Available online at www.eccomasproceedia.org
Eccomas Proceedia COMPDYN (2021) 5331-5341

ECCOMAS Proceedia

COMPDYN 2021
8th ECCOMAS Thematic Conference on
Computational Methods in Structural Dynamics and Earthquake Engineering
M. Papadrakakis, M. Fragiadakis (eds.)
Streamed from Athens, Greece, 28 - 30 June 2021

GEOMETRICALLY NONLINEAR FREE VIBRATIONS OF FULLY CLAMPED MULTI-STEPPED BEAMS CARRYING MULTIPLE MASSES

I. El Hantati[1], A. Adri[1], H. Fakhreddine[1], S. Rifai[1], R. Benamar[2]

[1] Laboratory of Production Mechanics and Industrial Engineering (LMPGI), Higher School of Technology, Hassan II University of Casablanca, Km 7, Route El Jadida, B.P. 8012 Oasis, Casablanca, Morocco
hantati.issam@gmail.com, ahmedadri@gmail.com, hatim.fakhreddine@gmail.com, said57.rifai@gmail.com

[2]Doctoral Studies Centre of National High School of Electricity and Mechanics (ENSEM)
Route d'ElJadida, Km 7, Oasis, Casablanca 8018, Morocco
hantati.issam@gmail.com,hatim.fakhreddine@gmail.com

[2]Mohammed V University in Rabat, EMI-Rabat, LERSIM, Agdal,
B.P. 765, Rabat, Morocco
rhali.benamar@gmail.com

Abstract

The main objective of this work is to study the geometrically non-linear free vibration of stepped beams carrying multiple masses. These beams are studied on the basis of the Euler-Bernoulli beam theory and the Von Karman geometrical nonlinearity assumptions. The discrete expressions for the beam total strain and kinetic energies are derived. By applying Hamilton's principle, the problem is reduced to a non-linear algebraic system solved by an approximate method (the so-called second formulation). A parametric study is performed to explore the effect of non-linearity on the dynamic behaviour of stepped beams with several added masses. The free vibration case is discussed by considering three types of stepped beams which differ in the cross-section type.

Keywords: Stepped beam, transverse vibrations, nonlinear vibrations, free vibrations

ISSN:2623-3347
Peer-review under responsibility of the organizing committee of COMPDYN 2021.
doi: 10.7712/120121.8869.19328

1 INTRODUCTION

Stepped beams are commonly encountered in practical structures and extensively used in many domains, including mechanical and civil engineering, such as high-rise buildings, robot arms, etc. These structures, often subjected during their lifetime to transverse vibrations, may contain concentric elements that can be assimilated to point masses or to linear or rotational springs. Considerable work has been done to study the dynamic behaviour of these types of beams by means of linear theory, making the analysis relatively easy, but leading to inaccurate results when the beams are subjected to large vibration amplitudes. In such situation, non-linear analysis is needed to obtain with accuracy the deformation of the beam for each amplitude level considered. A review of the literature goes back to Naguleswaran who presents an overview of the existing studies performed to analyse stepped beams using linear approaches[1]. El Hantati[2], [3] presents a literature review of works related to non-uniform beams in the case of linear and non-linear vibrations. A significant number of studies have been conducted on the dynamic response of single and multi-mass beams, using linear and non-linear analyses for both free and forced vibrations. These studies have been mostly reported in an extensive literature review presented by Fakhreddine[4]. Regarding non-uniform beams with concentric masses, many studies have been carried out to analyse their response to free and forces vibrations, considering both the Euler-Bernoulli and the Timoshenko beam theories. Karami[5] investigated free vibrations of arbitrary non-uniform Timoshenko beams with their concentrated mass and rotary inertia and resting on elastic supports based on the differential quadrature element method. Based on the same method, Torabi et al. studied in [6] free vibrations of a non-uniform cantilever Timoshenko beam with multiple concentrated masses. Kohan et al. [7]studied a non-uniform beam carrying a concentrated masses and supported by linear and rotational springs at both sides. The analytical formulation exposed in this work was based on the Ritz method and the orthogonal polynomials within the framework of the first order shear deformation beam theory. Jong-Shyong[8]used the continuous-mass transfer matrix method to investigate free vibrations of a multi-step Timoshenko beam axially loaded and carrying arbitrary concentrated elements. K.Torabi et al [9] investigated multiple-stepped Bernoulli Euler and Timoshenko beams, carrying concentrated masses with rotary inertia at arbitrary points based on the same method. Yusuf Yesilce[10]studied free vibrations of a Timoshenko multiple-step beam under axial load, carrying a number of intermediate lumped masses and rotary inertias based on the Differential transform method and numerical assembly technique. Farghaly and El-Sayedpresented in[11] parametric studies for an axially loaded stepped Timoshenko beam carrying several attachments. All of the aforementioned studies were carried out assuming linearity and none of them took into account the effect of geometrical non-linearity. This work, which is a continuation of the work previously carried out by Benamar et al.[12]–[19], is intended to contribute to a non-linear modal analysis of structural vibration by studying the geometrically non-linear free vibration of stepped beams carrying multiple masses using the Euler-Bernoulli beam theory and the Von Karman geometrical nonlinearity assumptions. The beam total strain and kinetic energies are presented as discrete expressions and then derived. By applying Hamilton's principle, the problem is reduced to a non-linear algebraic system solved by an approximate method previously applied to similar problems (the so-called second formulation) [16]. A parametric study is performed to explore the effect of non-linearity on the dynamic behaviour of stepped beams with several added masses. The free vibration case is discussed by considering three types of stepped beams which differ in the cross-section type.

1- General formulation

1.1 Linear formulation

The current study deals with the three homogeneous Euler-Bernoulli multi-step beams shown in Figure (1), characterised by a thin rectangular cross-section. The first beam rectangular cross-section (a) varied in breadth, the second beam (b) varied in depth, and the third beam (c) varied in both. For the remainder of the work, the beams examined are fully clamped, with a concentrated mass at the middle, and subjected to transverse vibrations.

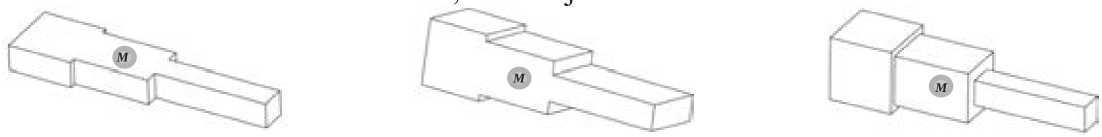

Figure 1: Three representative cases of the stepped beam.

Each of the beams shown in Figure 2 carries a concentric mass in the middle. They are also partitioned into three uniform cross-section parts and are subjected to transverse vibrations.

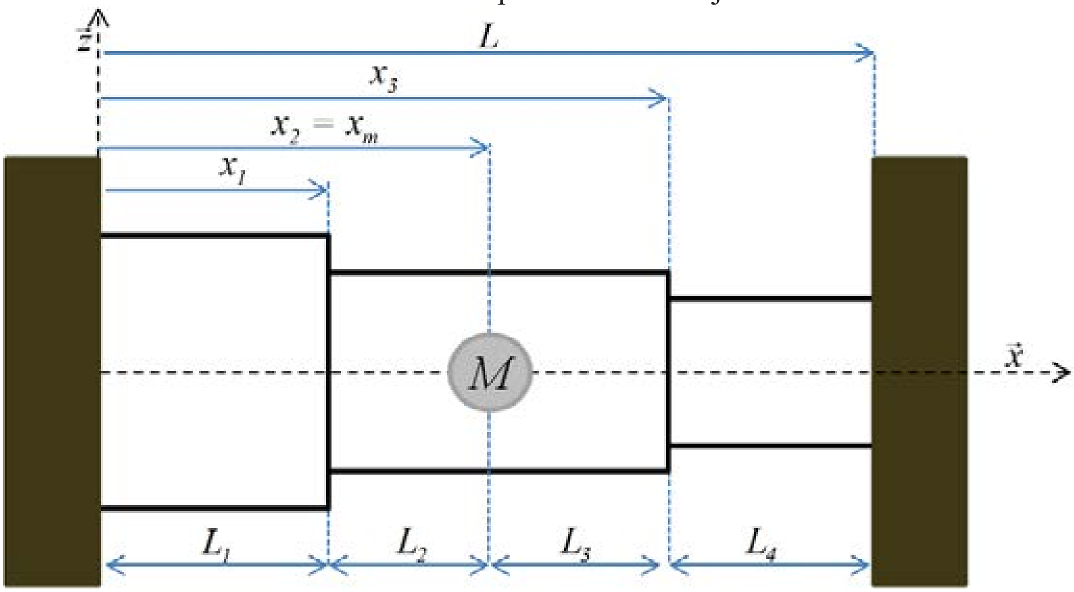

Figure 2: The coordinate system for a multi-stepped beam.

The transverse free vibration in each span is governed by the following equation:

$$\frac{d^4W_i(x)}{dx^4} - \beta_i^4 W_i(x) = 0 \qquad ; x \epsilon [0 \ L] \text{(For i=1,2..., 12)} \tag{1}$$

Where

$$\beta_{ij}^4 = \omega_i^2 \frac{\rho_j S_j}{E_j I_j} \text{(For j=1,2 and 4)} \tag{2}$$

After putting the parameters of equation (1) into a non-dimensional form, Equation (1) can be written in the following form:

$$\frac{d^4W_i\left(x^*\right)}{dx^4}-\beta_i^4\,W_i\left(x^*\right)=0 \qquad ;x^*\epsilon\left[0\ 1\right] \tag{3}$$

Where

$$x_j^{\ *}=\frac{x_j}{L};R_j^{\ *}=\frac{L_j}{L}\quad ;\ z_j=\frac{\beta_{ij}}{\beta_{i1}};u_j=\frac{S_j}{S_1};v_j=\frac{I_j}{I_1} \tag{4}$$

Each step of the beam is characterized by a ratio of sections, moments of inertia and eigen-value parameters which are denoted by (u_j, v_j and z_j).

The solution of Eq. (1) presents the linear modes form which can be expressed at each span as:

$$W_i\left(x^*\right)=\begin{cases} W_{ij}\left(x^*\right)\rightarrow\left\{x_{j-1}^{\ *}\le x^*\le x_j^{\ *}\right\} \\ \dots \\ \dots \end{cases}$$

$$W_{ij}\left(x^*\right)=A_j\cosh\left(\beta_{ij}L\left(x^*-x_{j-1}^{\ *}\right)\right)+B_j\sinh\left(\beta_{ij}L\left(x^*-x_{j-1}^{\ *}\right)\right)+C_j\cos\left(\beta_{ij}L\left(x^*-x_{j-1}^{\ *}\right)\right)+D_j\sin\left(\beta_{ij}L\left(x^*-x_{j-1}^{\ *}\right)\right) \tag{5}$$

In equation (5), the constants A_j, B_j, C_j and D_j can be solved from the compatibility conditions and the boundary conditions.

The boundary conditions at the left side:

$$W_{i1}\left(x^*\right)\Big|_{x^*=0}=0 \quad ;\left.\frac{dW_{i1}\left(x^*\right)}{dx^*}\right|_{x^*=0}=0 \tag{6,7}$$

The boundary conditions at the right side:

$$W_{i4}\left(x^*\right)\Big|_{x^*=0}=0 \quad ;\left.\frac{dW_{i4}\left(x^*\right)}{dx^*}\right|_{x^*=1}=0 \tag{8,9}$$

The compatibility conditions at each change in the cross-section are given by:

$$W_{ij}\left(x^*\right)\Big|_{x^*=x_j^*}=W_{ij+1}\left(x^*\right)\Big|_{x^*=x_j^*} \tag{10}$$

$$\left.\frac{dW_{ij}\left(x^*\right)}{dx^*}\right|_{x^*=x_j^*}=\left.\frac{dW_{ij+1}\left(x^*\right)}{dx^*}\right|_{x^*=x_j^*} \tag{11}$$

$$EI_j\left.\frac{d^2W_{ij}\left(x^*\right)}{dx^{*2}}\right|_{x^*=x_j^*}=EI_{j+1}\left.\frac{d^2W_{ij+1}\left(x^*\right)}{dx^{*2}}\right|_{x^*=x_j^*} \tag{12}$$

$$EI_j \frac{d^3W_{ij}\left(x^*\right)}{dx^{*3}}\bigg|_{x^*=x_j^*} = EI_{j+1} \frac{d^3W_{ij+1}\left(x^*\right)}{dx^{*3}}\bigg|_{x^*=x_j^*} \tag{13}$$

The compatibility conditions for an attached mass with a rotary inertia are given by:

$$W_{ji}(x^*)\Big|_{x^*=x_m^*} = W_{(j+1)i}(x^*)\Big|_{x^*=x_m^*} \tag{14}$$

$$\frac{dW_{ji}(x^*)}{dx^*}\bigg|_{x^*=x_m^*} = \frac{dW_{(j+1)i}(x^*)}{dx^*}\bigg|_{x^*=x_m^*} \tag{15}$$

$$\frac{d^2W_{ji}(x^*)}{dx^{*2}}\bigg|_{x^*=x_m^*} = \frac{d^2W_{(j+1)i}(x^*)}{dx^{*2}}\bigg|_{x^*=x_m^*} + J_j\boldsymbol{\omega}^2 \frac{dW_{ji}(x^*)}{dx^*}\bigg|_{x^*=x_m^*} \tag{16}$$

$$\frac{d^3W_{ji}(x^*)}{dx^{*3}}\bigg|_{x^*=x_m^*} = \frac{d^3W_{(j+1)i}(x^*)}{dx^{*3}}\bigg|_{x^*=x_m^*} - \frac{M_j\boldsymbol{\omega}^2W_{ji}(x^*)}{\mathrm{EI}}\bigg|_{x^*=x_m^*} \tag{17}$$

Non-trivial solutions corresponding to the natural frequencies are derived by stating that the determinant of the homogeneous system obtained from the boundary and compatibility conditions vanishes. The transcendental equation obtained is solved iteratively by the Newton-Raphson method and the constants A_j, B_j, C_j and D_j are determined by classical algebra.

2.3. Non-linear formulation

2.2.1 Free vibration

The kinetic energy T of the beam is given by:

$$T = \frac{\rho}{2} \int_0^L S\left(x\right)\left(\frac{\delta w\left(x^*,t\right)}{\delta t}\right)^2 dx + \frac{1}{2}M_c\left(\frac{\delta w\left(x_m,t\right)}{\delta t}\right)^2 \tag{18}$$

The total deformation energy V is the sum of the deformation energy due to bending and the axial deformation energy due to the non-linear stretching forces induced by the large deflections, respectively denoted by V_f and V_a and written as follows:

$$V = V_f + V_a \tag{19}$$

$$V_f = \frac{1}{2}\int_0^L E\ I\left(x\right)\left(\frac{d^2w_i}{dx^2}\right)\left(\frac{d^2w_j}{dx^2}\right)dx \tag{20}$$

$$V_a = \frac{1}{2}\frac{N_x^{\,2}}{E}\int_0^L \frac{dx}{S\left(x\right)} \tag{21}$$

Where N_x present the non-linear stretching forces in the beam. The dynamic behaviour of the system is studied by applying Hamilton's principle, stated formally as follows:

$$\delta\int_0^{\frac{2\omega}{t}} \left(V - T\right)dt = 0 \tag{22}$$

Assuming that the system has a harmonic motion, the transverse displacement can be written as:

$$w(x,t) = a_i W_i(x) sin(\omega t) \qquad (For\ i = 1,2 \ldots,n) \tag{23}$$

The linear modes, the basic function contribution coefficients and the associated frequency are denoted by $W_i(x)$, a_i and ω. After Substitution of equation (23) into equations (18, 20 and 21), the expressions for the kinetic energy, the axial deformation energy and the bending strain energy can be written as follows:

$$T = \frac{1}{2}\ \omega^2\ a_i a_i \left(\cos(\omega t)\right)^2 m_{ij}\ ; \tag{24}$$

$$V_a = \frac{1}{2}\ a_i a_j\ a_k\ a_l\ (\sin(\omega t))^4 b_{ijkl}\ ; V_f = \frac{1}{2}\ a_i a_i \left(\sin(\omega t)\right)^2 k_{ij} \tag{25,26}$$

Where k_{ij}, b_{ijkl} and m_{ij} denote the stiffness tensor due to V_f, the non-linearity tensor due to V_a and the mass tensor attributable to T. Equation (23) is extended by adding equations (24 to 26) into it, which leads to the following new form:

$$2 a_i k_{ir} + 3 a_i a_j a_k b_{ijkr} - 2\omega^2\ a_i m_{ir} = 0 \qquad ; \quad \mathrm{r} = 1,\ldots.,\mathrm{n} \tag{28}$$

Equation (28) can be expressed in a matrix form as follows:

$$2[K]\{A\} + 3\left[B(\{A\})\right]\{A\} - 2\omega^2\ [M]\{A\} = 0 \tag{29}$$

In which $(k_{ir}),(m_{ir}),(a_j a_k b_{ijkl})$ and (a_i) are the general terms of the matrices $[K],[M],\left[B(\{A\})\right]$ and $\{A\}$. Setting the dimensionless parameters as follows:

$$\frac{W_i(x^*)}{W_i^*(x^*)} = h_0; \frac{\omega^2}{\omega^{*2}} = \frac{E\,I_1}{\rho S_1 L^4}\ ; \frac{M_{ij}}{M_{ij}^{\ *}} = \rho\, S_1 h_0^{\ 2} \tag{30,31,32}$$

$$k_{ij} = \frac{E\,I_1 h_0^{\ 2}}{L^3} k_{ij}^{\ *}; \ b_{ijkl} = \frac{E\,I_1 h_0^{\ 2}}{L^3}\ b_{ijkl}^{\ \ *} \tag{33,34}$$

After substitution of the dimensionless parameters, Equation (29) becomes:

$$\left[K^*\right]\{A\} + \frac{3}{2}\left[B^*(\{A\})\right]\{A\} - \omega^{*2}\ [M^*]\{A\} = 0 \tag{35}$$

In order to determine the frequency ω and the column vector of the contribution coefficients {A}, which are the unknowns of equation (35), the so-called second formulation, previously developed by EL KADIRI et al. in [16], is used to solve equation (35).

NUMERICAL RESULTS AND DISCUSSION:

1.2 Free vibration:

In order to validate the numerical method mentioned above using the available data, the results corresponding to a uniform beam carrying a point mass at the middle are compared to those obtained in [20], where the section ratios of the the 1st and 2nd steps, designated by u_1 and u_2, are equal to $u_1 = u_2 = 1$ and the mass magnitude $M_c = 1$

W_{max}/r	Present work	Reference [20]	Rel Diff[20]
0.4	1.0035	1.0034	0.01%
0.8	1.0142	1.0137	0.05%
1	1.0221	1.0213	0.08%
2	1.0851	1.0814	0.34%

Table 1: $\left(\omega_{NL}/\omega\right)$ for clamped beams, in which a concentred mass is located at its middle with $r=\sqrt{I/S}$.

The results show an excellent agreement, with a relative difference not exceeding 0, 34%. The cases considered in all figures below are for three stepped beams with two cross-section changes and carrying a concentrated mass at the middle as shown in figure (1). The mass magnitude $M_c = 1$, the cross-section ratios at the 1st and 2nd step u_1 and u_2 equal to u_1=0.8 and u_2= 0.6. The three beams keep the same ratios u1 and u2, but each of them is expressed differently. The figures below show the effect of the cross-section variations on the frequency ratio.

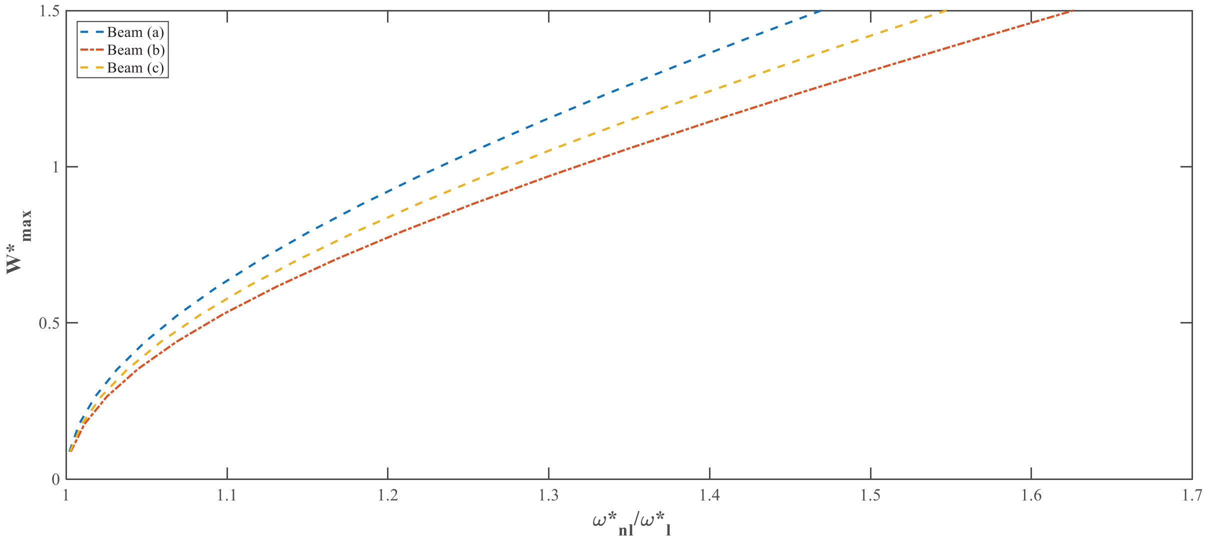

Figure (3): Backbone curves of the three stepped beams studied, corresponding to the first nonlinear mode.

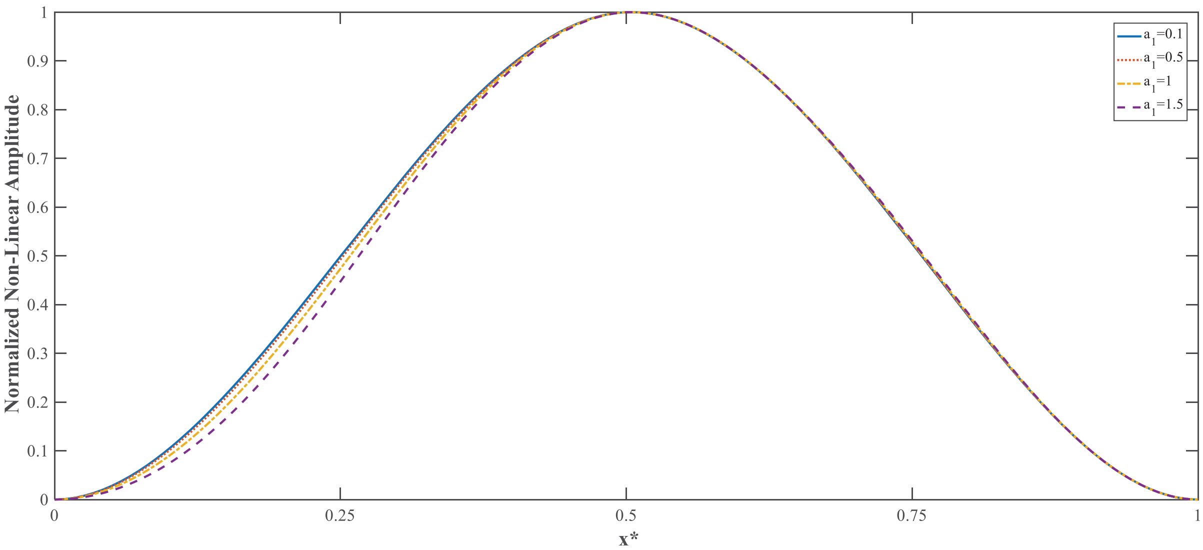

Figure (4): The normalized first non-linear mode of beam (a) for different values of a_1.

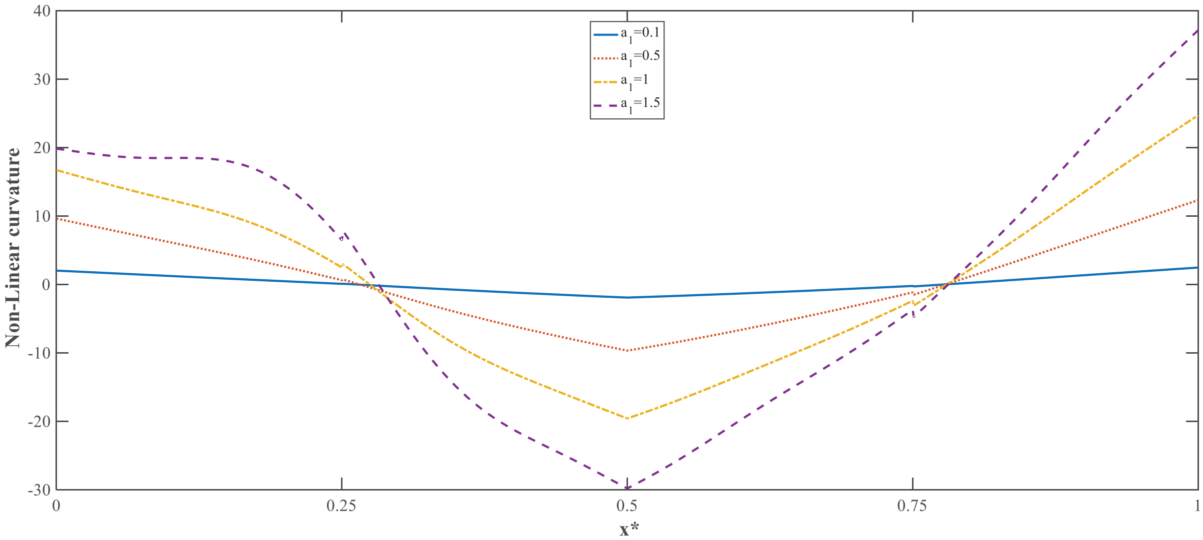

Figure (5): The non-linear curvature distribution of beam (a) associated to the normalized first non-linear mode for different values of a_1.

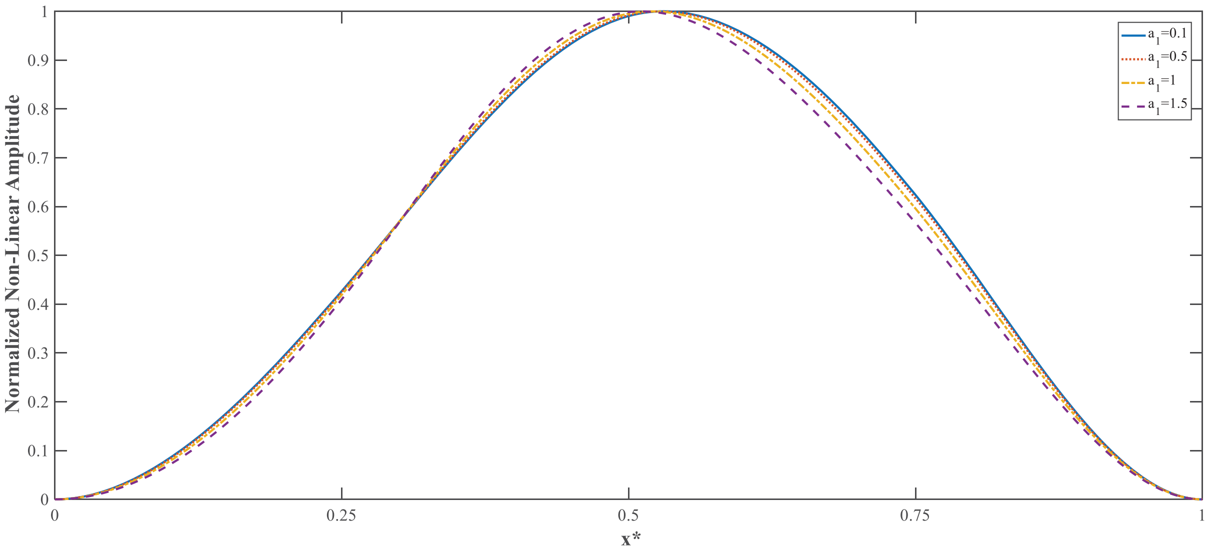

Figure (6): The normalized first non-linear mode of beam (b) for different values of a_1.

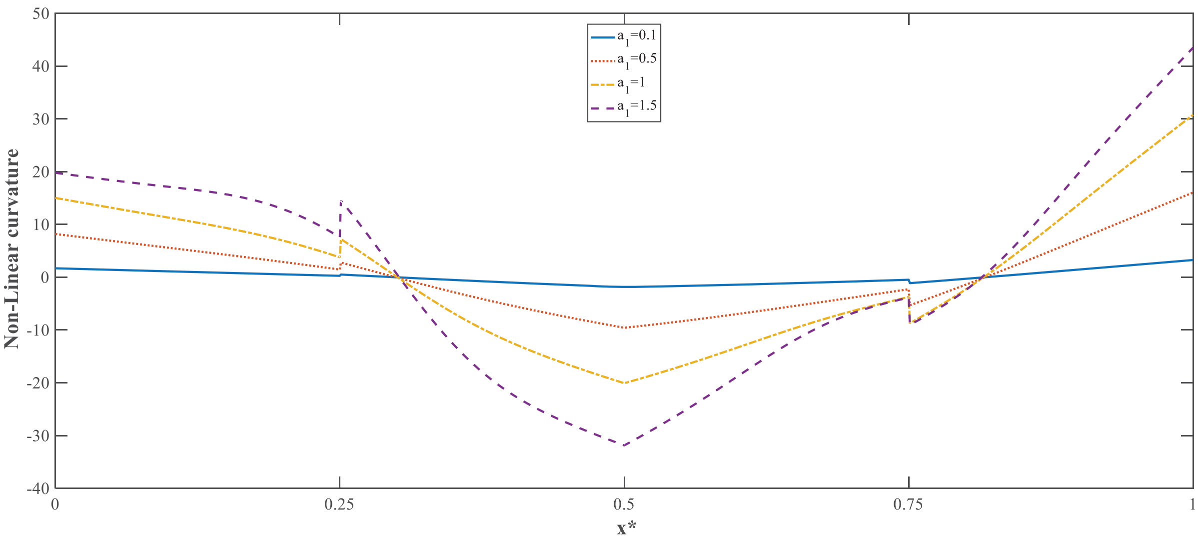

Figure (7): The non-linear curvature distribution of beam (b) associated to the normalized first non-linear mode for different values of a_1.

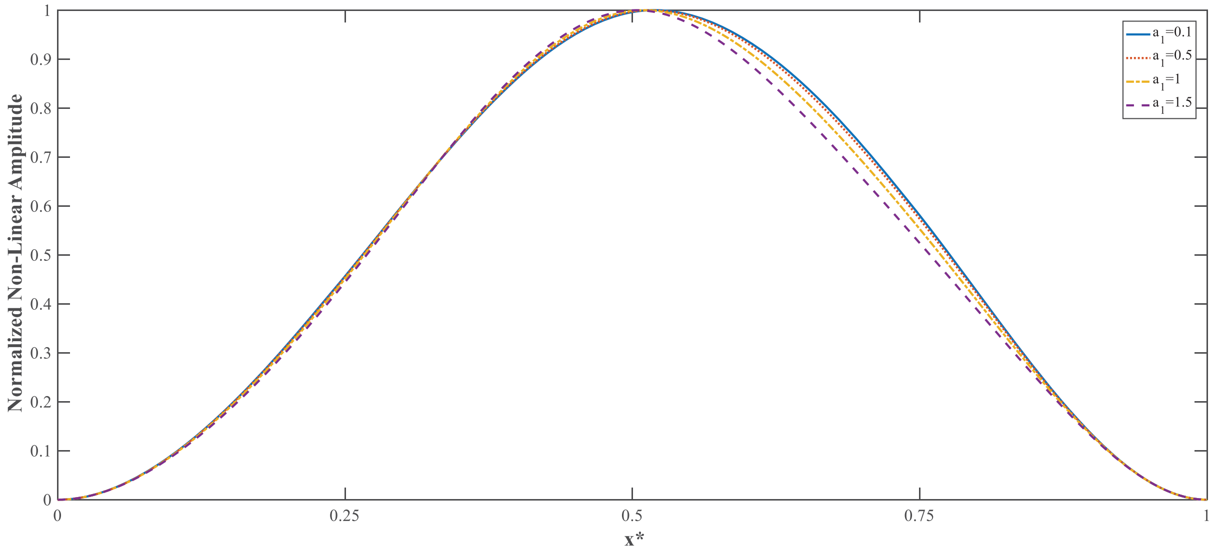

Figure (8): The normalized first non-linear mode of beam (c) for different values of a_1.

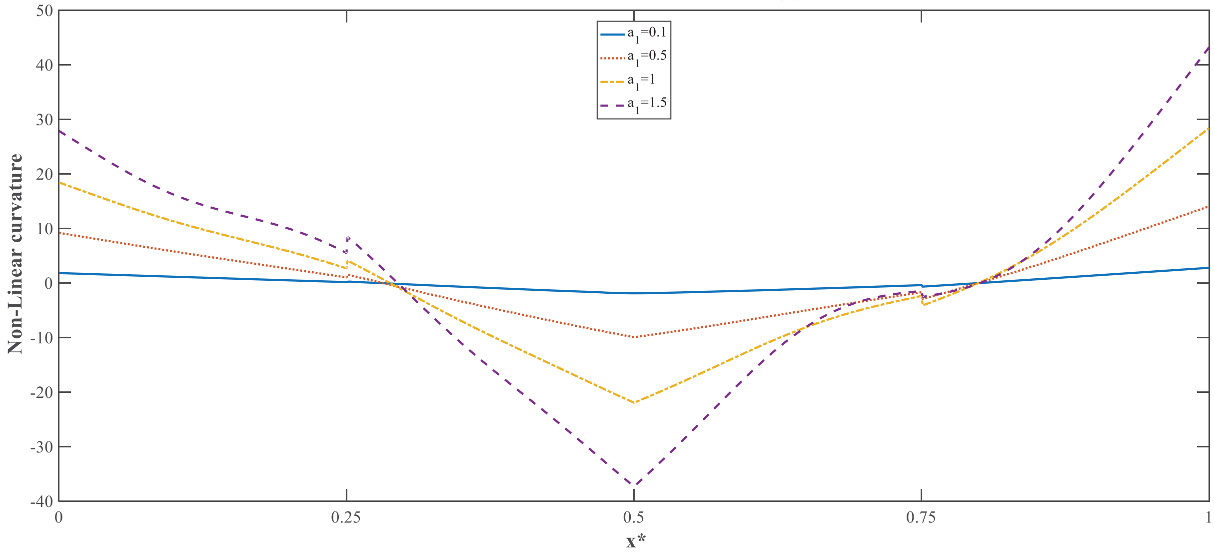

Figure (9): The non-linear curvature distribution of beam (c) associated to the normalized first non-linear mode for different values of a_1.

	Curvature		
beam	Non-linear theory	Linear theory	percentage correction %
a	49,33	61,43	19,69
b	54,18	81,44	33,47
c	58,56	70,12	19,74

Table 2: The percentage correction between the curvatures estimated at the right end via the linear and nonlinear theories for a1=2.

The backbone curves in Figure (3) show that by increasing the maximum non-dimensional amplitude W^*_{max}, the frequency ratios also increase. Also, the hardening effect of non-linearity is more accentuated in beam (b) compared to the other. It can be noticed that the geometrical non-linearity is not only related to the section ratio but also to the way it varies. For instance, the results obtained from Table (2) show that the percentage correction of the linear and non-linear bending moment of beam (a) at the right end with a contribution coefficient a1=2 is 19.69%, and remains higher than the two others, where the percentage correction of beam (c) and beam (a) are respectively 19.74% and 19.69%. Once more, these results reveal that the assumption of neglecting geometrical non-linearity for the same beam can be misleading.

2 CONCLUSION

The geometrically non-linear transverse vibrations of multi-stepped fully clamped beams carrying masses at the middle was studied analytically based on the Euler-Bernoulli's beams theory and the Von Karman's nonlinearity assumptions. The solution of the linear problem was obtained for the stepped beams considered. After determining the linear mode shapes, these were employed as basic functions for non-linear vibration analysis. To solve the non-linear algebraic system, the approximate method called the second formulation was applied. A comparison was performed between the result obtained from the numerical method used in the present work and those of Reference [18]. This comparison showed an excellent agreement. Then, the non-linear dynamic behaviour of multi-stepped beams in the free vibration case was examined by considering three types of fully clamped multi-stepped beams carrying masses at the middle. The results were illustrated by backbone curves, amplitude dependent non-linear modes and curvatures, showing a clear hardening non-linear behaviour. The assumption of neglecting the geometrical non-linearity is also illustrated and discussed.

3 REFERENCES

[1] S. Naguleswaran, “Vibration of an Euler-Bernoulli beam on elastic end supports and with up to three step changes in cross-section,” *Int. J. Mech. Sci.*, vol. 44, no. 12, pp. 2541–2555, 2002, doi: 10.1016/S0020-7403(02)00190-X.

[2] I. El Hantati, A. Adri, H. Fakhreddine, S. Rifai, and R. Benamar, “Geometrically nonlinear forced vibrations of multiple-stepped euler-bernoulli beams,” *Proc. Int. Conf. Struct. Dyn. , EURODYN*, vol. 1, no. June, pp. 193–204, 2020, doi: DOI: 10.47964/1120.9015.19055.

[3] I. El Hantati, A. Adri, H. Fakhreddine, S. Rifai, and R. Benamar, “A Multimode Approach to Geometrically Nonlinear Free and Forced Vibrations of Multistepped Beams,” *Shock Vib.*, vol. 2021, no. 4, pp. 1–18, Apr. 2021, doi: 10.1155/2021/6697344.

[4] H. Fakhreddine, A. Adri, M. Chajdi, S. Rifai, and R. Benamar, “A multimode approach to geometrically non-linear forced vibration of beams carrying point masses,” *Diagnostyka*, vol. 21, no. 4, pp. 23–33, Nov. 2020, doi: 10.29354/diag/128603.

[5] G. Karami, P. Malekzadeh, and S. A. Shahpari, “A DQEM for vibration of shear deformable nonuniform beams with general boundary conditions,” vol. 25, pp. 1169–1178, 2003, doi: 10.1016/S0141-0296(03)00065-8.

[6] K. Torabi and H. Afshari, “Free vibration analysis of a non-uniform cantilever Timoshenko beam with multiple concentrated masses using DQEM Free vibration analysis of a non-uniform cantilever Timoshenko beam with multiple concentrated masses using DQEM,” no. January, 2013, doi: 10.5267/j.esm.2013.06.002.

[7] P. H. Kohan, L. G. Nallim, and S. B. Gea, “Dynamic characterization of beam type structures : Analytical , numerical and experimental applications,” *Appl. Acoust.*, vol. 72, no. 12, pp. 975–981, 2011, doi: 10.1016/j.apacoust.2011.06.007.

[8] J. Wu and B. Chang, “European Journal of Mechanics A / Solids Free vibration of axial-loaded multi-step Timoshenko beam carrying arbitrary concentrated elements using continuous-mass transfer matrix method,” *Eur. J. Mech. / A Solids*, vol. 38, pp.

20–37, 2013, doi: 10.1016/j.euromechsol.2012.08.003.

[9] K. Torabi, H. Afshari, and H. Najafi, "Vibration analysis of multi-step Bernoulli-Euler and Timoshenko beams carrying concentrated masses," *J. Solid Mech.*, vol. 5, no. 4, pp. 336–349, 2013.

[10] S. Engineering, "Differential transform method and numerical assembly technique for free vibration analysis of the axial-loaded Timoshenk ....," no. February, 2015, doi: 10.12989/sem.2015.53.3.537.

[11] S. H. Farghaly and T. A. El-sayed, "Exact free vibration of multi-step Timoshenko beam system with several attachments," *Mech. Syst. Signal Process.*, vol. 72–73, pp. 525–546, 2016, doi: 10.1016/j.ymssp.2015.11.025.

[12] R. Benamar, M. M. K. Bennouna, and R. G. White, "The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic structures part I: Simply supported and clamped-clamped beams," *J. Sound Vib.*, vol. 149, no. 2, pp. 179–195, Sep. 1991, doi: 10.1016/0022-460X(91)90630-3.

[13] R. Benamar, M. M. K. Bennouna, and R. G. White, "The Effects of Large Vibration Amplitudes on the Mode Shapes and Natural Frequencies of Thin Elastic Structures, Part II: Fully Clamped Rectangular Isotropic Plates," *J. Sound Vib.*, vol. 164, no. 2, pp. 295–316, Jun. 1993, doi: 10.1006/jsvi.1993.1215.

[14] L. Azrar, R. Benamar, and R. G. White, "Semi-analytical approach to the non-linear dynamic response problem of s–s and c–c beams at large vibration amplitudes part i: general theory and application to the single mode approach to free and forced vibration analysis," *J. Sound Vib.*, vol. 224, no. 2, pp. 183–207, Jul. 1999, doi: 10.1006/jsvi.1998.1893.

[15] K. El Bikri, R. Benamar, and M. Bennouna, "Geometrically non-linear free vibrations of clamped simply supported rectangular plates. Part I: The effects of large vibration amplitudes on the fundamental mode shape," *Comput. Struct.*, vol. 81, no. 20, pp. 2029–2043, 2003, doi: 10.1016/S0045-7949(03)00152-4.

[16] M. EL KADIRI, R. BENAMAR, and R. G. WHITE, "Improvement of the semi-analytical method, for determining the geometrically non-linear response of thin straight structures. Part i: application to clamped–clamped and simply supported–clamped beams," *J. Sound Vib.*, vol. 249, no. 2, pp. 263–305, Jan. 2002, doi: 10.1006/jsvi.2001.3808.

[17] E. bekkaye Merrimi, K. El bikri, and R. Benamar, "Geometrically non-linear steady state periodic forced response of a clamped-clamped beam with an edge open crack," *Comptes Rendus - Mec.*, vol. 339, no. 11, pp. 727–742, 2011, doi: 10.1016/j.crme.2011.07.008.

[18] H. Fakhreddine, A. Adri, S. Rifai, and R. Benamar, "A Multimode Approach to Geometrically Non-linear Forced Vibrations of Euler–Bernoulli Multispan Beams," *J. Vib. Eng. Technol.*, vol. 8, no. 2, pp. 319–326, 2020, doi: 10.1007/s42417-019-00139-8.

[19] O. Outassafte, A. Adri, Y. El Khouddar, S. Rifai, and R. Benamar, "Geometrically Non-Linear Free In-Plane Vibration Of Circular Arch Elastically Restrained Against Rotation At The Two Ends," *Int. J. Eng. Trends Technol.*, vol. 69, no. 3, pp. 85–95, Mar. 2021, doi: 10.14445/22315381/IJETT-V69I3P215.

[20] K. K. Raju and G. Venkateswara Rao, "Non-linear vibrations of beams carrying a concentrated mass," *J. Sound Vib.*, vol. 48, no. 3, pp. 445–449, Oct. 1976, doi: 10.1016/0022-460X(76)90073-0.