to the critical, corresponding to a Courant number $C = 0.98C_{cr}^p$. Two cases are solved, the standard stiffness penalty case $r \to \infty$ and the bi-penalty case with stiffness-mass penalty ratio $r = 2$. The results for exclusively stiffness penalty are represented in Figure 5, demonstrating that the standard stiffness penalty method is not able to produce stable results for time-steps close to the critical time-step with penalization. On the other hand, see results of Figure 6, the bi-penalty method with ratio $r = 1$ produces completely stable solutions and results very similar to those obtained with a stable time step, represented in Figure 6.

Another aspect to comment is the extreme computational efficiency of the contact algorithm used, where only matrix-vector products are needed to produce the solution, thanks to the proposed direct computation of the inverse mass matrix.

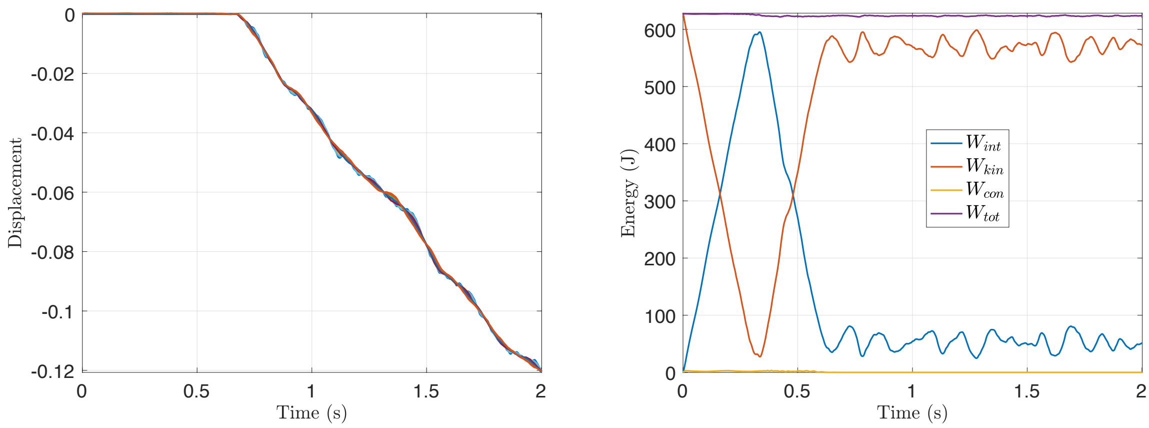

Figure 4: Results for Courant number $C = 0.7\, C_{cr}^{p}$ and bi-penalty parameter $r = 1$. Horizontal displacement of nodes located in the contact zone (left) and internal, kinetic, contact and total energy of the cylinder (right).

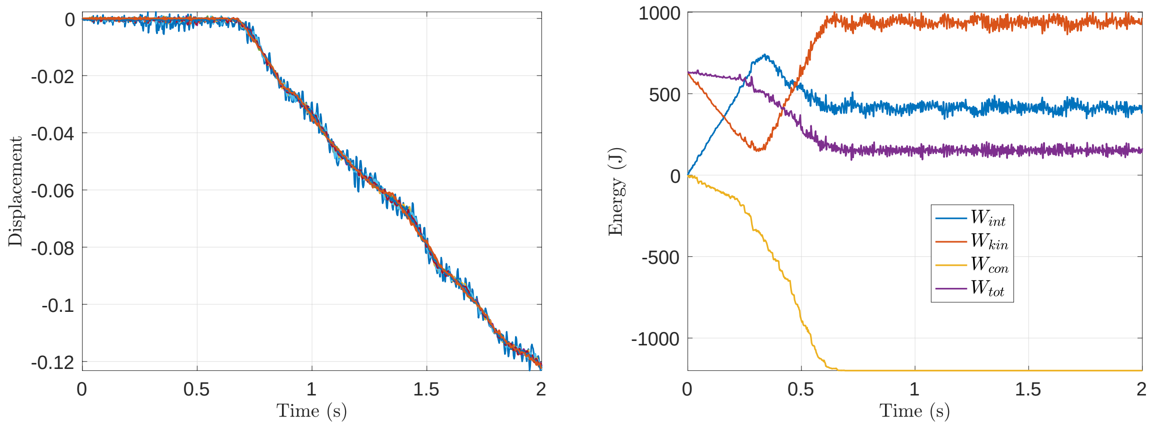

Figure 5: Results for Courant number $C = 0.98\, C_{cr}^{p}$ and stiffness penalty ($r \to \infty$). Horizontal displacement of nodes located in the contact zone (left) and internal, kinetic, contact and total energy of the cylinder (right).

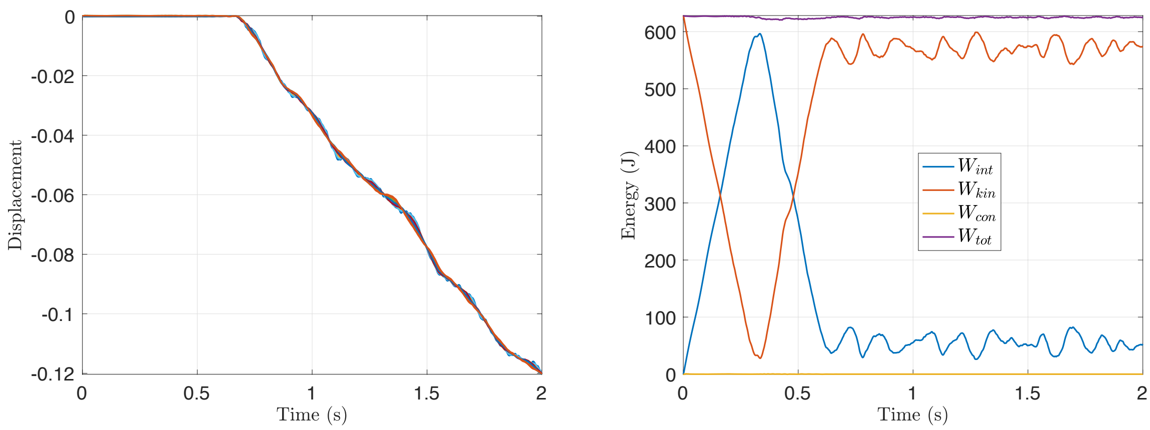

Figure 6: Results for Courant number $C = 0.98\, C_{cr}^{p}$ and bi-penalty parameter $r = 1$. Horizontal displacement of nodes located in the contact zone (left) and internal, kinetic, contact and total energy of the cylinder (right).

5 CONCLUSIONS

A new methodology for direct generation of the inverse mass-matrix for discrete finite element equations of elastodynamics is presented and combined with a bi-penalty contact formulation for impact problems. The methodology is completely element-independent and does not require any special treatment of the elements adjacent to the contact boundaries. In the present method, the equations of motion of the solids are isolated from the contact constraints using localized Lagrange multipliers. This permits to solve the impact problem using a direct inverse mass matrix obtained for the solids, combined with the solution of a small system reduced to the contact interface. This process is found to be computationally efficient, because matrices are sparse and only the factorization of a small matrix associated with the contact DOFs is required.

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