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IMPACT FACTORS DERIVED FROM INTERNAL FORCES IN TIMOSHENKO BEAMS SUBJECTED TO CONCENTRATED MOVING LOADS

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Abstract. *The aim of this work is to analyse the impact factors in the Timoshenko beams response under moving loads in terms of both displacements and internal forces, particularly bending moments. The dynamic response is determined by modal analysis, which consist in representing the absolute displacement field by means of a weighted combination of the undamped system eigenfunctions. This paper presents a finite-element based approach that can be used to compute the dynamic internal forces, both in Bernoulli–Euler and Timoshenko beams models under moving loads. Firstly, this methodology has been validated against an analytical solution for a single-span bridge. Afterwards, the impact factors in a real four-span bridge are presented with a discussion of several issues which come up with the resolution of this kind of problems, such as the damping level and the difference between positive and negative bending moments. Finally, conclusions about the similarities in the dynamic amplification factors for different magnitudes are discussed, including one case of visible resonant behaviour.*

Keywords: Structural Dynamics, Moving Loads, Impact Factors, Continuous Beams.

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1 INTRODUCTION

The idea of substituting the dynamic analysis of structures under moving loads by static analyses where the results are amplified by an *impact factor* is as old as the moving load problem itself. Certainly, static analysis is more amenable to practical application by engineers who may be specialists in various aspects of structural analysis and design, but with no particular background in dynamics. Moreover, before the speeds of trains were high enough to raise concerns about resonance problems, carrying out full dynamic analyses with the sole purpose of obtaining a certain percentage of increase in structural displacements and internal forces was not justified: a simple amplification coefficient is generally perceived as sufficient in that case, also because vibrations levels (accelerations) are not of importance. With the advent of high-speeds railways, the perspective has changed noticeably.

Presently this issue is tackled in some building codes in a way that leaves margin for interpretation. We consider EN 1991-2 [1] as a reference standard for railway bridge design. Under certain conditions, an impact coefficient or *dynamic enhancement* φ' is to be calculated according to EN 1991-2, as a function of the "maximum dynamic response" and "maximum static response", where the meaning of the term "response" is not unambiguous. If the designer is to provide a solution that complies with the standard employing the least effort, "response" can be interpreted as "displacement"—particularly, "vertical displacement" when the deformation is characterised by such type of motion, as it is the case during bending of beam-like structures.

Therefore, the question can be set forth as follows: is it possible to obtain a good approximation to the dynamic enhancement of internal forces (bending moments, shear force, etc.) by using the dynamic enhancement corresponding to displacements?

Since the computation of dynamic displacements is simpler than for internal forces, an affirmative answer to this question would have certain advantages: a coarser mesh could be used and, principally, less modes are needed for a good convergence to the dynamic displacements than to internal forces. The difficulties of convergence can be circumvented by using solution methods that combine *quasi-static* analysis plus a *dynamic increment* with respect to the quasi-static solution, as González *et al.* employed for the shear forces in [2]. However, this approach is not usually available in commercial Finite Element (FEM) codes. Additionally, less information is generated and manipulated if only displacements are postprocessed to obtain the impact factors, which results in faster analyses and reduced possibilities of unnoticed errors.

For these reasons, this paper explores the possibility of approximating impact coefficients for internal forces by means of displacement-based coefficients. The problem is addressed for bending moments in continuous beams, which is a structural type of great interest in bridge engineering. Timoshenko beam theory is used as a basis. While this theory is not expected to produce results significantly different from Bernoulli–Euler (B-E) theory for maximum resonant response of railway bridges —due to the usual slenderness of such structures—, there are other applications of interest where Timoshenko beam theory could be required, as for instance Hyperloop^{TM} tubes of large diameter and small thickness, recently analysed by the authors under the main expected design loads [3].

The dynamic response of continuous beams is determined here by modal analysis. The paper presents a systematic, finite-element based approach to compute the dynamic internal forces, both in B-E and Timoshenko beams models under moving loads. In section 2 the closed-form solution is presented for the vertical displacements and rotations according to Timoshenko's theory. Subsequently, the FEM approach is described in section 3. Section 4 presents a validation example of the FEM modal solution against the closed-form version. From there, obtaining the internal forces is straightforward. Section 5 shows an application to the computation of impact factors in a real four-span continuous bridge; the results are presented with a discussion of several issues that come up with the resolution of this kind of problems, such as the poor convergence of some magnitudes and damping modelisation. Finally, conclusions about the similarities in the dynamic amplification factors for different magnitudes are discussed.

2 CLOSED-FORM SOLUTION

Analytical methods for dynamic response of one-span Bernoulli-Euler beam under concentrated moving loads has been extensively developed in the last few decades. Regarding the solution to this problem in the framework of Timoshenko's beam theory, current existing publications address the question employing several methods such as modal expansion, integral transforms or wave propagation functions to name just a few. A complete treatment is given in the work of Kim *et al.* [4].

For the sake of completeness, the essential aspects of the solution corresponding to Timoshenko's beam theory are summarised in this section. Classical modal expansion approach is adopted here, in which the complete displacement field is represented by means of a weighted combination of the undamped system eigenfunctions, typically truncated for computational efficiency.

Aside from the beam model selected, as is well known, a deflection function convergence is achieved with relatively low modes whereas bending moment and shear forces require far more terms to attain a reasonable accuracy. Since conventional methods are not able to capture well the discontinuities in shear diagram and bending moment slope generated at the concentrated moving load location, some attempts has been made to overcome this limitations, see [5]. For our purpose, the accuracy achieved with the modal superposition technique is enough to gain insight into the impact factor properties for the bending moments.

Consider the governing differential equations of a Timoshenko beam model:

$$\begin{aligned} m\frac{\partial^2 w}{\partial t^2} - k_s GA\left(\frac{\partial \theta_y}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) &= q_z \\ m r_y^2 \frac{\partial^2 \theta_y}{\partial t^2} + k_s GA\left(\theta_y + \frac{\partial w}{\partial x}\right) - EI_y \frac{\partial^2 \theta_y}{\partial x^2} &= m_y \end{aligned} \tag{1}$$

Once the separation of variables has been imposed $w(x,t) = W(x)e^{i\omega t}$, $\theta_y(x,t) = \Psi(x)e^{i\omega t}$ on the related homogeneous equation, the spatial displacement function $W(x)$ will be the solution of the following compact equation,

$$W^{IV}(x) + \frac{m(\omega r)^2}{EI}\left(1 + \frac{E}{k_s G}\right)W^{II}(x) - \frac{m\omega^2}{EI}\left(1 - \frac{m(\omega r)^2}{k_s GA}\right)W(x) = 0 \tag{2}$$

being the roots of this spatial differential equation,

$$r_{1,2,3,4} = \pm\frac{\overline{\omega}}{\sqrt{2}\lambda}\sqrt{\sqrt{\left(1 - \frac{\eta}{12}\lambda^2\right)^2 + \left(\frac{2\lambda^2}{\overline{\omega}}\right)^2} \pm \left(1 + \frac{\eta}{12}\lambda^2\right)} \tag{3}$$

where the following dimensionless parameters has been defined,

$\lambda = \frac{L}{\sqrt{I/A}}$	slenderness ratio
$\overline{\omega} = \omega\sqrt{m/EI}\; L^2$	adimensional frequency
$\eta = \dfrac{12EI}{k_s GAL^2}$	bending to shear stiffness ratio

Except for the critical frequency $\overline{\omega}_c = \sqrt{12/\eta}\,\lambda$ the general solution is in the form,

$$W(x) = A_1 e^{r_1 x/L} + A_2 e^{r_2 x/L} + A_3 e^{r_3 x/L} + A_4 e^{r_4 x/L} \tag{4}$$

which in turn can be reformulated in terms of trigonometric and hyperbolic functions,

$$W(x) = C_1 \sin(ax) + C_2 \cos(ax) + C_3 \sinh(bx) + C_4 \cosh(ax) \tag{5}$$

being,

$$a = \frac{\overline{\omega}}{\sqrt{2}\lambda L}\sqrt{\sqrt{\left(1 - \frac{\eta}{12}\lambda^2\right)^2 + \left(\frac{2\lambda^2}{\overline{\omega}}\right)^2} - \left(1 + \frac{\eta}{12}\lambda^2\right)}$$
$$b = \frac{\overline{\omega}}{\sqrt{2}\lambda L}\sqrt{\sqrt{\left(1 - \frac{\eta}{12}\lambda^2\right)^2 + \left(\frac{2\lambda^2}{\overline{\omega}}\right)^2} + \left(1 + \frac{\eta}{12}\lambda^2\right)}$$

Additionally, taking into account the boundary conditions of a simply supported beam, these coefficients C_i can be established as a function of $\overline{\omega}$ except for a constant of proportionality. In order to let exist a non-trivial solution, the frequency equation for frequencies lower than $\overline{\omega}_c$ is as follows,

$$\sin(aL) = 0 \qquad \longrightarrow \qquad aL = h_k \pi \tag{6}$$

Sorting the eigenvalues in increasing order $(w_k;\ k = 1, n)$ and preserving the number of its spatial half-waves h_k (wave number), their associate eigenfuctions turn out to be,

$$W_k(x) = \sin\left(h_k \pi \frac{x}{L}\right) \tag{7}$$
$$\Psi_k(x) = \frac{-h_k \pi}{L}\left[1 - \frac{\eta}{12}\left(\frac{\overline{\omega}_k}{h_k \pi}\right)^2\right]\cos\left(h_k \pi \frac{x}{L}\right)$$

in virtue of the orthogonality property of modes, the projection onto the modal space with the displacement vector $N_k(x) = (W_k(x)\ \ \Psi_k(x))^T$ results in a set of independent non-homogeneous second order differential equation.

$$\ddot{q}_k(t) + 2\,\zeta_k\,\omega_k\,\dot{q}_k(t) + \omega_k^2\,q_k(t) = \frac{P\sin\left(h_k \pi \frac{x}{L}\right)}{\frac{mL}{2} M_{eq_k}} \tag{8}$$

with the equivalent modal mass

$$M_{eq_k} = m\int_0^L N_k^T \begin{pmatrix} 1 & 0 \\ 0 & \left(\frac{L}{\lambda}\right)^2 \end{pmatrix} N_k\,dx \tag{9}$$

its solution for an underdamped system will be,

$$q_k(t) = \begin{cases} \frac{1}{\omega_{d_k}} \int_0^t e^{-\omega_k \zeta_k (t-\tilde{t})} \sin\left(\omega_{d_k}(t-\tilde{t})\right) \frac{P \sin\left(h_k \pi \frac{v\tilde{t}}{L}\right)}{\frac{mL}{2} M_{eq_k}} d\tilde{t}, & \text{if } t \leq t_e \\ e^{-\omega_k \zeta_k (t-t_e)} \left(\frac{\dot{q}_k(t_e)+\omega_k \zeta_k q_k(t_e)}{\omega_{d_k}} \sin\left(\omega_{d_k}(t-t_e)\right) + q_k(t_e) \cos\left(\omega_{d_k}(t-t_e)\right)\right), & \text{otherwise.} \end{cases} \tag{10}$$

where,

$\omega_{d_k} = \omega_k \sqrt{1-\zeta^2}$	damped angular frequency
v	load constant velocity
$t_e = L/v$	time when the load leaves the bridge

Adding up the contribution of all retained modes, the displacement and rotation response will be respectively,

$$w(x,t) = \sum_1^n q_k(t) W_k(x) \tag{11}$$

$$\theta_y(x,t) = \sum_1^n q_k(t) \Psi_k(x)$$

3 FINITE ELEMENT IMPLEMENTATION

An analytical formulation for a one-span simple supported beam has been presented in previous section. However, analytical solution for multiple-span bridges with general boundary conditions is so arduous and cumbersome to obtain analytically that it is of common use to employ numerical methods like, in most cases, the Finite Element Method [6]. The FEM permits a systematic implementation to solve general problems, so the implementation to solve the moving load problem is particularly presented here in this section.

A Timoshenko beam element has been adopted to model the bridge behaviour, as well as the Bernoulli-Euler to allow a comparison of results. Element mass and stiffness matrices follow the formulation in [7], while the concentrated moving loads are assembled as equivalent nodal loads to obtain the global matrices. The equation of motion of the bridge is given in terms of the displacement vector $\mathbf{x}$, and its derivatives $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$, the force vector $\mathbf{f}$, mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ as

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f} \tag{12}$$

As well as in the analytical formulation, displacements are represented by a truncated base of eigenvectors $\mathbf{\Phi}_k$, previously determined by a modal analysis.

$$\mathbf{x}(t) = \sum_1^k q_k(t)\ \mathbf{\Phi}_k \qquad \begin{cases} |\omega^2 \mathbf{M} - \mathbf{K}| = 0 \\ [\omega_k^2 \mathbf{M} - \mathbf{K}]\, \mathbf{\Phi}_k = \mathbf{0} \end{cases} \tag{13}$$

The orthogonality property of the undamped eigenvectors permit the projection of the global matrix equation onto the modal space turning into a set of independent second order differential equations in which viscous modal damping ζ_k is then assigned.

$$\ddot{q}_k(t) + 2\,\zeta_k\,\omega_k\,\dot{q}_k(t) + \omega_k^2\,q_k(t) = \frac{\mathbf{\Phi}_k^T \mathbf{f}}{\mathbf{\Phi}_k^T \mathbf{M} \mathbf{\Phi}_k} \tag{14}$$

After each mode coordinate $q_k(t)$ is obtained, the displacements are expanded as previously stated 13. Internal forces, namely bending moment and shear force, are afterwards recovered by means of the element stiffness matrices $\mathbf{K^e}$ and the displacement vector $\mathbf{u^e}$ of their related nodes.

$$\mathbf{F^e} = \mathbf{K^e}\,\mathbf{u^e} \tag{15}$$

Actually, element nodal loads consist of a contribution of stiffness, damping an inertial forces. Inertial loads at nodes are a result of the space discretization and tends to vanish with the mesh refinement. Internal damping forces depends on the physical phenomena from which they are originated. Introducing modal damping leaves the distribution of that forces undetermined and a hypothesis on how they are generated is required to define damping forces. In any case, for a conventional structure damping ζ is usually below 3%, which would represent a negligible contribution to the total internal force. Thus, if the mesh is refined enough, the stiffness part

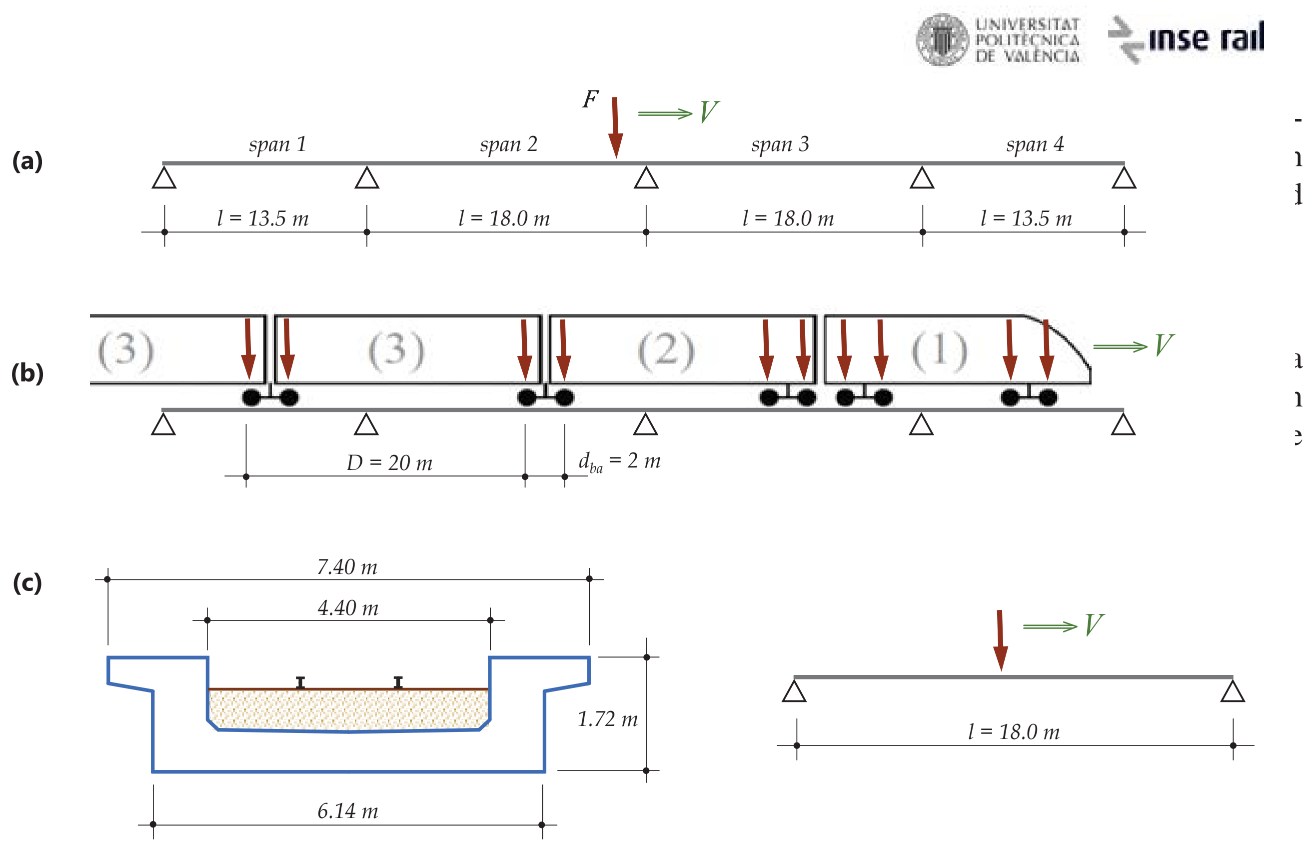

Figure 1: Bridge Section and one-span moving load model.

Figuras Articulo HL 1

Section properties are as follows: cross-section area $A = 6.46\,\mathrm{m}^2$, second moment of area about an horizontal axis passing through the c.o.g. $I = 1.69\,\mathrm{m}^4$ with a linear mass density $m = 21400\,\mathrm{kg/m}$ including self and death weight. Flexural to shear stiffness factor is estimated in $k_s = 0.461$ to be used in the Timoshenko beam model. Material properties are $E = 36.0\,\mathrm{GPa}$, $\nu = 0.3$ with a viscous critical damping ratio of $\zeta = 1.14\%$ for prestressed structures according to [1].

4.2 FEM RESULTS vs. CLOSED-FORM SOLUTION

In a single-span SS beam the firsts few lower modes are the major contributors to the displacement response, with a fast convergence to the complete solution. Therefore, in this section both analytical and FEM displacement variables are expanded with the contribution of the first five modes.

As can be seen in Table 1, with a mesh of 100 elements, relative error of FEM compared to the analytical solution is negligible. Even with 0.75m element length (24 elem) is enough to accurately capture up to 5 modes with a relative error less than 1.0%.

An interesting fact is that the first two eigenfrequencies from B-E theory are quite similar to that of Timoshenko theory, while higher modes are appreciably stiffer with the B-E modelisation.

	frequency (Hz)					
	Timoshenko			Bernoulli-Euler		
Mode no.	Analytic	FEM	$\epsilon_r(\%)$	Analytic[1]	FEM	$\epsilon_r(\%)$
1	7.968	7.968	0.000%	$\pi^2 k$	8.175	0.000%
2	29.76	29.76	0.000%	$(2\pi)^2 k$	32.70	0.000%
3	60.97	60.97	0.000%	$(3\pi)^2 k$	73.57	0.000%
4	97.62	97.62	0.001%	$(4\pi)^2 k$	130.8	0.000%
5	137.1	137.1	0.002%	$(5\pi)^2 k$	204.4	0.000%

[1] Conversion factor from dimensionless frequency to the angular frequency $k = \sqrt{EI/m}/(2\pi L^2)$=0.828Hz

Table 1: Timoshenko vs Bernoulli-Euler eigenvalues.

Figure 2 illustrates the displacement transient response of the mid-span section when the beam is traversed by a concentrated load of $100\,\mathrm{kN}$. An excellent agreement of the FEM solutions compared with the analytical one is obtained with a 100 elements mesh for two different load velocities in a one-span simple supported bridge. Changes to the spatial discretisation show that even with a coarse mesh of 4 elements reasonable accuracy is achieved. The use of different beam models arise some perceivable differences in the transient response, but the maximum response is quite similar in this example.

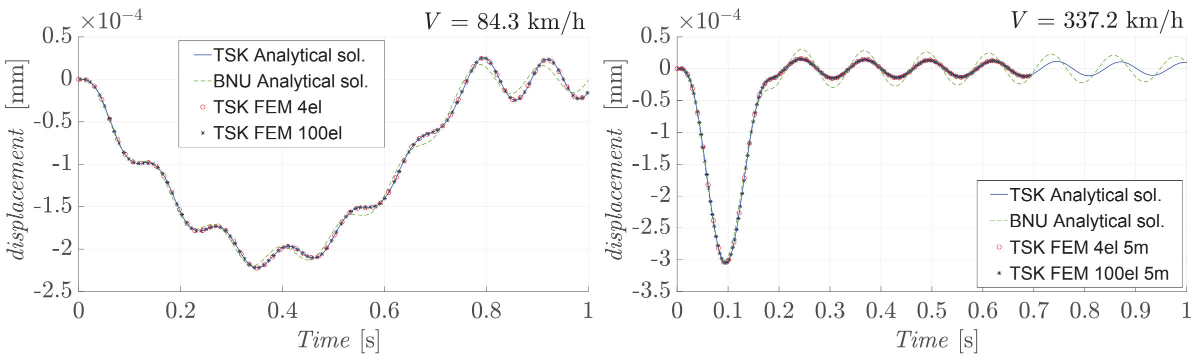

Figure 2: Analytical and FEM displacements (TSK - Timoshenko; BNU - Bernoulli-Euler beam model).

5 REAL CASE STUDY. FOUR-SPAN BRIDGE

As mentioned before, one four-span continuous bridge from the Swedish railway network has been selected as a real case study. Figure 3 shows the spans length and Figure 1 depicts the beam section dimensions whose properties has been described in Section 4.1

5.1 LOAD CASES

Two cases of constant velocity moving loads are considered, and presented in Figure 3. Case (a) is a single concentrated moving load $100\,\mathrm{kN}$, while (b) is an articulated train (HSLM-A Train A-3, as defined in [1]) comprising a series of coaches with axle loads of $180\,\mathrm{kN}$.

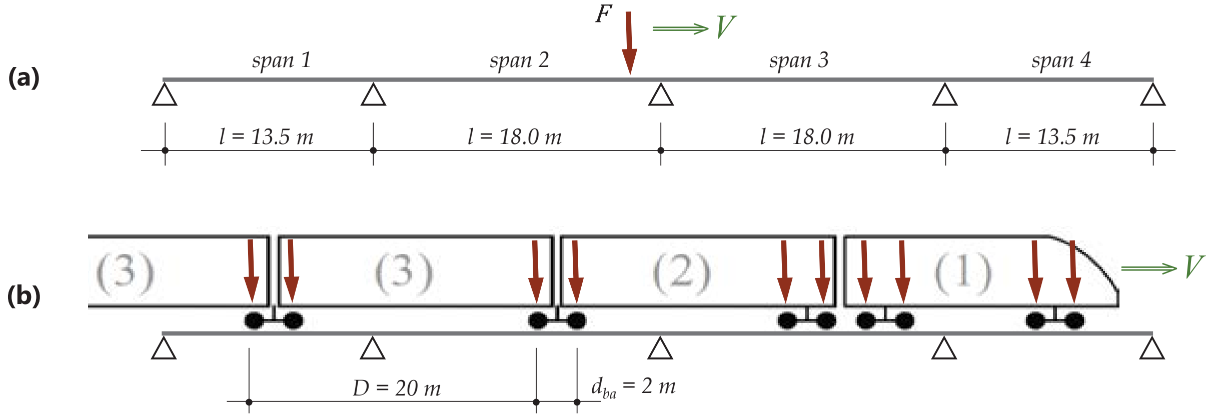

Figure 3: Bridge dimensions, support conditions and load set (a) single load, (b) HSLM-A3 trainset.

5.2 IMPACT FACTORS

In order to compare the impact factors (IF) of displacements and bending moments, both are evaluated at the mid-span sections. The IF is defined simply as the ratio between the maximum dynamic effect divided by the maximum static effect. First, positive bending moments at mid-span are compared with downward displacements in order to analyse possible similarities. Later, negative bending moments will be also discussed. A 0.75m mesh element length has been used and response variables are expanded with the contribution of 5 *clusters* of modes. A cluster is considered as group of (similar) frequencies equal to the number of spans, which is typical in continuous beams.

IFs for a single load, Figure 4, does not display too prominent peaks, and values remain lower than 1.2 in the analysed velocity range. Its aspect reminds of the classical dynamic influence lines, obtained for one load in a single span [8]. While for central spans maximum value appears at 300-350 km/h, outer spans seems to increase their level at velocities higher than 400 km/h. Also, the curves for spans 1-2 are smoother than for spans 3-4. Of great interest is that IFs for positive bending moment follow in general the same tendency than the displacement ones for each span, with somewhat lower values.

When a train of loads is considered, the system can be expected to develop resonance at certain speeds. Figure 5 presents a noticeable peak around 340 km/h that is more prominent in spans 1-2-3. Interestingly, as it happened for the single concentrated load, positive bending moment IFs follow the same tendency than the displacement ones, being slightly lower.

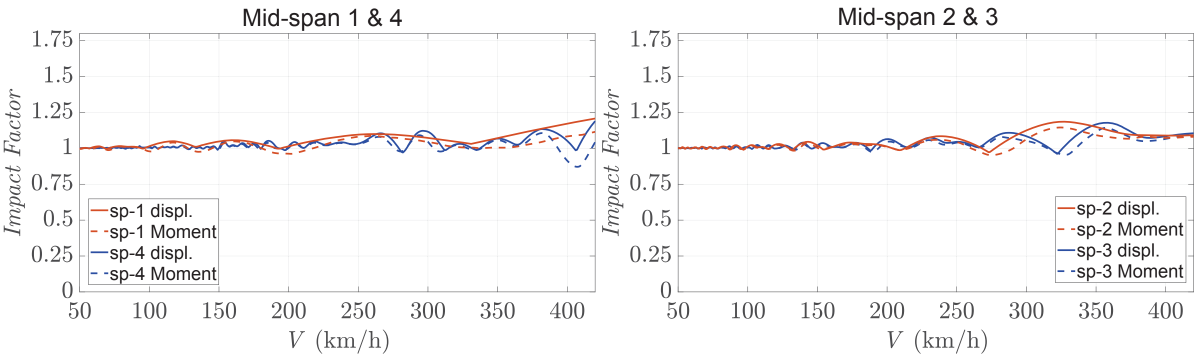

Figure 4: IF for a single concentrated moving load, $\zeta = 1.14\%$, outer spans (left) and inner spans (right).

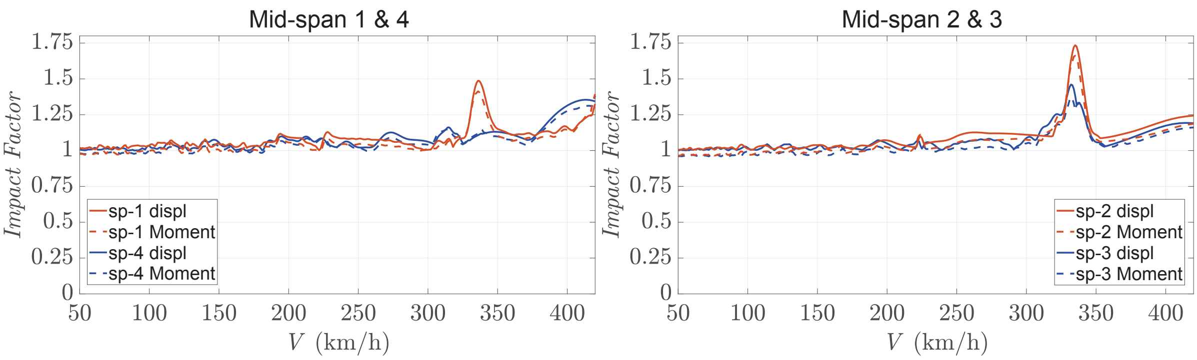

Figure 5: IF for HSLM-A3 train, $\zeta = 1.14\%$, outer spans (left) and inner spans (right).

As it was expected, a decrement in the damping factor produce a higher amplification of the IF as can be seen in Figure 6. All the conclusions from the previous analyses are applicable to this case as well.

In this case, the fact that IFs for positive bending moment follow the trends of the displacement IF, being slightly lower, is in accordance with the usual procedure followed in many technical offices, in which the displacement IF is also applied for the strength verification at ULS, as it was mentioned in the introductory section.

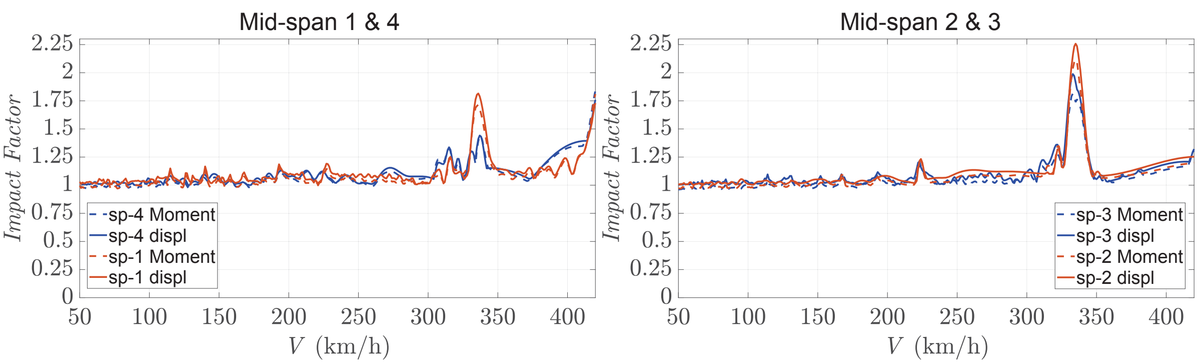

Figure 6: IF for HSLM-A3 train, $\zeta = 0.50\%$, outer spans (left) and inner spans (right).

However, the situation is not so clear when negative bending moments at the supports are analysed. Figures 7 to 9 show the IF for the negative moment in all three supports, along with the IFs corresponding, in each case, to the mid-span sections of both adjacent spans, as well as the average displacement IF of both mid spans. We designate *support 2* the one located at the left end of *span 2*, according to Figure 3, and similarly for supports 3 and 4. Focussing on the resonant speed determinant for the bridge design, one first conclusion is that IFs for negative bending moments are lower compared to positive bending moments. Moreover, the approximation via displacement IFs is not satisfactory: in supports 2 and 3 the displacement coefficients overestimate the negative bending moment impact, see Figures 7 and 8; besides, in support 4 the correlation of the peak is good with the average of the displacements, but the curves are irregular in that speed range and no clear trend can be established (Figure 9).

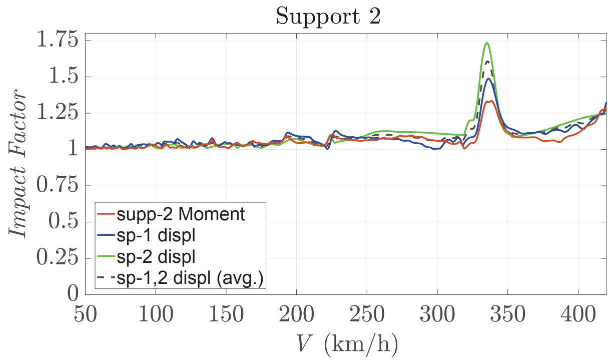

Figure 7: IF for HSLM-A3 train, $\zeta = 1.14\%$, support 2.

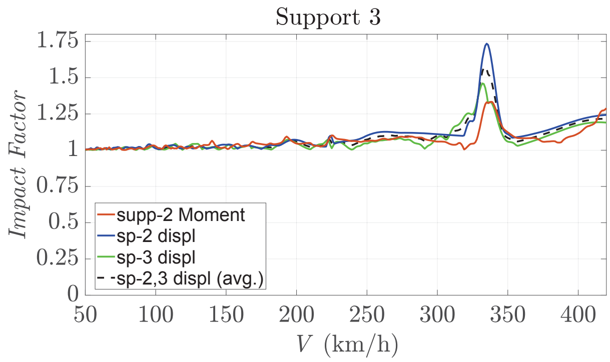

Figure 8: IF for HSLM-A3 train, $\zeta = 1.14\%$, support 3.

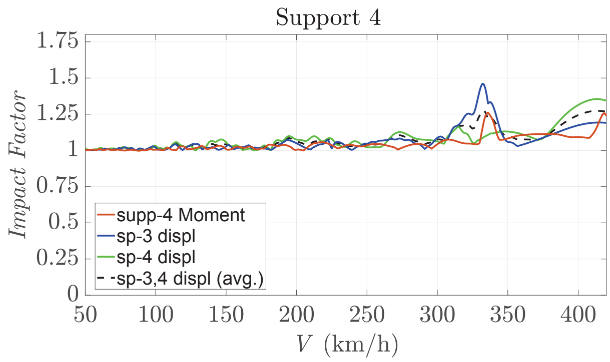

Figure 9: IF for HSLM-A3 train, $\zeta = 1.14\%$, support 4.

These findings suggest that using displacement-based impact factors to amplify static negative bending moments may be less accurate than for positive bending moments. Nevertheless, such conclusion is based in the analysis of one representative example and needs further verification.

6 CONCLUSIONS

In this paper a Finite Element approach based on Timoshenko beam theory for the calculation of dynamic impact factors in continuous beams has been presented. The time-varying amplitudes of the modal coordinates have been validated against a closed-form solution of the equations of motion. The influence of mesh refinement has been discussed.

Subsequently, the impact factors for positive bending moments at mid-span have been compared with those of downwards displacements. Impact factors for negative bending moments have been scrutinised as well. First, a single travelling load has been considered; secondly, the effects of a high-speed train capable of producing resonance have been analysed for an actual four-span, single-track, continuous railway bridge. In light of the outcomes of these analyses it can be concluded that:

- Though it is sometimes believed that continuous bridges are largely immune to resonance, results for the four-span bridge studied here under the HSLM-A3 train display a prominent peak around 340 km/h, with impact factors as high as 1.6 (for damping $\zeta = 1.14\%$), or 2.1 (for damping $\zeta = 0.5\%$).
- Inner spans present higher resonant peaks than the end spans.
- Impact factors for positive bending moment follow the same tendencies of the displacement ones for each span, with slightly lower values.
- The preceding conclusions related to the impact factors are valid for different levels of structural damping.
- The use of displacement-based impact factors to amplify static negative bending moments may be less accurate than for positive bending moments.
- As it is known, Bernoulli-Euler beam theory overestimates natural frequencies with respect to the Timoshenko beam model. Particulary, the first two natural frequencies of a single-span railway bridge have been found to be quite similar—specially the first one. The difference increases noticeably for the higher modes.

7 ACKNOWLEDGMENTS

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