Available online at www.eccomasproceedia.org
Eccomas Proceedia COMPDYN (2021) 28-41

ECCOMAS
Proceedia

COMPDYN 2021
8th ECCOMAS Thematic Conference on
Computational Methods in Structural Dynamics and Earthquake Engineering
M. Papadrakakis, M. Fragiadakis (eds.)
Streamed from Athens, Greece, 28 - 30 June 2021

SOME RECENT ADVANCES IN THE WAVE FINITE ELEMENT METHOD

D. Duhamel[1], B. Claudet[1], T. Hoang[1], G. Foret[1] and J.-M. Mencik[2]

[1]Ecole des Ponts, laboratoire Navier,
6 et 8 Avenue Blaise Pascal, Cité Descartes, Champs sur Marne,
77 455 Marne la Vallée, cedex 2, France
e-mail: {denis.duhamel,benjamin.claudet,tien.hoang,gilles.foret}@enpc.fr

[2] INSA Centre Val de Loire, Université d'Orléans,
Université de Tours, Laboratoire de Mécanique Gabriel Lamé,
Rue de la Chocolaterie, 41000 Blois, France
e-mail: jean-mathieu.mencik@insa-cvl.fr

Abstract. *The wave finite element (WFE) method is now an established numerical method for obtaining the structural response of periodic structures. From a model of a substructure obtained from any finite element software, it allows to get dispersion curves and responses of finite periodic structures with a low calculation cost. Here, we consider some recent improvements of the method. First of all, the original WFE is often formulated with some point loads on the structure, but we show that it is possible to extend this to the consideration of general loads as pressure waves or moving loads for which external loads are applied on each substructure. Second, the classical WFE deals with structures in the frequency domain. It would be interesting to consider the analysis of periodic structures in the time domain, for instance to deal with blast loads. We present here one possibility to do so by computing absorbing boundary conditions in the time domain. By considering supplementary variables at the boundary, a new formulation can be obtained and a classical equation with extended mass, damping and stiffness matrices can be formulated in the time domain and solved by classical algorithms like the Newmark scheme.*

Keywords: Wave Finite Element, Periodic Structures, Structural Dynamics, Distributed Loads, Numerical Methods, Time Response, Absorbing Boundary Conditions.

ISSN:2623-3347
Peer-review under responsibility of the organizing committee of COMPDYN 2021.
doi: 10.7712/120121.8461.18920

1 INTRODUCTION

The wave finite element method (WFE) is often used to predict the dynamic response of periodic structures under harmonic loading like railway tracks, pipelines, ribbed plates, tires, reinforced panels or metamaterials. This consists in computing wave modes (propagation constants, wave shapes) of a periodic structure from the finite element (FE) model of a substructure and its related mass, damping and stiffness matrices which can be obtained from any FE software. Afterwards, these wave modes can be used to calculate the harmonic response of periodic structures in an efficient way, i.e., by computing small matrix systems for one substructure, or a few of them, see for instance the works in [1, 2, 3, 4].

First of all, the original WFE considers mainly some points loads on the structure, but we show here that it is possible to extend this to the consideration of general loads as pressure waves or moving loads for which external loads are applied on each substructure. Several applications of this are possible. At first is the computation of civil engineering structures such as bridges which have extended loads and various displacement and reaction forces at supports, or the computation of structures under moving loads such as railways tracks under the dynamic loads of a train.

Second, it is a matter of fact that the classical WFE deals with structures in the frequency domain. But it would also be interesting to consider the analysis of periodic structures in the time domain, for instance to deal with shocks or blast loads. We present here one possibility to do so by computing absorbing boundary conditions in the time domain. For dispersive media like beams [5], multi-layered systems [6] or general periodic structures [7], absorbing BCs can be formulated in terms of boundary operators involving complicated functions of the frequency — e.g., square roots of the frequency — which, as such, cannot be converted to simple functions of time after inverse Fourier transforms. To solve this issue, the impedance relation at the boundary of a periodic structure is first written in the frequency domain, then this impedance frequency function is decomposed as a rational function for which poles and residues are computed. By considering supplementary variables at the boundary, a formulation in the time domain can be obtained. Finally, a classical equation with extended mass, damping and stiffness matrices can be formulated in the time domain and solved by classical algorithms like the Newmark scheme.

The rest of this paper is organized as follows. In section 2, the main steps of the WFE method are recalled. In section 3, the case of complex and moving loads are considered. Then section 4 is concerned with the formulation of time domain absorbing boundary conditions for periodic structures and the solution of time domain structural dynamics problems. Numerical comparisons with analytical or an equivalent infinite full FE model are presented. Concluding remarks are finally brought in section 5.

2 WFE method

The present paper investigates the dynamic response of infinite 1D periodic structures subject to harmonic or time-dependent loadings. For instance, a schematic of a periodic structure made up of identical substructures is shown in Fig. 1. The substructures under concern can be of arbitrary shape and are supposed to be linear, elastic, isotropic and damped (viscous damping). Also, the excitations are applied on the structure encompassing those substructures which can be infinite or bounded by two left and right boundaries.

The wave propagation analysis in 1D periodic structures can be conducted with the WFE method [2, 4, 8]. The basics of the method are recalled hereafter. Let us consider infinite structures under harmonic disturbance $e^{i\omega t}$ which are built from identical substructures as shown

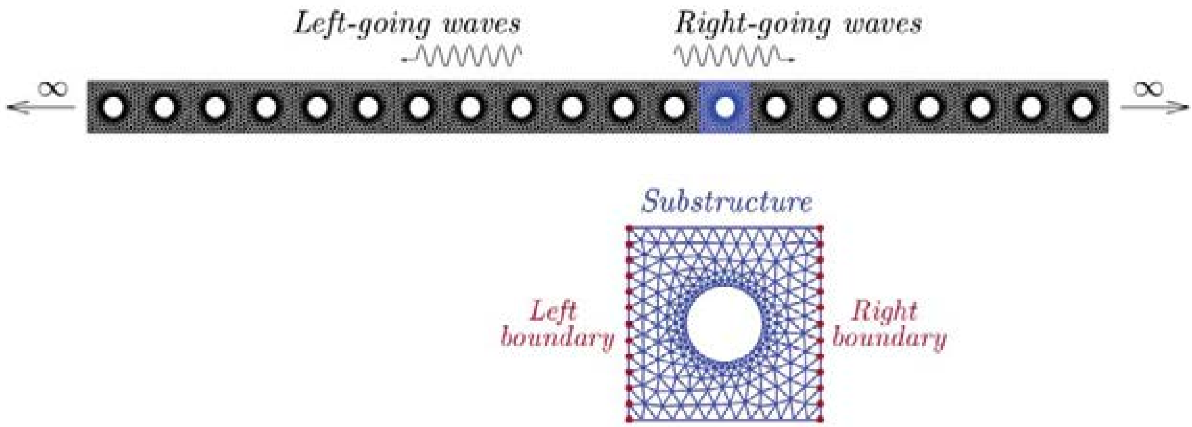

Figure 1: Periodic structure of infinite length, and FE mesh of a substructure.

in Fig. 1, are supposed to share the same FE mesh, and are modelled by means of identical mass, damping and stiffness matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$. The related dynamic equilibrium equation is given by:

$$\begin{bmatrix} \mathbf{D}_{II} & \mathbf{D}_{IL} & \mathbf{D}_{IR} \\ \mathbf{D}_{LI} & \mathbf{D}_{LL} & \mathbf{D}_{LR} \\ \mathbf{D}_{RI} & \mathbf{D}_{RL} & \mathbf{D}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{q}_L \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{F}_I \\ \mathbf{F}_L \\ \mathbf{F}_R \end{bmatrix} \tag{1}$$

where $\mathbf{q}$ and $\mathbf{F}$ refer to the displacement vector and the force vector (respectively), and where $\mathbf{D}$ is the dynamic stiffness matrix of the substructures (similar for all the substructures) expressed by $\mathbf{D} = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$. The FE mesh of a substructure is shown in Fig. 1, and involves internal (I) degrees of freedom (DOFs), left (L) and right (R) boundaries which are described with the same number n of DOFs. If we assume that the internal DOFs are free from excitation, e.g., $\mathbf{F}_I = \mathbf{0}$, we can remove $\mathbf{q}_I$ and rearrange Eq. (1) to yield the following transfer matrix relation between the right and left boundaries of the substructure:

$$\mathbf{u}_R = \mathbf{S}\mathbf{u}_L, \tag{2}$$

where $\mathbf{u}_R$ and $\mathbf{u}_L$ are $2n \times 1$ state vectors expressed by:

$$\mathbf{u}_R = \begin{bmatrix} \mathbf{q}_R \\ \mathbf{F}_R \end{bmatrix} \quad , \quad \mathbf{u}_L = \begin{bmatrix} \mathbf{q}_L \\ -\mathbf{F}_L \end{bmatrix}. \tag{3}$$

Also, $\mathbf{S}$ is a symplectic $2n \times 2n$ matrix (also called transfer matrix) expressed by:

$$\mathbf{S} = \left[\begin{array}{c|c} -(\mathbf{D}^\star_{LR})^{-1}\mathbf{D}^\star_{LL} & -(\mathbf{D}^\star_{LR})^{-1} \\ \hline \mathbf{D}^\star_{RL} - \mathbf{D}^\star_{RR}(\mathbf{D}^\star_{LR})^{-1}\mathbf{D}^\star_{LL} & -\mathbf{D}^\star_{RR}(\mathbf{D}^\star_{LR})^{-1} \end{array} \right], \tag{4}$$

where $\mathbf{D}^\star$ refers to the dynamic stiffness matrix of the substructure condensed on the left and right boundaries [9].

The eigenvalues and eigenvectors of the transfer matrix $\mathbf{S}$ occur in pairs as $(\mu_j, \boldsymbol{\phi}_j)$ and $(\mu^\star_j = 1/\mu_j, \boldsymbol{\phi}^\star_j)$ with $|\mu_j| < 1$ (see [3] for further details about the computation of the eigensolutions of $\mathbf{S}$). Also, the eigenvectors $\boldsymbol{\phi}_j$ (resp. $\boldsymbol{\phi}^\star_j$) have the meaning of wave shapes, for the waves traveling to the right and left directions (respectively) of the periodic structure. Those vectors of wave shapes are of size $2n \times 1$ and are usually partitioned as follows:

$$\boldsymbol{\phi}_j = \begin{bmatrix} \boldsymbol{\phi}_{qj} \\ \boldsymbol{\phi}_{Fj} \end{bmatrix} \quad , \quad \boldsymbol{\phi}^\star_j = \begin{bmatrix} \boldsymbol{\phi}^\star_{qj} \\ \boldsymbol{\phi}^\star_{Fj} \end{bmatrix}, \tag{5}$$

where $\boldsymbol{\phi}_{qj}$ and $\boldsymbol{\phi}^{\star}_{qj}$ (resp. $\boldsymbol{\phi}_{Fj}$ and $\boldsymbol{\phi}^{\star}_{Fj}$) are $n \times 1$ vectors involving displacement (resp. force) components. The related $n \times n$ matrices of wave shapes — namely, $\boldsymbol{\Phi}_q$, $\boldsymbol{\Phi}^{\star}_q$, $\boldsymbol{\Phi}_F$ and $\boldsymbol{\Phi}^{\star}_F$ — are given by:

$$\boldsymbol{\Phi}_q = [\boldsymbol{\phi}_{q1} \cdots \boldsymbol{\phi}_{qn}], \quad \boldsymbol{\Phi}^{\star}_q = [\boldsymbol{\phi}^{\star}_{q1} \cdots \boldsymbol{\phi}^{\star}_{qn}], \quad \boldsymbol{\Phi}_F = [\boldsymbol{\phi}_{F1} \cdots \boldsymbol{\phi}_{Fn}], \quad \boldsymbol{\Phi}^{\star}_F = [\boldsymbol{\phi}^{\star}_{F1} \cdots \boldsymbol{\phi}^{\star}_{Fn}]. \tag{6}$$

A state vector at any substructure interface can then be decomposed on these wave modes by

$$\mathbf{u} = \sum_{j=1}^{j=n} Q_j \boldsymbol{\phi}_j + \sum_{j=1}^{j=n} Q^{\star}_j \boldsymbol{\phi}^{\star}_j \tag{7}$$

where Q_j and $Q^{\star}_j$ are respectively the amplitudes of the right and left propagating waves in the considered section.

3 WFE with complex loads

3.1 Equations

We consider now the case where the structure is under a distributed load that can be applied on each substructure. For loads inside a substructure, relation (1) yields

$$\mathbf{q}_I = \mathbf{D}^{-1}_{II} \left[\mathbf{F}_I - \mathbf{D}_{IL}\mathbf{q}_L - \mathbf{D}_{IR}\mathbf{q}_R \right] \tag{8}$$

and so relation (2) is transformed to a relation linking the state vector in sections n (between substructures $n-1$ and n) and the state vector in section $n+1$ (see [10] for details)

$$\mathbf{u}^{(n+1)} = \mathbf{S}\mathbf{u}^{(n)} + \mathbf{b}^{(n)}, \tag{9}$$

with

$$\mathbf{u}^{(n)} = \begin{bmatrix} \mathbf{q}^{(n)} \\ -\mathbf{F}^{(n)}_L \end{bmatrix}, \quad \mathbf{b}^{(n)} = \begin{bmatrix} \mathbf{D}_{qI}\mathbf{F}_I \\ \mathbf{D}_{fI}\mathbf{F}_I - \mathbf{F}^{ext}_R \end{bmatrix} \tag{10}$$

where $\mathbf{F}^{ext}_R$ is the external load applied in section $n+1$ and

$$\begin{bmatrix} \mathbf{D}_{qI} \\ \mathbf{D}_{fI} \end{bmatrix} = \begin{bmatrix} -(\mathbf{D}^{\star}_{LR})^{-1}\mathbf{D}^{\star}_{LI} \\ \mathbf{D}^{\star}_{RI} - \mathbf{D}^{\star}_{RR}(\mathbf{D}^{\star}_{LR})^{-1}\mathbf{D}^{\star}_{LI} \end{bmatrix} \tag{11}$$

where the superscript * means that the matrix is condensed on the boundary, for instance,

$$\begin{aligned} \mathbf{D}^{\star}_{LR} &= \mathbf{D}_{LR} - \mathbf{D}_{LI}\mathbf{D}^{-1}_{II}\mathbf{D}_{IR} \\ \mathbf{D}^{\star}_{LI} &= \mathbf{D}_{LI}\mathbf{D}^{-1}_{II} \end{aligned} \tag{12}$$

Finally, the state vector in section n is related to the state vectors of sections 1 and $N+1$ by

$$\mathbf{u}^{(n)} = \mathbf{S}^{n-1}\mathbf{u}^{(1)} + \sum_{k=1}^{n-1} \mathbf{S}^{n-k-1}\mathbf{b}^{(k)} \tag{13}$$

or

$$\mathbf{u}^{(N+1)} = \mathbf{S}^{N-n+1}\mathbf{u}^{(n)} + \sum_{k=n}^{N} \mathbf{S}^{N-k}\mathbf{b}^{(k)} \tag{14}$$

where the sum including $\mathbf{b}^{(k)}$ allows to take into account all the loads applied between the two sections. Projecting on the wave modes, the state vector $\mathbf{u}^{(n)}$ in section n and load vector $\mathbf{b}^{(k)}$ in section k can also be written in terms of wave amplitudes by

$$\begin{aligned} \mathbf{u}^{(n)} &= \boldsymbol{\Phi}\mathbf{Q}^{(n)} + \boldsymbol{\Phi}^{\star}\mathbf{Q}^{\star(n)} \\ \mathbf{b}^{(k)} &= \boldsymbol{\Phi}\mathbf{Q}^{(k)}_E + \boldsymbol{\Phi}^{\star}\mathbf{Q}^{\star(k)}_E \end{aligned} \tag{15}$$

3.2 Case of moving loads

An interesting case concerns moving loads. For a moving load at speed v, with a substructure of length L, the amplitude of the load in section (k) is related to the amplitude of the load in section (0) by

$$\begin{aligned} \mathbf{Q}_E^{\star(k)} &= e^{-ik\frac{\omega L}{v}}\mathbf{Q}_E^{\star(0)} \\ \mathbf{Q}_E^{(-k)} &= e^{ik\frac{\omega L}{v}}\mathbf{Q}_E^{(0)} \end{aligned} \tag{16}$$

because the loads on the different substructures are related by $f_n(t) = f_{n-1}(t - L/v)$. From relations (13) and (15), one gets

$$\begin{aligned} \mathbf{u}^{(n)} &= \mathbf{\Phi}\mathbf{Q}^{(n)} + \mathbf{\Phi}^\star\mathbf{Q}^{\star(n)} \\ &= \mathbf{\Phi}\boldsymbol{\mu}^n\left(\mathbf{Q} + \sum_{k=0}^{n-1}\boldsymbol{\mu}^{\star k+1}\mathbf{Q}_E^{(k)}\right) + \mathbf{\Phi}^\star\boldsymbol{\mu}^{-n}\left(\mathbf{Q}^\star + \sum_{k=0}^{n-1}\boldsymbol{\mu}^{k+1}\mathbf{Q}_E^{\star(k)}\right) \end{aligned} \tag{17}$$

with $\boldsymbol{\mu}$ the diagonal matrix made of the propagation constants μ_j on the diagonal and $\mathbf{Q}$, $\mathbf{Q}^\star$ are free wave amplitudes in the structure determined by the boundary conditions. As the precedent relation should be bounded for $n \to \infty$, we get

$$\mathbf{Q}^\star = -\sum_{k=0}^{k=\infty}\boldsymbol{\mu}^{k+1}\mathbf{Q}_E^{\star(k)} = -\frac{\boldsymbol{\mu}}{1 - \boldsymbol{\mu}e^{-i\frac{\omega L}{v}}}\mathbf{Q}_E^{\star(0)} \tag{18}$$

For the left side, we get

$$\mathbf{Q} = \sum_{k=1}^{k=\infty}\boldsymbol{\mu}^{k-1}\mathbf{Q}_E^{(-k)} = \frac{e^{i\frac{\omega L}{v}}}{1 - \boldsymbol{\mu}e^{i\frac{\omega L}{v}}}\mathbf{Q}_E^{(0)} \tag{19}$$

Finally, one gets

$$\mathbf{u}^{(0)} = \frac{e^{i\frac{\omega L}{v}}}{1 - \boldsymbol{\mu}e^{i\frac{\omega L}{v}}}\mathbf{Q}_E^{(0)}\mathbf{\Phi} - \frac{\boldsymbol{\mu}}{1 - \boldsymbol{\mu}e^{-i\frac{\omega L}{v}}}\mathbf{Q}_E^{\star(0)}\mathbf{\Phi}^\star \tag{20}$$

3.3 A simple example

Consider the very simple example of a bar undergoing a longitudinal displacement subject to a moving force

$$\rho\frac{\partial^2 u}{\partial t^2} = Eu'' + f \tag{21}$$

We have $k = \sqrt{\frac{\rho\omega^2}{E}} = \frac{\omega}{c}$, $c = \sqrt{\frac{E}{\rho}}$, $f(x,t) = \delta(x - vt)$ and the wave modes are

$$\Phi = \begin{bmatrix} 1 \\ -ikE \end{bmatrix}, \quad \Phi^\star = \begin{bmatrix} 1 \\ ikE \end{bmatrix} \tag{22}$$

The propagation constant is $\mu = e^{-ikL}$ and the load amplitudes are obtained from the load in the period $[0, L]$. The solution of (21) such that $u(0) = u'(0) = 0$ is given by

$$u(x) = -\int_0^x \frac{1}{kE}\sin(k(x-t))f(t)dt \tag{23}$$

So that for $x = L$ and $f(t) = \frac{1}{v}e^{-i\omega t/v}$, one gets

$$\begin{aligned} u(L) &= -\frac{1}{2ikEv}\int_0^L (e^{ik(L-t)} - e^{-ik(L-t)})e^{-i\omega t/v}dt \\ &= -\frac{1}{2ikEv}\left(\frac{e^{-i\omega L/v} - e^{ikL}}{-ik - i\omega/v} - \frac{e^{-i\omega L/v} - e^{-ikL}}{ik - i\omega/v}\right) \end{aligned} \quad (24)$$

and

$$Eu'(L) = -\frac{1}{2ikv}\left(\frac{\frac{-i\omega}{v}e^{-i\omega L/v} - ike^{ikL}}{-ik - i\omega/v} - \frac{\frac{-i\omega}{v}e^{-i\omega L/v} + ike^{-ikL}}{ik - i\omega/v}\right) \quad (25)$$

Using relation (9) and the solution of (21) such that $\mathbf{b} = {}^T[u(L) \;\; Eu'(L)]$, the load amplitudes are obtained by decomposing the solution at $x = L$ on the wave modes by

$$\begin{aligned} Q_E^{(0)} &= \frac{{}^T\Phi^\star \mathbf{J}\mathbf{b}}{{}^T\Phi^\star \mathbf{J}\Phi} \\ &= -\frac{e^{-i\omega\frac{L}{v}} - e^{-ikL}}{2kEv(k - \frac{\omega}{v})} \\ Q_E^{\star(0)} &= \frac{{}^T\Phi\mathbf{J}\mathbf{b}}{{}^T\Phi\mathbf{J}\Phi^\star} \\ &= -\frac{e^{-i\omega\frac{L}{v}} - e^{ikL}}{2kEv(k + \frac{\omega}{v})} \end{aligned} \quad (26)$$

with

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (27)$$

The amplitudes of the right-going and left-going waves are thus given by, using (20),

$$\begin{aligned} Q &= -\frac{1}{2kEv(k - \frac{\omega}{v})} \\ Q^\star &= -\frac{1}{2kEv(k + \frac{\omega}{v})} \end{aligned} \quad (28)$$

and we recover the displacement at $x = 0$ by

$$\begin{aligned} u &= -\frac{1}{2kEv(k - \frac{\omega}{v})} - \frac{1}{2kEv(k + \frac{\omega}{v})} \\ &= -\frac{1}{Ev(k^2 - \frac{\omega^2}{v^2})} \\ &= -\frac{1}{Ev\omega^2(\frac{1}{c^2} - \frac{1}{v^2})} \end{aligned} \quad (29)$$

which is the analytical solution of (21). Following the same procedure, this can be extended to more complex structures with many DOFs in each section.

4 Absorbing boundary conditions in the time domain

4.1 WFE formulation

Consider now a periodic structure involving a finite number N of substructures which is enclosed between two left and right boundaries S_L and S_R where absorbing BCs are considered (see Fig. 2). Such BCs are used to describe the coupling effect between the periodic structure and two semi-infinite periodic structures, that would expand to the left and right directions. The related impedance matrices for the left and right boundaries S_L and S_R — namely, $\mathbf{Z}_L$ and $\mathbf{Z}_R$ — can be defined, in the frequency domain, as follows:

$$\mathbf{F}_L = \mathbf{Z}_L \mathbf{q}_L \quad , \quad \mathbf{F}_R = \mathbf{Z}_R \mathbf{q}_R, \tag{30}$$

where $\mathbf{q}_L$ and $\mathbf{q}_R$ (resp. $\mathbf{F}_L$ and $\mathbf{F}_R$) are the displacement (resp. force) vectors (size $n \times 1$) for the periodic structure on S_L and S_R. Following the WFE procedure and expanding those vectors on the basis of wave shapes, this yields [9]:

$$\mathbf{q}_L = \mathbf{\Phi}_q^\star \mathbf{Q}_L^\star \quad , \quad \mathbf{q}_R = \mathbf{\Phi}_q \mathbf{Q}_R, \tag{31}$$

and

$$\mathbf{F}_L = -\mathbf{\Phi}_F^\star \mathbf{Q}_L^\star \quad , \quad \mathbf{F}_R = \mathbf{\Phi}_F \mathbf{Q}_R, \tag{32}$$

where $\mathbf{Q}_L^\star$ (resp. $\mathbf{Q}_R$) is the vector of wave amplitudes, at position S_L (resp. S_R), for the waves traveling to the left (resp. right) direction of the structure. The fact that left-going (resp. right-going) waves are only considered at S_L (resp. S_R) results from the absorbing BC, i.e., the fact that no wave comes from infinity. So we get the following expressions for the impedance matrices:

$$\mathbf{Z}_L = -\mathbf{\Phi}_F^\star (\mathbf{\Phi}_q^\star)^{-1} \quad , \quad \mathbf{Z}_R = \mathbf{\Phi}_F (\mathbf{\Phi}_q)^{-1}. \tag{33}$$

Let us decompose the impedance matrices $\mathbf{Z}_L$ and $\mathbf{Z}_R$ which are complicated functions of frequency that do not generally possess an analytical expression, via rational approximations like in [7]:

$$\mathbf{Z}_L = \sum_{k=1}^{P} \frac{\mathbf{R}_{Lk}}{i\omega - p_{Lk}} + \mathbf{K}_L \quad , \quad \mathbf{Z}_R = \sum_{k=1}^{P} \frac{\mathbf{R}_{Rk}}{i\omega - p_{Rk}} + \mathbf{K}_R, \tag{34}$$

where p_{Lk} and p_{Rk} denote poles ($k = 1, \ldots, P$), and $\mathbf{R}_{Lk}$ and $\mathbf{R}_{Rk}$ denote matrices of residues. Some of these usually appear in conjugate pairs, i.e., $(p_{Lk}, \overline{p_{Lk}})$ and $(p_{Rk}, \overline{p_{Rk}})$, and $(\mathbf{R}_{Lk}, \overline{\mathbf{R}_{Lk}})$ and $(\mathbf{R}_{Rk}, \overline{\mathbf{R}_{Rk}})$. As a result, Eq. (34) leads to:

$$\mathbf{Z}_L = \sum_{k=1}^{Q} 2 \frac{i\omega \Re\{\mathbf{R}_{L(2k)}\} - \Re\{\overline{p_{L(2k)}} \mathbf{R}_{L(2k)}\}}{-\omega^2 - 2i\omega \Re\{p_{L(2k)}\} + |p_{L(2k)}|^2} + \sum_{k=2Q+1}^{P} \frac{\mathbf{R}_{Lk}}{i\omega - p_{Lk}} + \mathbf{K}_L \tag{35}$$

where $2Q < P$. Only the left impedance is described as the right impedance follows similar expression by replacing the subscript L by R. To remove the denominator terms in Eq. (35), let us introduce $n \times 1$ vectors of supplementary variables $\mathbf{X}_{Lk}$ and $\mathbf{X}_{Rk}$, and let us rewrite Eq. (30) by means of Eq. (35) as follows:

$$\mathbf{F}_L = \sum_{k=1}^{Q} 2 \left(i\omega \Re\{\mathbf{R}_{L(2k)}\} - \Re\{\overline{p_{L(2k)}} \mathbf{R}_{L(2k)}\right) \mathbf{X}_{Lk} + \sum_{k=2Q+1}^{P} \mathbf{R}_{Lk} (i\omega) \mathbf{X}_{L(k-Q)} + \mathbf{K}_L \mathbf{q}_L \tag{36}$$

where:

$$\begin{array}{rcll} \left(-\omega^2 - 2i\omega\Re\{p_{L(2k)}\} + |p_{L(2k)}|^2\right)\mathbf{X}_{Lk} & = & \mathbf{q}_L & \text{for } k = 1, \ldots, Q, \\ (-\omega^2 - i\omega p_{Lk})\mathbf{X}_{L(k-Q)} & = & \mathbf{q}_L & \text{for } k = (2Q+1), \ldots, P, \end{array} \quad (37)$$

Finally, let us introduce the following $(P-Q)n \times 1$ vector $\mathbf{X}_L$ defined by:

$$\mathbf{X}_L = \left[\begin{array}{c} \mathbf{X}_{L1} \\ \vdots \\ \mathbf{X}_{LQ} \\ \hline \mathbf{X}_{L(Q+1)} \\ \vdots \\ \mathbf{X}_{L(P-Q)} \end{array}\right]. \quad (38)$$

The block components of the matrices occurring in Eqs. (36) and (37) represent polynomials of $i\omega$ of orders 0, 1 or 2, which as such can be simply and quickly converted to the time domain (see hereafter). By separating the terms of identical powers of $i\omega$, and by invoking the classical time-frequency transforms $\mathbf{q}(\omega) \to \mathbf{q}(t)$, $i\omega\mathbf{q} \to \dot{\mathbf{q}}$, $-\omega^2\mathbf{q} \to \ddot{\mathbf{q}}$ and $\mathbf{X}(\omega) \to \mathbf{X}(t)$, $i\omega\mathbf{X} \to \dot{\mathbf{X}}$, $-\omega^2\mathbf{X} \to \ddot{\mathbf{X}}$ (where dot and double-dot notations mean single and double time derivatives, respectively), this yields:

$$\mathbb{M}_L \begin{bmatrix} \ddot{\mathbf{q}}_L \\ \ddot{\mathbf{X}}_L \end{bmatrix} + \mathbb{C}_L \begin{bmatrix} \dot{\mathbf{q}}_L \\ \dot{\mathbf{X}}_L \end{bmatrix} + \mathbb{K}_L \begin{bmatrix} \mathbf{q}_L \\ \mathbf{X}_L \end{bmatrix} = \begin{bmatrix} \mathbf{F}_L \\ \mathbf{0} \end{bmatrix} \quad (39)$$

where:

$$\begin{array}{rcl} \mathbb{M}_L & = & \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \texttt{blkdiag}\left\{\mathbf{I}_n\right\}_{k=1}^{Q} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \texttt{blkdiag}\left\{\mathbf{I}_n\right\}_{k=2Q+1}^{P} \end{array}\right], \\ \mathbb{C}_L & = & \left[\begin{array}{c|c|c} \mathbf{0} & 2\Re\left\{\left[\mathbf{R}_{L(2)} \cdots \mathbf{R}_{L(2Q)}\right]\right\} & \left[\mathbf{R}_{L(2Q+1)} \cdots \mathbf{R}_{LP}\right] \\ \hline \mathbf{0} & \texttt{blkdiag}\left\{-2\Re\{p_{L(2k)}\}\mathbf{I}_n\right\}_{k=1}^{Q} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \texttt{blkdiag}\left\{-p_{Lk}\mathbf{I}_n\right\}_{k=2Q+1}^{P} \end{array}\right], \\ \mathbb{K}_L & = & \left[\begin{array}{c|c|c} \mathbf{K}_L & -2\Re\left\{\left[\overline{p_{L(2)}}\mathbf{R}_{L(2)} \cdots \overline{p_{L(2Q)}}\mathbf{R}_{L(2Q)}\right]\right\} & \mathbf{0} \\ \hline -\mathbb{1}_{Q\times 1} \otimes \mathbf{I}_n & \texttt{blkdiag}\left\{|p_{L(2k)}|^2\mathbf{I}_n\right\}_{k=1}^{Q} & \mathbf{0} \\ \hline -\mathbb{1}_{(P-2Q)\times 1} \otimes \mathbf{I}_n & \mathbf{0} & \mathbf{0} \end{array}\right], \end{array} \quad (40)$$

where $\otimes$ denotes the Kronecker product.

Let us denote by $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ the mass, damping and stiffness matrices of the periodic structure (N substructures), and let us write the related equation of motion as follows:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}, \quad (41)$$

where $\mathbf{q} = \mathbf{q}(t)$ and $\mathbf{F} = \mathbf{F}(t)$ are the displacement and force vectors, respectively. In this case, the force vector is expressed by $\mathbf{F} = [\mathbf{F}_I^T \mathbf{F}_L^T \mathbf{F}_R^T]$ where $\mathbf{F}_L$ and $\mathbf{F}_R$ refer to the force vectors on S_L and S_R (absorbing BCs), and $\mathbf{F}_I$ refers the force vector for the internal DOFs (I) of the structure. Also, the displacement vector is expressed by $\mathbf{q} = [\mathbf{q}_I^T \mathbf{q}_L^T \mathbf{q}_R^T]$ where $\mathbf{q}_I$ is the displacement vector for the internal DOFs. By considering the absorbing BCs (Eq. (39)), this yields:

$$\mathbf{M}_{\mathtt{tot}}\ddot{\mathbf{y}} + \mathbf{C}_{\mathtt{tot}}\dot{\mathbf{y}} + \mathbf{K}_{\mathtt{tot}}\mathbf{y} = \mathbf{f}. \tag{42}$$

with

$$\mathbf{y} = \begin{bmatrix} \mathbf{q}_I \\ \mathbf{q}_L \\ \mathbf{q}_R \\ \mathbf{X}_L \\ \mathbf{X}_R \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \mathbf{F}_I \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{43}$$

In (42), the matrices $\mathbf{M}_{\mathtt{tot}}$, $\mathbf{C}_{\mathtt{tot}}$ and $\mathbf{K}_{\mathtt{tot}}$ are given by:

$$\begin{aligned}
\mathbf{M}_{\mathtt{tot}} &= \begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{IL} & \mathbf{M}_{IR} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{LI} & \mathbf{M}_{LL} & \mathbf{M}_{LR} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{RI} & \mathbf{M}_{RL} & \mathbf{M}_{RR} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{M}_{L(XX)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{M}_{R(XX))} \end{bmatrix}, \\
\mathbf{C}_{\mathtt{tot}} &= \begin{bmatrix} \mathbf{C}_{II} & \mathbf{C}_{IL} & \mathbf{C}_{IR} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{LI} & \mathbf{C}_{LL} & \mathbf{C}_{LR} & -\mathbb{C}_{L(qX)} & \mathbf{0} \\ \mathbf{C}_{RI} & \mathbf{C}_{RL} & \mathbf{C}_{RR} & \mathbf{0} & -\mathbb{C}_{R(qX)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{C}_{L(XX)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{C}_{R(XX))} \end{bmatrix}, \\
\mathbf{K}_{\mathtt{tot}} &= \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IL} & \mathbf{K}_{IR} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{LI} & \mathbf{K}_{LL} - \mathbb{K}_{L(qq)} & \mathbf{K}_{LR} & -\mathbb{K}_{L(qX)} & \mathbf{0} \\ \mathbf{K}_{RI} & \mathbf{K}_{RL} & \mathbf{K}_{RR} - \mathbb{K}_{R(qq)} & \mathbf{0} & -\mathbb{K}_{R(qX)} \\ \mathbf{0} & \mathbb{K}_{L(Xq)} & \mathbf{0} & \mathbb{K}_{L(XX)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{K}_{R(Xq)} & \mathbf{0} & \mathbb{K}_{R(XX)} \end{bmatrix},
\end{aligned} \tag{44}$$

where:

$$
\begin{aligned}
\mathbb{M}_{L(XX)} &= \left[\begin{array}{c|c} \texttt{blkdiag}\{\mathbf{I}_n\}_{k=1}^{Q} & \mathbf{0} \\ \hline \mathbf{0} & \texttt{blkdiag}\{\mathbf{I}_n\}_{k=2Q+1}^{P} \end{array}\right], \\
\mathbb{C}_{L(qX)} &= \left[\ 2\Re\left\{\left[\mathbf{R}_{L(2)} \cdots \mathbf{R}_{L(2Q)}\right]\right\} \,\middle|\, \left[\mathbf{R}_{L(2Q+1)} \cdots \mathbf{R}_{LN}\right]\ \right], \\
\mathbb{C}_{L(XX)} &= \left[\begin{array}{c|c} \texttt{blkdiag}\left\{-2\Re\{p_{L(2k)}\}\mathbf{I}_n\right\}_{k=1}^{Q} & \mathbf{0} \\ \hline \mathbf{0} & \texttt{blkdiag}\left\{-p_{Lk}\mathbf{I}_n\right\}_{k=2Q+1}^{P} \end{array}\right], \\
\mathbb{K}_{L(qq)} &= \mathbf{K}_L, \\
\mathbb{K}_{L(qX)} &= \left[\ -2\Re\left\{\left[\overline{p_{L(2)}}\mathbf{R}_{L(2)} \cdots \overline{p_{L(2Q)}}\mathbf{R}_{L(2Q)}\right]\right\} \,\middle|\, \mathbf{0}\ \right], \\
\mathbb{K}_{L(Xq)} &= \left[\begin{array}{c} -\mathbb{1}_{Q\times 1} \otimes \mathbf{I}_n \\ \hline -\mathbb{1}_{(P-2Q)\times 1} \otimes \mathbf{I}_n \end{array}\right], \\
\mathbb{K}_{L(XX)} &= \left[\begin{array}{c|c} \texttt{blkdiag}\left\{|p_{L(2k)}|^2\mathbf{I}_n\right\}_{k=1}^{Q} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right],
\end{aligned}
\tag{45}
$$

and similar expressions for the right boundary.

Eq. (42) represents a second-order differential matrix equation for the displacement vector $\mathbf{q}$ and the vector of supplementary variables $\mathbf{X} = [\mathbf{X}_L^T \mathbf{X}_R^T]^T$. This indeed represents a classical dynamic equation, in the time domain, of a structure with absorbing BCs and subject to an input force vector $\mathbf{F}_I = \mathbf{F}_I(t)$, with the only modification that supplementary DOFs are added at the boundaries. Therefore, this equation can be solved in a standard way via a time integration numerical scheme (e.g., Newmark scheme).

4.2 Euler-Bernoulli beam on an elastic foundation

We consider the example of the dynamic response of an infinite Euler-Bernoulli beam lying on an elastic foundation as shown in Fig. 2 and subject to some forces $f(x,t)$. The governing equation of motion of the beam is given by:

$$
\rho S\ddot{v} + EI\left(\frac{\partial^4 v}{\partial x^4} + \xi\frac{\partial^4 \dot{v}}{\partial x^4}\right) + k_F v = f(x,t), \tag{46}
$$

where $v = v(x,t)$ represents the transverse displacement, ρ is the density, S is the cross-sectional area, E is the Young's modulus, I is the inertia moment, and ξ is a damping parameter. For harmonic disturbance of the form $f(x)e^{i\omega t}$, Eq. (46) leads to:

$$
(-\rho S\omega^2 + k_F)v + EI(1 + i\omega\xi)\frac{\partial^4 v}{\partial x^4} = f(x), \tag{47}
$$

For this simple case, there exist analytical expressions of the matrices of wave shapes $\mathbf{\Phi}_q$, $\mathbf{\Phi}_q^\star$, $\mathbf{\Phi}_F$ and $\mathbf{\Phi}_F^\star$, see Eq. (6). Hence, by expressing the transverse displacement v together with the rotation $\theta = \partial v/\partial x$, this yields:

$$
\begin{bmatrix} v \\ \theta \end{bmatrix} = \mathbf{\Phi}_q \begin{bmatrix} e^{-ikx} & 0 \\ 0 & e^{-kx} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} + \mathbf{\Phi}_q^\star \begin{bmatrix} e^{ikx} & 0 \\ 0 & e^{kx} \end{bmatrix} \begin{bmatrix} Q_1^\star \\ Q_2^\star \end{bmatrix}, \tag{48}
$$

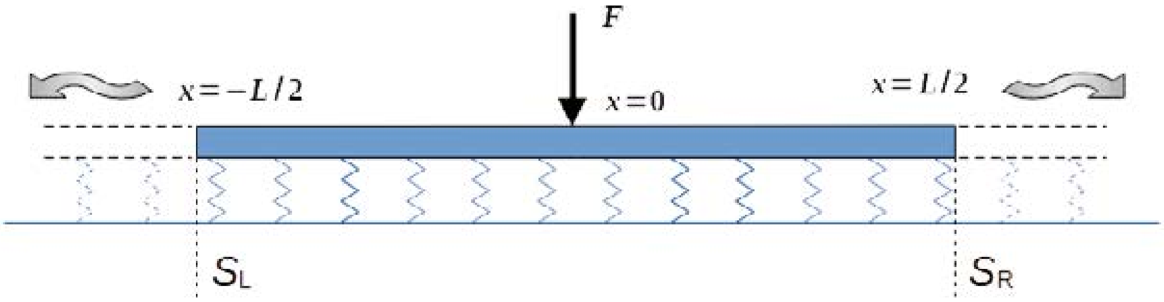

Figure 2: Infinite flexural beam on an elastic foundation.

where:

$$\mathbf{\Phi}_q = \begin{bmatrix} 1 & 1 \\ -ik & -k \end{bmatrix} \quad , \quad \mathbf{\Phi}_q^\star = \begin{bmatrix} 1 & 1 \\ ik & k \end{bmatrix} \quad , \quad k = \left(\frac{\rho S \omega^2 - k_F}{EI(1+i\omega\xi)} \right)^{1/4} . \tag{49}$$

Also, by expressing the shearing force $V = -EI(\partial^3 v/\partial x^3)$ and the bending moment $M = EI(\partial^2 v/\partial x^2)$, this yields:

$$\mathbf{\Phi}_F = EIk^2 \begin{bmatrix} -ik & k \\ -1 & 1 \end{bmatrix} \quad , \quad \mathbf{\Phi}_F^\star = EIk^2 \begin{bmatrix} ik & -k \\ -1 & 1 \end{bmatrix} . \tag{50}$$

Therefore, the impedance matrices are written as:

$$\mathbf{Z}_L = -\mathbf{\Phi}_F^\star (\mathbf{\Phi}_q^\star)^{-1} = -\frac{EIk^2}{1-i} \begin{bmatrix} 2ik & -(1+i) \\ -(1+i) & \frac{2}{k} \end{bmatrix} , \tag{51}$$

and

$$\mathbf{Z}_R = \mathbf{\Phi}_F (\mathbf{\Phi}_q)^{-1} = -\frac{EIk^2}{1-i} \begin{bmatrix} 2ik & 1+i \\ 1+i & \frac{2}{k} \end{bmatrix} . \tag{52}$$

These are analytical expressions of the impedance matrices that could also have been obtained numerically by means of the WFE method.

Consider an infinite beam of rectangular cross-section having the following parameters: height $h = 0.001$ m, width $b = 0.01$ m, Young's modulus $E = 2.2 \times 10^{11}$ Pa, density $\rho = 7800$ kg/m^3, damping parameter $\xi = 0.001$ s. Also, the lineic stiffness of the elastic foundation is $k_F = 1$ N/m^2. The system is at rest at time $t = 0$ — i.e., $v(x,0) = 0$ and $\dot{v}(x,0) = 0$ — and, for $t \geq 0$, it is excited by a point harmonic force of frequency $f_0 = 5$ Hz (at $x = 0$):

$$f(x,t) = \cos(2\pi f_0 t)\delta(x) \qquad \text{for } t \geq 0. \tag{53}$$

A beam of finite length L — i.e., $x \in [-L/2, L/2]$ where, for instance, $L = 5$ m — excited at $x = 0$ (Eq. (53)) is considered as shown in Fig. 2. Here, the system beam-foundation is modeled by means of 500 identical substructures that represent identical two-node Hermitian beam elements of length $d = 0.01$ m. The rational approximations of the impedance matrices $\mathbf{Z}_L$ and $\mathbf{Z}_R$ with $P = 12$ poles/residues (see Eqs. (34)) are computed with the MATLAB rationalfit function.

The differential matrix equation (42) is solved with the Newmark algorithm where $\Delta t = 0.01$ s, $\mathbf{y}^0 = \mathbf{0}$ and $\dot{\mathbf{y}}^0 = \mathbf{0}$. The related transverse displacement field, at $t = 20$ s, is shown in the left of Fig. 3 along with the analytical harmonic solution. In this case, the proposed solution closely matches the analytical one, as expected. Finally, the right of Fig. 3 shows the history of the displacement solution at position $x = L/2$. Again, it is seen that, after a certain time (transient period), the solution issued from the proposed approach stabilizes towards the harmonic solution.

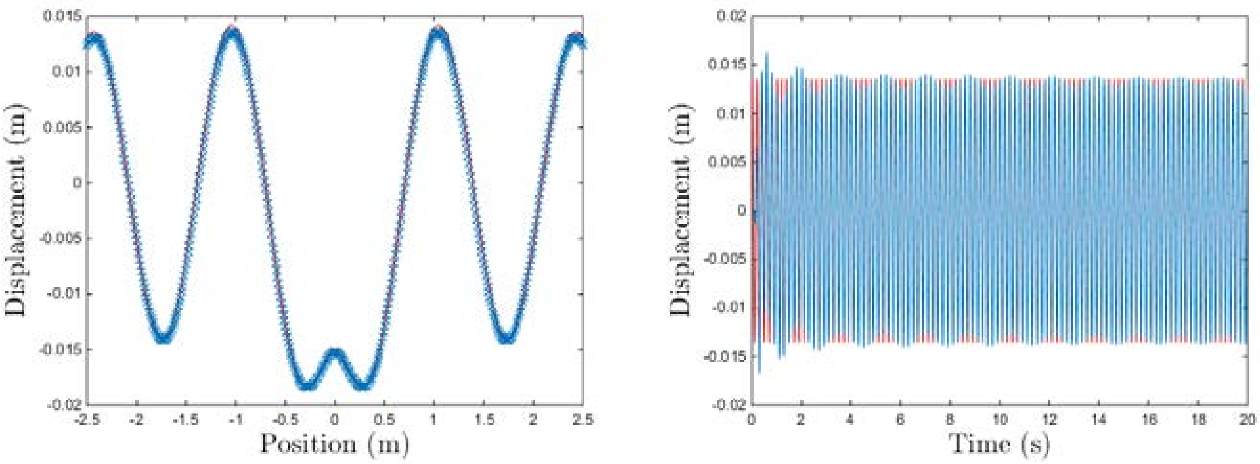

Figure 3: Transverse displacement field of the beam on the elastic foundation at $t = 20$ s (left): (blue crosses) Proposed approach; (red line) analytical harmonic solution and time response at x=L/2 for $t \in [0\,,\,20]$ s (right) (blue line and crosses) Proposed approach; (red line) analytical theory, harmonic response.

4.3 Periodic structure with 2D substructures

We consider now a 2D beam with periodic distributions of holes and elastic supports (springs of stiffness K_s) as shown in Fig. 4. Here, square substructures of dimensions 2×2 m^2 with holes of radius 0.4 m are considered which are similar to those depicted in Fig. 1. Regarding the modeling of the periodic supports, a nodal stiffness of $K_\mathrm{s}/2$ (vertical direction) is added to the FE model of the substructures at the left and right boundaries (bottom node). Other substructure parameters are: thickness $e = 0.005$ m, Young's modulus $E = 7 \times 10^{10}$ Pa, Poisson's ratio $\nu = 0.35$, density $\rho = 2700$ kg/m^3, and stiffness $K_\mathrm{s} = 10^5$ N/m. Rayleigh-type damping matrices $\mathbf{C} = a\mathbf{M} + b\mathbf{K}$ are also considered where $a = 0.01$ s^{-1} and $b = 5 \times 10^{-5}$ s.

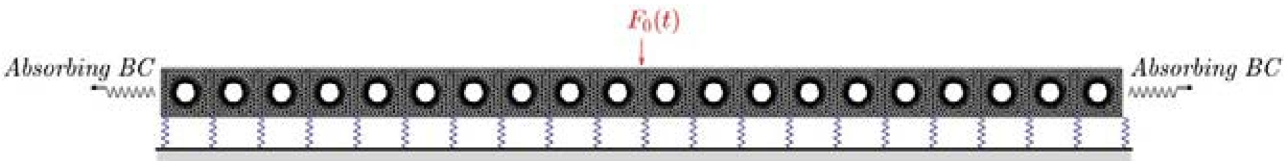

Figure 4: Schematic of an infinite periodic structure with periodic elastic supports.

The time response of the infinite periodic structure subject to a vertical point force $F_0(t)$ at $x = 0$ (top node) is analyzed. Within the framework of the proposed approach, a periodic structure involving $N = 20$ substructures and absorbing BCs is considered as shown in Fig. 4. In this case, the structure has a length of $L = 40$ m. The rational approximations of the impedance matrices $\mathbf{Z}_L$ and $\mathbf{Z}_R$ (Eq. (34)) are expressed by means of $P = 15$ poles/residues. The time response of the structure is computed over a time range of $[0\,,\,0.1]$ s by solving the differential matrix equation (42) with the Newmark algorithm where $\Delta t = 10^{-4}$ s, $\mathbf{y}^0 = \mathbf{0}$ and $\dot{\mathbf{y}}^0 = \mathbf{0}$. For comparison purpose, an equivalent FE model of an "infinite" structure with a larger number of substructures (200) is considered and simulated over the time range $[0\,,\,0.1]$ s which is supposed to be small enough to prevent wave reflections (free boundaries). Consider a harmonic point force of magnitude $F_0(t) = 10^4 \cos(2\pi \times 100t)$ acting at $x = 0$, and assume that the structure is at rest at $t = 0$. The time response is analyzed over a time range of $[0\,,\,0.1]$ s which is supposed to be broad enough to include several oscillations (10 in this case) and cover the transient phase. Especially, the time variation of the transverse displacement at $x = 20$ m (right boundary, top node) can be computed as shown in Fig. 5. It is shown that the proposed solution perfectly matches the reference one over the whole time range. It is also numerically stable, i.e., a smooth curve that well predicts the oscillating nature of the signal.

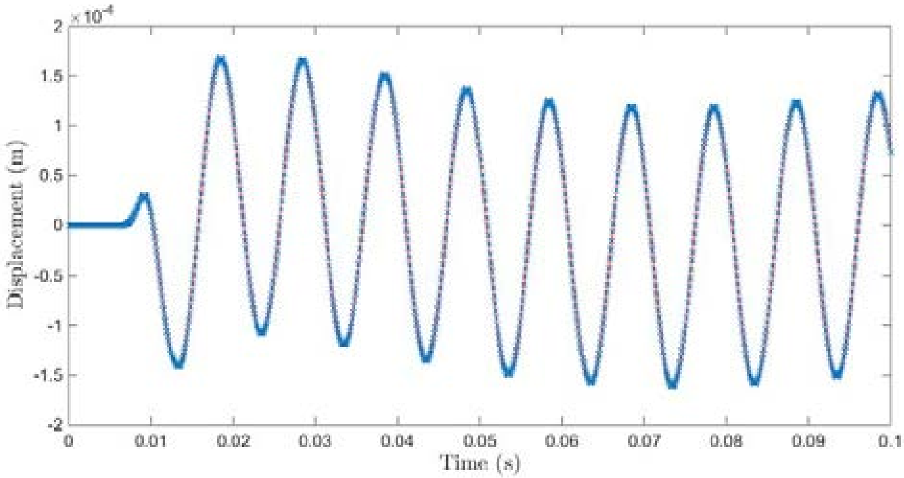

Figure 5: Harmonic force: time response at x=L/2 (top node, vertical displacement). (blue crosses) Proposed approach; (red line) reference FE method.

5 Conclusion

A FE procedure has been proposed to model infinite periodic structures subject to distributed or moving loads, or localized time-dependent excitations. For instance, in the case of moving loads, the load is applied on each substructure. Concerning time domain problems, first, using the WFE method to express the absorbing BCs by means of impedance matrices, in the frequency domain, impedance matrices have been rewritten in terms of polynomials of the frequency $i\omega$ up to order 2. Then, they are simply converted to the time domain and the global dynamic equation is solved by the Newmark algorithm. Follow-on works could include the analysis of infinite periodic structures with localized nonlinear effects.

REFERENCES

[1] B. Mace, D. Duhamel, M. Brennan, L. Hinke, Finite element prediction of wave motion in structural waveguides, Journal of the Acoustical Society of America 117 (2005) 2835–2843.

[2] D. Duhamel, B. R. Mace, M. J. Brennan, Finite element analysis of the vibrations of waveguides and periodic structures, Journal of Sound and Vibration 294 (1-2) (2006) 205–220.

[3] J.-M. Mencik, D. Duhamel, A wave-based model reduction technique for the description of the dynamic behavior of periodic structures involving arbitrary-shaped substructures and large-sized finite element models, Finite Elements in Analysis and Design 101 (2015) 1–14.

[4] J.-M. Mencik, M. N. Ichchou, Multi-mode propagation and diffusion in structures through finite elements, European Journal of Mechanics - A/Solids 24 (5) (2005) 877–898.

[5] P. Ruge, C. Birk, A comparison of infinite Timoshenko and Euler-Bernoulli beam models on Winkler foundation in the frequency- and time-domain, Journal of Sound and Vibration 304 (3) (2007) 932 – 947.

[6] M. Zhao, L. Wu, X. Du, Z. Zhong, C. Xu, L. Li, Stable high-order absorbing boundary condition based on new continued fraction for scalar wave propagation in unbounded multilayer media, Computer Methods in Applied Mechanics and Engineering 334 (2018) 111 – 137.

[7] D. Duhamel, J. Mencik, Time response analysis of periodic structures via wave-based absorbing boundary conditions, European Journal of Mechanics - A/Solids (submitted).

[8] W. X. Zhong, F. W. Williams, On the direct solution of wave propagation for repetitive structures, Journal of Sound and Vibration 181 (3) (1995) 485–501.

[9] J.-M. Mencik, New advances in the forced response computation of periodic structures using the wave finite element (WFE) method, Computational Mechanics 54 (3) (2014) 789–801.

[10] T. Hoang, D. Duhamel, G. Foret, Wave finite element method for waveguides and periodic structures subjected to arbitrary loads, Finite Elements in Analysis and Design 179 (2020) 103437.