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ANALYTICAL/NUMERICAL SOLUTIONS TO THE PROBLEM OF THE DYNAMIC RESPONSE OF AN ELASTIC PLATE ON A CONTINUOUSLY NON-HOMOGENEOUS CROSS-ANISOTROPIC VISCOELASTIC SOIL TO A MOVING LOAD

Niki D. Beskou[1] and Edmond V. Muho[2]

[1]Department of Civil Engineering, University of the Peloponnese
Patras 26334, Greece
email: nikidiane@gmail.com

[2]Department of Disaster Mitigation for Structures, College of Civil Engineering, Tongji University
200092 Shanghai, China
email: edmondmuho@gmail.com

Abstract

In this work the problem of the dynamic response of a flexural elastic plate on a continuously non-homogeneous viscoelastic cross-anisotropic half-plane or half-space soil medium to a line or rectangular load moving with constant speed is determined analytically/numerically. Soil non-homogeneity is associated with elastic moduli increasing with depth. Plate viscous damping is considered, and the viscoelastic effects of the soil are introduced via hysteric damping. The solution of the problem is obtained with the aid of the complex Fourier series method involving the horizontal coordinates and the time. Employment of this method reduces the partial differential equations of motion for the soil and the plate to ordinary differential equations with variable coefficients and an algebraic equation, respectively. Those ordinary differential equations are solved by the method of Frobenius. Verification of the solutions is done by means of comparisons with known existing analytical solutions for the special cases of isotropy and homogeneity with or without the plate.

Keywords: Moving loads, dynamic response, elastic plate, soil non-homogeneity, Soil cross-anisotropy.

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1 INTRODUCTION

In this work, the dynamic response of a flexural elastic plate to a distributed load moving with constant speed is determined analytically. The plate is supported by a cross-anisotropic, non-homogeneous, viscoelastic half-plane or half-space soil medium. Soil viscoelasticity is considered in order to take into account dissipation of energy [1]. In the context of linear dynamic analysis, viscoelastic material behavior is usually modeled in the frequency domain by replacing the elastic moduli by their complex counterparts, which are functions of hysteretic damping [2]. Soil anisotropy is considered in order to take into account different moduli values along different directions. In soils, the simple but realistic model of transverse isotropy (or cross-anisotropy) is considered with elastic moduli being different in horizontal and vertical planes due to the soil layers deposition process [3-5]. Soil nonhomogeneity is characterized by different physical properties (density, elastic moduli) at different depths because of the soil overburden. It can be modeled in a discrete manner by layers or in a continuous manner by assuming density and/or elastic moduli to vary continuously with depth. In the latter case, it is usually assumed that density is constant and the elastic moduli vary non-linearly with depth [6-12].

Use is made of the complex Fourier series method involving the horizontal coordinate and the time to reduce the governing equations of motion of the plate-soil system to algebraic and ordinary differential equations, respectively. The latter equations because they have variable coefficients are solved by the method of Frobenius.

The following literature review is restricted to analytical works closely related to the present work. One can place those works in the following categories:

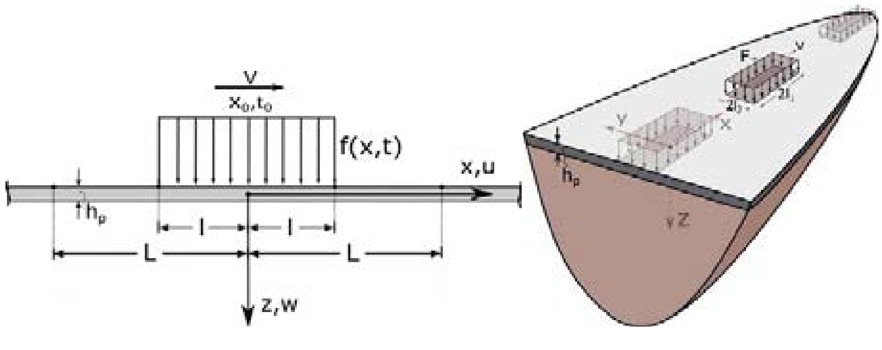
1) Works associated with a homogeneous or layered, isotropic elastic half-space under point or distributed, constant or time harmonic loads moving with constant speed, e.g., [13-22].
2) Works associated with elastic beams or plates on a homogeneous or layered, isotropic half-space under point or distributed, constant or time harmonic loads moving with constants speed, e.g., [23-26].
3) Works associated with or without plates on a homogeneous or layered, transversely isotropic (or cross-anisotropic) elastic half-spaces under point or distributed, constant or time harmonic loads moving with constant speed, e.g., [27-28].
4) Works associated with homogeneous, transversely isotropic (or cross-anisotropic) elastic half-spaces under dynamic but stationary loads, e.g. [29-31].
5) Works associated with an isotropic, continuously non-homogeneous elastic half-space or half-plane under dynamic stationary or moving with constant speed [32-34].
6) Works associated with a transversely isotropic (or cross-anisotropic), continuously non-homogeneous, viscoelastic half-space [35,36] under stationary dynamic.

From the above, one can easily conclude that the only works dealing with all three features of soil behavior (viscoelasticity, anisotropy and continuous nonhomogeneity) are [35, 36] and the present work. However, works [35, 36] are associated with stationary and not moving loads as it is the case with the present work. In addition, no plate effect is considered in [35, 36] as in the present work. Furthermore, use is made here of the complex Fourier series method, which is able, because the load speed is constant, to handle both the horizontal coordinate and the time using a single summation and provide a closed form solution in series form. Most of all the other methods used in the aforementioned works employ integral transform methods (mainly Fourier and Hankel transforms) and this requires numerical integration or complicated analytical inversion techniques instead of simple series evaluation in the present case. This complex Fourier series method has been successfully used by Siddharthan et al [37] and Theodorakopoulos [38] in connection with moving loads on poroelastic half-planes and Beskou et al [39] and Muho and Beskou [40] in connection with moving loads on anisotropic homogeneous half-plane and isotropic nonhomogeneous half-plane, respectively.

The obtained analytical solution of the problem considered here is first verified against other existing analytical solutions pertaining to simpler special cases.

2 PROBLEM DESCRIPTION

Consider a homogeneous isotropic, elastic, thin flexural plate, supported by a cross-anisotropic (or transversely isotropic) nonhomogeneous viscoelastic half-plane or half-space soil medium under conditions of plane strain, as depicted in Fig. 1 in the framework of a Cartesian system x and z or x,y,z, respectively. On the surface of the plate, a uniformly distributed load of intensity F over a rectangle with sides $2l_1$ along the x direction or $2l_1$ and $2l_2$ along the x and y directions, respectively, is assumed to move with constant velocity V along the x-direction, as shown in Fig.1.



(a) (b)

Figure 1: A rectangular distributed load moving with constant speed V on a plate resting on a transversely isotropic viscoelastic (a) half-plane or (b) half-space with variable with depth moduli.

The equations of motion of the half-plane soil medium read as [41]

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \ddot{u} \tag{1}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \ddot{w} \tag{2}$$

and for the half-space as [34]

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho\, \ddot{u} \tag{3}$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} = \rho\, \ddot{v} \tag{4}$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = \rho\, \ddot{w} \tag{5}$$

where, ρ is the soil mass density, $u = u(x, y, z, t)$, $v = v(x, y, z, t)$ and $w = w(x, y, z, t)$ are the soil displacements along the x, y and z directions, respectively, and overdots denote differentiation with respect to time t.

The equation of lateral motion of the plate for the half-plane case has the form [42]

$$D\frac{\partial^4 w_p}{\partial x^4} + m_p\frac{\partial^2 w_p}{\partial t^2} + c_p\frac{\partial w_p}{\partial t} = f(x, t) - q(x, t) \tag{6}$$

where $w_p = w_p(x,t)$ is the lateral plate deflection, m_p is the mass per unit length of the plate, c_p is the viscous damping coefficient of the plate, q(x,t) is the interaction of the soil on the plate and D is the flexural rigidity of the plate defined as

$$D = E_p\, h_p^3/12(1 - \nu_p^2) \tag{7}$$

with h_p being the thickness of the plate and E_p and ν_p the modulus of elasticity and Poisson's ratio of the plate, respectively. For the half-space case Eq. (6) is replaced by

$$D\,\nabla^4 w_p + m_p\ddot{w}_p + c_p\dot{w}_p += f(x, y, t) - q(x, y, t) \tag{8}$$

where, $\nabla^4 = \frac{\partial^4}{\partial_x^4} + \frac{2\,\partial^4}{\partial_x^4\,\partial_y^4} + \frac{\partial^4}{\partial_y^4}$ is the biharmonic differential operator.

The soil stresses are expressible in terms of displacements for the present anisotropic and nonhomogeneous half-plane case as [41]

$$\sigma_{xx} = c_{11}(z)\frac{\partial u}{\partial x} + c_{13}(z)\frac{\partial w}{\partial z} \tag{9}$$

$$\sigma_{zz} = c_{13}(z)\frac{\partial u}{\partial x} + c_{33}(z)\frac{\partial w}{\partial z} \tag{10}$$

$$\sigma_{xz} = c_{44}(z)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{11}$$

and for the half-space as [34]

$$\sigma_{xx} = c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} \tag{12}$$

$$\sigma_{yy} = c_{12}\frac{\partial u}{\partial x} + c_{11}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} \tag{13}$$

$$\sigma_{zz} = c_{13}\frac{\partial u}{\partial x} + c_{13}\frac{\partial v}{\partial y} + c_{33}\frac{\partial w}{\partial z} \tag{14}$$

$$\sigma_{yz} = c_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{15}$$

$$\sigma_{xz} = c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{16}$$

$$\sigma_{xy} = c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{17}$$

for the half-space soil medium [34].

In the above equations, $c_{11}, c_{12}, c_{13}, c_{33}, c_{44}$ and c_{66} are elastic constants which are expressible in terms of the engineering constants as [34]

$$c_{11} = \frac{E\left(E' - Ev'^2\right)}{(1+v)\left(E' - E'v - 2Ev'^2\right)} \tag{18}$$

$$c_{12} = \frac{E\left(E'v + Ev'^2\right)}{(1+v)\left(E' - E'v - 2Ev'^2\right)} \tag{19}$$

$$c_{13} = \frac{EE'v'}{E' - E'v - 2Ev'^2} \tag{20}$$

$$c_{33} = \frac{E'^2(1-v)}{E' - E'v - 2Ev'^2} \tag{21}$$

$$c_{44} = G' \tag{22}$$

$$c_{66} = G = \frac{c_{11} - c_{12}}{2} = \frac{E}{2(1+v)} \tag{23}$$

where, E, G, and v denote elastic modulus, shear modulus, and Poisson's ratio, respectively and the primes refer to the vertical axis of rotational material symmetry, which coincides with the z-axis of Fig.1. Those E and G without prime refer to the horizontal (x, y) plane of isotropy, while it has been assumed here that the value of the Poisson's ratio is the same along the vertical and horizontal direction.

Substitution of Eqs. (9)-(11) into Eqs. (1) and (2) for the half-plane model or Eqs. (12)-(17) into (3) and (4) for the half-space model, results in the governing equations of motion of the soil medium in terms of displacements of the form

$$c_{11}\frac{\partial^2 u}{\partial x^2} + c_{44}\frac{\partial^2 u}{\partial z^2} + c_{44}'\frac{\partial u}{\partial z} + (c_{13} + c_{44})\frac{\partial^2 w}{\partial x \partial z} + c_{44}'\frac{\partial w}{\partial x} = \rho\ddot{u} \tag{24}$$

$$c_{33}\frac{\partial^2 w}{\partial z^2} + c_{44}\frac{\partial^2 w}{\partial x^2} + c_{33}'\frac{\partial w}{\partial z} + (c_{13} + c_{44})\frac{\partial^2 u}{\partial x \partial z} + c_{13}'\frac{\partial u}{\partial x} = \rho\ddot{w} \tag{25}$$

for the half-plane and

$$c_{11}\frac{\partial^2 u}{\partial x^2} + c_{66}\frac{\partial^2 u}{\partial y^2} + c_{44}\frac{\partial^2 u}{\partial z^2} + c_{44}'\frac{\partial u}{\partial z} + (c_{12} + c_{66})\frac{\partial^2 v}{\partial x\,\partial y} + (c_{13} + c_{44})\frac{\partial^2 w}{\partial x\,\partial z} + c_{44}'\frac{\partial w}{\partial x} = \rho\,\ddot{u} \tag{26}$$

$$c_{66}\frac{\partial^2 v}{\partial x^2} + c_{11}\frac{\partial^2 v}{\partial y^2} + c_{44}\frac{\partial^2 v}{\partial z^2} + c_{44}'\frac{\partial v}{\partial z} + (c_{12} + c_{66})\frac{\partial^2 u}{\partial x\,\partial y} + (c_{13} + c_{44})\frac{\partial^2 w}{\partial y\,\partial z} + c_{44}'\frac{\partial w}{\partial y} = \rho\,\ddot{v} \tag{27}$$

$$c_{44}\frac{\partial^2 w}{\partial x^2}+c_{44}\frac{\partial^2 w}{\partial y^2}+c_{33}\frac{\partial^2 w}{\partial z^2}+c'_{33}\frac{\partial w}{\partial z}+(c_{13}+c_{44})\frac{\partial^2 u}{\partial x\,\partial z}+(c_{13}+c_{44})\frac{\partial^2 v}{\partial y\,\partial z} + c'_{13}\frac{\partial u}{\partial x}+c'_{13}\frac{\partial v}{\partial y}=\rho\,\ddot{w} \tag{28}$$

for the half-space, respectively, where primes on $c_{ij} = c_{ij}(z)$ denote differentiation with respect to the depth z.

The boundary conditions of the half-plane case consist of dynamic equilibrium and compatibility at the plate-soil interface (where smooth contact is assumed) of the form

$$\sigma_{xz}(x,0,t)=0 \tag{29}$$

$$\sigma_{zz}(x,0,t)=-q(x,t) \tag{30}$$

$$w(x,0,t)=w_p(x,t) \tag{31}$$

The corresponding boundary conditions for the half-space case have the form

$$\sigma_{xz}(x,y,0,t)=0 \tag{32}$$

$$\sigma_{yz}(x,y,0,t)=0 \tag{33}$$

$$\sigma_{zz}(x,y,0,t)=-q(x,y,t) \tag{34}$$

$$w(x,y,0,t)=w_p(x,y,t) \tag{35}$$

Finally, at a depth $z = H$ approaching infinity, all soil displacements become zero due to the radiation conditions at infinity and one has

$$u(x,H,t)=0 \text{ for half-plane or } u(x,y,H,t)=0 \text{ for half-space} \tag{36}$$

$$v(x,y,H,t)=0 \text{ for half-space} \tag{37}$$

$$w(x,H,t)=0 \text{ for half-plane or } w(x,y,H,t)=0 \text{ for half-space} \tag{38}$$

Viscoelasticity in the soil medium is considered in the frequency domain, utilized during the solution procedure of the next section, by replacing the elastic constants c_{ij} by their complex counterparts ${c_{ij}}^*$according to the relation [2]

$$c^*_{ij}=c_{ij}(1+2i\xi) \tag{39}$$

resulting from the correspondence principle, where ξ is the hysteretic damping coefficient and i=$\sqrt{-1}$.

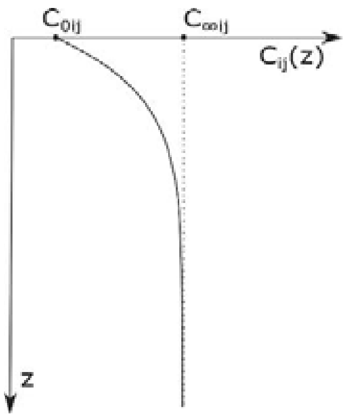


Figure 2: Soil elastic constants $c_{i,j}$ increasing with depth z

The nonhomogeneity of the soil is associated here only with a variability of the elastic constants with depth, i.e., $c_{ij} = c_{ij}(z)$. It is assumed here that these constants increase with depth z in accordance with the relation

$$c_{ij}(z) = c_{0ij} + (c_{\infty ij} - c_{0ij})(1 - e^{-az}) \tag{40}$$

due to Vrettos [7, 8] as shown in Fig. 2, which is based on tests and provides bounded values of c_{ij} at infinity. In the above, c_{0ij}, $c_{\infty ij}$ and $c_{ij}(z)$ are elastic constants at the soil surface, at an infinite depth and at depth z from the soil surface, respectively, and α is a constant with dimensions of inverse length.

3 SOLUTION PROCEDURE

The problem to be solved consists of the governing partial differential equation of motion of the plate (Eq. (6) or Eq. (8)) and the governing partial differential equations of motion of the soil (Eqs. (24), (25) or (26)-(28)) subject to the boundary conditions (29)-(31) or (32)-(35). The solution procedure consists of two major steps: i) reduction of the partial differential equations of the plate-soil system to an algebraic and ordinary differential equations, respectively, by using the complex Fourier series approach [37-40] and ii) solution of the ordinary differential equations with variable coefficients for the soil by the method of Frobenius [43] and employment of the boundary conditions to obtain the final response of the plate-soil system to the moving load.

Thus, using the method of complex Fourier series, the load f(x,t)=f(x-Vt) is expanded in series of the form [39, 40]

$$f(x - Vt) = Re \sum_{n=0}^{N} F_n e^{i\lambda_n (x - Vt)} \tag{41}$$

where the load amplitude F_n is given by

$$F_n = \begin{cases} \dfrac{Fl}{L}, & for\ n = 0 \\ 2\dfrac{F}{n\,\pi}\sin\left(\dfrac{n\pi l}{L}\right), & for\ n = 1,2,\ldots,N \to \infty \end{cases} \tag{42a}$$

for the half-plane model or

$$F_{nm}=\begin{cases} F\dfrac{l_1 l_2}{L_1 L_2} & \text{for } n=m=0 \\ 2F\dfrac{l_2}{n\pi L_2}\sin\dfrac{n\pi l_1}{L_1} & \text{for } m=0, n=1,2,\dots,N \\ 2F\dfrac{l_1}{m\pi L_1}\sin\dfrac{m\pi l_2}{L_2} & \text{for } n=0, m=1,2,\dots,M \\ 4F\dfrac{1}{nm\pi^2}\sin\dfrac{n\pi l_1}{L_1}\sin\dfrac{m\pi l_2}{L_2} & \text{for } n=1,2,\dots,N, m=1,2,\dots,M \end{cases} \tag{42b}$$

for the half-space model, respectively.

In Eqs. (41) and (42) and with reference Fig.1, λ_n=nπ/L are the wavenumber parameters, 2l is the length of the distributed load, 2L is the wavelength of the system, Re denotes the real part and i=$\sqrt{-1}$. Because the plate-soil system is linear, its response to the moving load f(x, t) will be of the form

$$w_p(w,t)=Re\sum_{n=0}^{N}W_{pn}e^{i\lambda_n(x-Vt)} \tag{43a}$$

$$u(x,z,t)=Re\sum_{n=0}^{N}U_n(z)e^{i\lambda_n(x-Vt)} \tag{44a}$$

$$w(x,z,t)=Re\sum_{n=0}^{N}W_n(z)e^{i\lambda_n(x-Vt)} \tag{45a}$$

$$q(x,z,t)=Re\sum_{n=0}^{N}Q_n e^{i\lambda_n(x-Vt)} \tag{46a}$$

for the half-plane model or

$$w_p(w,t)=Re\sum_{n=0}^{N}\sum_{m=0}^{M}W_{pnm}e^{i\lambda_n(x-Vt)}e^{i\mu_m y} \tag{43b}$$

$$u(x,z,t)=Re\sum_{n=0}^{N}\sum_{m=0}^{M}U_{nm}e^{i\lambda_n(x-Vt)}e^{i\mu_m y} \tag{44b}$$

$$v(x,z,t)=Re\sum_{n=0}^{N}\sum_{m=0}^{M}V_{nm}e^{i\lambda_n(x-Vt)}e^{i\mu_m y} \tag{44c}$$

$$w(x,z,t) = Re\sum_{n=0}^{N}\sum_{m=0}^{M} W_{nm} e^{i\lambda_n(x-Vt)} e^{i\mu_m y} \tag{45b}$$

$$q(x,z,t) = Re\sum_{n=0}^{N}\sum_{m=0}^{M} Q_{nm} e^{i\lambda_n(x-Vt)} e^{i\mu_m y} \tag{46b}$$

for the half-space model, respectively, where W_p, U, V, W and Q denote response amplitudes to be determined and n, m=1,2,…,N,M→ ∞.

In order to avoid numerical problems associated with n=0 or m=0 implying $\lambda_n=0$ or $\mu_m=0$, respectively, solutions can be derived for each of the following four cases of the load amplitudes of Eq. (42), i.e., i) n=m=0, ii) m=0, n>0, iii) n=0, m>0 and iv) n>0, m>0. Only the solution for n>0, m>0 will be derived here. Solutions for the other three cases can be very easily determined following the same procedure.

Substituting Eqs. (43)-(46) in Eqs. (6), (24), (25) or (26)-(28), omitting the summation symbol and the common factors $e^{i\lambda_n(x-Vt)}$ and introducing the proposed by Vrettos [7, 8] dimensionless parameters

$$\begin{gathered}\frac{\lambda_n^2}{a^2} = \beta, \quad \frac{\lambda_n}{a} = -\gamma, \quad \frac{\lambda_n^2 \rho V^2}{a^2 c_{\infty 44}} = \theta \\ \frac{c_{\infty 11}}{c_{\infty 44}} = \delta_1, \quad \frac{c_{\infty 13}}{c_{\infty 44}} = \delta_2, \quad \frac{c_{\infty 33}}{c_{\infty 44}} = \delta_3,\end{gathered} \tag{47}$$

for the half-plane or

$$\begin{gathered}\frac{\lambda_n^2}{\alpha^2} = \beta_1, \frac{\mu_m}{\alpha^2} = \beta_2, \frac{\lambda_n}{\alpha} = -\gamma_1, \frac{\mu_m}{\alpha} = \gamma_2 \\ \frac{\rho\lambda_n^2 V^2}{\alpha^2 c_{\infty 44}} = \theta, \frac{c_{\infty 11}}{c_{\infty 44}} = \delta_1, \frac{c_{\infty 13}}{c_{\infty 44}} = \delta_2, \frac{c_{\infty 33}}{c_{\infty 44}} = \delta_3, \frac{c_{\infty 12}}{c_{\infty 44}} = \delta_4, \frac{c_{\infty 66}}{c_{\infty 44}} = \delta_5\end{gathered} \tag{48}$$

for the half space model, respectively, one finally receives the form

$$\begin{aligned}\zeta^2(1-\zeta)U_n'' + \zeta(1-2\zeta)U_n' + [\theta - \beta\delta_1(1-\zeta)]U_n + i\gamma\zeta(1-\zeta)(1+\delta_2)W_n' \\ - i\gamma\zeta W_n = 0\end{aligned} \tag{49}$$

$$\begin{aligned}\delta_3\zeta^2(1-\zeta)W_n'' + \delta_3\zeta(1-2\zeta)W_n' + [\theta - \beta(1-\zeta)]W_n + i\gamma\zeta(1-\zeta)(1+\delta_2)U_n' \\ - i\gamma\zeta\delta_2 U_n = 0\end{aligned} \tag{50}$$

for the half-plane or

$$\begin{aligned}(1-\zeta)\zeta^2 U_{nm}'' + \zeta(1-2\zeta)U_{nm}' + [\theta - (1-\zeta)(\beta_1\delta_1 + \beta_2\delta_5)]U_{nm} \\ - (\delta_4+\delta_5)\gamma_1\gamma_2(1-\zeta)V_{nm} + (1+\delta_2)i\gamma_1\zeta(1-\zeta)W_{nm}' - i\gamma_1\zeta W_{nm} \\ = 0\end{aligned} \tag{51}$$

$$\begin{aligned}(1-\zeta)\zeta^2 V_{nm}'' + \zeta(1-2\zeta)V_{nm}' + [\theta - (1-\zeta)(\beta_1\delta_5 + \beta_2\delta_1)]V_{nm} \\ - (\delta_4+\delta_5)\gamma_1\gamma_2(1-\zeta)U_{nm} + (1+\delta_2)i\gamma_2\zeta(1-\zeta)W_{nm}' - i\gamma_2\zeta W_{nm} \\ = 0\end{aligned} \tag{52}$$

$$\begin{aligned}\delta_3(1-\zeta)\zeta^2 W_{nm}'' + \delta_3\zeta(1-2\zeta)W_{nm}' + [\theta - (1-\zeta)(\beta_1+\beta_2)]W_{nm} \\ + (1+\delta_5)i\gamma_1\zeta(1-\zeta)U_{nm}' - \delta_2 i\gamma_1\zeta U_{nm} + (1+\delta_2)i\gamma_2\zeta(1-\zeta)V_{nm}' \\ - \delta_2 i\gamma_2\zeta V_{nm} = 0\end{aligned} \tag{53}$$

for the half-space model, respectively, where primes indicate differentiation with respect to ζ.

The above system of two or three coupled ordinary differential equations with variable coefficients is solved by the method of Frobenius [43].

A power series solution is assumed of the form

$$U_n(\zeta) = \sum_{r=0}^{R} a_r \zeta^{r+m} \tag{54}$$

$$W_n(\zeta) = \sum_{r=0}^{R} b_r \zeta^{r+m} \tag{55}$$

for the half-plane or

$$U_{nm} = \sum_{r=0}^{\infty} a_r \zeta^{r+k} \tag{56}$$

$$V_{nm} = \sum_{r=0}^{\infty} b_r \zeta^{r+k} \tag{57}$$

$$W_{nm} = \sum_{r=0}^{\infty} c_r \zeta^{r+k} \tag{58}$$

for the half-space model, respectively, where α_r, b_r and m or a_r, b_r, c_r and k are constants to be determined and r=0,1,2,…,R→ ∞. Substitution of the expressions (54) and (55) for U_n and W_n or (56)-(58) for U_{nm} , V_{nm} and W_{nm}, respectively, into Eqs. (49) and (50) or (51)-(53) after some manipulations results in the equations

$$\begin{aligned} ([m^2 + (\theta - \beta\delta_1)]\alpha_0 + [i\gamma(1+\delta_2)m]b_0)\zeta^m + \sum_{r=1}^{R}[(r+m)^2 + (\theta - \beta\delta_1)]\alpha_r\zeta^{r+m} \\ + \sum_{r=1}^{R}[i\gamma(1+\delta_2)(r+m)]b_r\zeta^{r+m} \\ - \sum_{r=1}^{R}[(r+m)(r+m-1) - \beta\delta_1]\alpha_{r-1}\zeta^{r+m} \\ - \sum_{r=1}^{R}[i\gamma(1+\delta_2)(r+m-1) + i\gamma]b_{r-1}\zeta^{r+m} = 0 \end{aligned} \tag{59}$$

$$\begin{aligned}
&([\delta_3 m^2 + (\theta-\beta)]b_0 + [i\gamma(1+\delta_2)m]a_0)\zeta^m + \sum_{r=1}^{R}[\delta_3(r+m)^2 + (\theta-\beta)]b_r\zeta^{r+m} \\
&\qquad + \sum_{r=1}^{R}[i\gamma(1+\delta_2)(r+m)]a_r\zeta^{r+m} \\
&\qquad - \sum_{r=1}^{R}[\delta_3(r+m)(r+m-1)-\beta]b_{r-1}\zeta^{r+m} \\
&\qquad - \sum_{r=1}^{R}[i\gamma(1+\delta_2)(r+m-1)+i\gamma\delta_2]a_{r-1}\zeta^{r+m} = 0
\end{aligned} \tag{60}$$

for the half-plane or

$$\begin{aligned}
&([k^2+\theta-(\beta_1\delta_1+\beta_2\delta_5)]a_0 - [(\delta_4+\delta_5)\gamma_1\gamma_2]b_0 + [(1+\delta_2)i\gamma_1 k]c_0)\zeta^k \\
&\qquad + \left(\sum_{r=1}^{\infty}[(r+k)^2+\theta-(\beta_1\delta_1+\beta_2\delta_5)]a_r - \sum_{r=1}^{\infty}[(\delta_4+\delta_5)\gamma_1\gamma_2]b_r\right. \\
&\qquad \left. + \sum_{r=1}^{\infty}[(1+\delta_2)i\gamma_1(r+k)]c_r\right)\zeta^{r+k} \\
&\qquad - \left(\sum_{r=1}^{\infty}[(r+k-1)(r+k)-(\beta_1\delta_1+\beta_2\delta_5)]a_{r-1}\right. \\
&\qquad - \sum_{r=1}^{\infty}[(\delta_4+\delta_5)\gamma_1\gamma_2]b_{r-1} \\
&\qquad \left. + \sum_{r=1}^{\infty}[(1+\delta_2)i\gamma_1(r+k-1)+i\gamma_1]c_{r-1}\right)\zeta^{r+k} = 0
\end{aligned} \tag{61}$$

$$\begin{aligned}
&([k^2+\theta-(\beta_1\delta_5+\beta_2\delta_1)]b_0 - [(\delta_4+\delta_5)\gamma_1\gamma_2]a_0 + [(1+\delta_2)i\gamma_2 k]c_0)\zeta^k \\
&\qquad + \left(\sum_{r=1}^{\infty}[(r+k)^2+\theta-(\beta_1\delta_5+\beta_2\delta_1)]b_r - \sum_{r=1}^{\infty}[(\delta_4+\delta_5)\gamma_1\gamma_2]a_r\right. \\
&\qquad \left. + \sum_{r=1}^{\infty}[(1+\delta_2)i\gamma_2(r+k)]c_r\right)\zeta^{r+k} \\
&\qquad - \left(\sum_{r=1}^{\infty}[(r+k-1)(r+k)-(\beta_1\delta_5+\beta_2\delta_1)]b_{r-1}\right. \\
&\qquad - \sum_{r=1}^{\infty}[(\delta_4+\delta_5)\gamma_1\gamma_2]a_{r-1} \\
&\qquad \left. + \sum_{r=1}^{\infty}[(1+\delta_2)i\gamma_2(r+k-1)+i\gamma_2]c_{r-1}\right)\zeta^{r+k} = 0
\end{aligned} \tag{62}$$

$$\begin{aligned}
&([\delta_3 k^2+\theta-(\beta_1+\beta_2)]c_0-[(1+\delta_2)i\gamma_1 k]a_0+[(1+\delta_2)i\gamma_2 k]b_0)\zeta^k \\
&\quad+\left(\sum_{r=1}^{\infty}[\delta_3(r+k)^2+\theta-(\beta_1+\beta_2)]c_r+\sum_{r=1}^{\infty}[(1+\delta_2)i\gamma_1(r+k)]a_r\right. \\
&\quad\left.+\sum_{r=1}^{\infty}[(1+\delta_2)i\gamma_2(r+k)]b_r\right)\zeta^{r+k} \\
&\quad-\left(\sum_{r=1}^{\infty}[\delta_3(r+k-1)(r+k)-(\beta_1+\beta_2)]c_{r-1}\right. \\
&\quad-\sum_{r=1}^{\infty}i\gamma_1[(r+k)(1+\delta_2)-1]a_{r-1} \\
&\quad\left.+\sum_{r=1}^{\infty}i\gamma_2[(r+k)(1+\delta_2)-1]b_{r-1}\right)\zeta^{r+k}=0
\end{aligned} \tag{63}$$

for the half-space model, respectively.

For the case of the half-plane model, equations (59) and (60) are satisfied for all powers of ζ and hence one obtains a system to determine α_0 and b_0 of the form

$$[m^2+(\theta-\beta\delta_1)]\alpha_0+[i\gamma(1+\delta_2)m]b_0=0 \tag{64}$$

$$[i\gamma(1+\delta_2)m]\alpha_0+[\delta_3 m^2+(\theta-\beta)]b_0=0 \tag{65}$$

and a system of recursive relations to determine α_r and b_r in terms of α_{r-1} and b_{r-1} $(r>0)$ of the form

$$[(r+m)^2+(\theta-\beta\delta_1)]\alpha_r+[i\gamma(1+\delta_2)(r+m)]b_r=F_1 \tag{66}$$

$$[i\gamma(1+\delta_2)(r+m)]\alpha_r+[\delta_3(r+m)^2+(\theta-\beta)]b_r=F_2 \tag{67}$$

where

$$F_1=[(r+m)(r+m-1)-\beta\delta_1]\alpha_{r-1}+[i\gamma(1+\delta_2)(r+m-1)+1]b_{r-1} \tag{68}$$

$$F_1=i\gamma[(1+\delta_2)(r+m-1)+\delta_2]\alpha_{r-1}+[\delta_3(r+m)(r+m-1)-\beta]b_{r-1} \tag{69}$$

For the case of half-space model, equations (61)-(63) are satisfied for all powers of ζ and hence one obtains a system to determine α_0 and b_0 of the form

$$[k^2+\theta-(\beta_1\delta_1+\beta_2\delta_5)]a_0-[(\delta_4+\delta_5)\gamma_1\gamma_2]b_0+[(1+\delta_2)i\gamma_1 k]c_0=0 \tag{70}$$

$$-[(\delta_4+\delta_5)\gamma_1\gamma_2]a_0+[k^2+\theta-(\beta_1\delta_5+\beta_2\delta_1)]b_0+[(1+\delta_2)i\gamma_2 k]c_0=0 \tag{71}$$

$$[(1+\delta_2)i\gamma_1 k]a_0+[(1+\delta_2)i\gamma_2 k]b_0+[\delta_3 k^2+\theta-(\beta_1+\beta_2)]c_0=0 \tag{72}$$

and system of recursive relations to determine a_r, b_r and c_r in terms of a_{r-1}, b_{r-1} and c_{r-1} of the form

$$\begin{aligned}[(r+k)^2+\theta-(\beta_1\delta_1+\beta_2\delta_5)]a_r-[(\delta_4+\delta_5)\gamma_1\gamma_2]b_r+[(1+\delta_2)i\gamma_1(r+k)]c_r \\ =F_1\end{aligned} \tag{73}$$

$$-[(\delta_4+\delta_5)\gamma_1\gamma_2]a_r+[(r+k)^2+\theta-(\beta_1\delta_5+\beta_2\delta_1)]b_r+[(1+\delta_2)i\gamma_2(r+k)]c_r = F_2 \quad (74)$$

$$[(1+\delta_2)i\gamma_1 k]a_r+[(1+\delta_2)i\gamma_2(r+k)]b_r+[\delta_3(r+k)^2+\theta-(\beta_1+\beta_2)]c_r=F_3 \quad (75)$$

where

$$F_1=[(r+k-1)(r+k)-(\beta_1\delta_1+\beta_2\delta_5)]a_{r-1}-[(\delta_4+\delta_5)\gamma_1\gamma_2]b_{r-1} + i\gamma_1[(1+\delta_2)(r+k-1)+1]c_{r-1} \quad (76)$$

$$F_2=-[(\delta_4+\delta_5)\gamma_1\gamma_2]a_{r-1}+[(r+k-1)(r+k)-(\beta_1\delta_5+\beta_2\delta_1)]b_{r-1} + i\gamma_2[(1+\delta_2)(r+k-1)-1]c_{r-1} \quad (77)$$

$$F_3=i\gamma_1[(r+k)(1+\delta_2)-1]a_{r-1}+i\gamma_2[(r+k)(1+\delta_2)-1]b_{r-1} + [\delta_3(r+k-1)(r+k)-(\beta_1+\beta_2)]c_{r-1} \quad (78)$$

In order for the system of Eqs. (64) and (65) or (70)-(72) to have nonzero solutions, one should have

$$Am^4+Bm^2+C=0 \quad (79)$$

for the half-plane or

$$A\,k^6+B\,k^4+C\,k^2+J=0 \quad (80)$$

for the half-space model, respectively, where A, B, C, and J are functions of the dimensionless parameters of Eq. (47) or (48). One can observe that out of the four or six roots of Eq. (79) or (80), respectively, only the two or the three positive ones are acceptable.

Thus, in view of Eqs. (54), (55) or (56)-(58), the complete solution of Eqs. (49) and (50) or Eqs. (51) – (53) has the form

$$U_n(\zeta)=A_1U_n^{(1)}(\zeta)+A_2U_n^{(2)}(\zeta) \quad (81)$$

$$W_n(\zeta)=A_1W_n^{(1)}(\zeta)+A_2W_n^{(2)}(\zeta) \quad (82)$$

for the half-plane or

$$U_{nm}(\zeta)=A_1U_{nm}^{(1)}(\zeta)+A_2U_{nm}^{(2)}(\zeta)+A_3U_{nm}^{(3)}(\zeta) \quad (83)$$

$$V_{nm}(\zeta)=A_1V_{nm}^{(1)}(\zeta)+A_2V_{nm}^{(2)}(\zeta)+A_3V_{nm}^{(3)}(\zeta) \quad (84)$$

$$W_{nm}(\zeta)=A_1W_{nm}^{(1)}(\zeta)+A_2W_{nm}^{(2)}(\zeta)+A_3W_{nm}^{(3)}(\zeta) \quad (85)$$

for the half-space model, respectively, where A_1, A_2 and A_3 are constants of integration to be determined by the boundary conditions (29)-(31) and (32)-(35) where

$$U_n^{(i)}(\zeta)=\sum_{r=0}^{R}a_r\zeta^{r+m_i} \quad (86)$$

$$W_n^{(i)}(\zeta)=\sum_{r=0}^{R}b_r\zeta^{r+m_i} \quad (87)$$

$$\mathrm{U}_{\mathrm{nm}}^{(\mathrm{i})}(\zeta) = \sum_{\mathrm{r}=0}^{\infty} \mathrm{a}_{\mathrm{r}} \zeta^{\mathrm{r}+\mathrm{k}_{\mathrm{i}}} \tag{88}$$

$$\mathrm{V}_{\mathrm{nm}}^{(\mathrm{i})}(\zeta) = \sum_{\mathrm{r}=0}^{\infty} \mathrm{b}_{\mathrm{r}} \zeta^{\mathrm{r}+\mathrm{k}_{\mathrm{i}}} \tag{89}$$

$$\mathrm{W}_{\mathrm{nm}}^{(\mathrm{i})}(\zeta) = \sum_{\mathrm{r}=0}^{\infty} \mathrm{c}_{\mathrm{r}} \zeta^{\mathrm{r}+\mathrm{k}_{\mathrm{i}}} \tag{90}$$

with i=1,2, (or i=1,2,3) for the two (or three) roots of m (or k) given by Eq. (79) or Eq. (80), the α_r and b_r and c_r given recursively from the solution of Eqs. (66) and (67) or respectively (73) – (75) and α_0 and b_0 and c_0 obtained by assuming a_0=1 and solving Eq. (64) for b_0 or respectively the system of Eqs. (70) and (71) for b_0 and c_0.

Thus, the soil response solution is given by Eqs. (44) and (45) and the plate response solution is by Eq. (43) for both the half-plane and half-space cases. Once the plate deflection w_p is known, one can obtain the plate bending moment M_{xx} and shear force Q_x from [42]

$$M_{xx} = -D\frac{\partial^2 w_p}{\partial x^2} \tag{91}$$

$$Q_x = -D\frac{\partial^3 w_p}{\partial x^3} \tag{92}$$

for the half-plane or

$$M_{xx} = -D\left(\frac{\partial^2 w_p}{\partial x^2} + \upsilon_p \frac{\partial^2 w_p}{\partial y^2}\right) \tag{93}$$

$$M_{yy} = -D\left(\frac{\partial^2 w_p}{\partial y^2} + \upsilon_p \frac{\partial^2 w_p}{\partial x^2}\right) \tag{94}$$

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w_p}{\partial x^2} + \frac{\partial^2 w_p}{\partial y^2}\right) \tag{95}$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w_p}{\partial x^2} + \frac{\partial^2 w_p}{\partial y^2}\right) \tag{96}$$

for the half-space model, respectively.

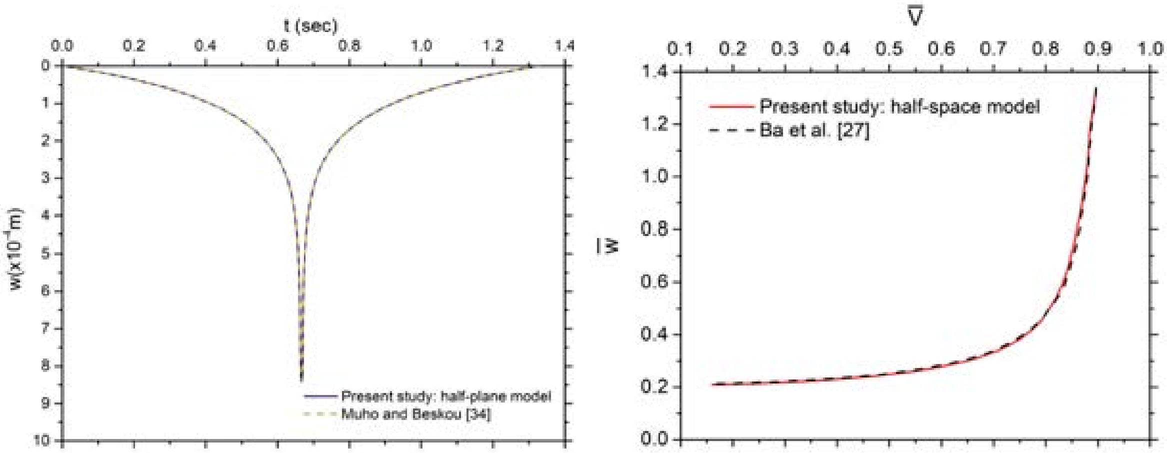
4 VERIFICATION AND CONVERGENCE STUDIES

In this final section, the accuracy of the proposed method is validated through comparison with two special cases for which there exist solutions in the literature involving homogeneous soil with and without a plate on its surface. The homogeneous soil is approximated by the proposed method by assigning a very small value to Ξ, e.g., $\Xi = 10^{-4}$.

For the first verification it is assumed a load P=80 kN distributed on a line-length *l*=0.15 m, which implies a value of F=266.66 kN/m, moving with constant speed V=40 m/s on the surface of a half-plane isotropic soil medium. For this soil medium one has mass density $\rho = 1816\,\text{kg/m}^3$, Poisson's ratio $v = 0.35$ and shear modulus varying with depth according to Eq. (40) with G_0=0.077 GPa (or $E_0 = 2G_0(1+v)$) with parameters $\Xi = 0.5$ and $\alpha = 1\, m^{-1}$. The absence of the plate was simulated by giving very small values to the plate parameters. Other parameters used here are L=80 m and x=L/3. Using the present solution, the soil surface vertical displacement w versus time t was calculated and the results are given in Fig. 3a together with those from [34]. One can clearly see in that figure that the two solutions coincide.

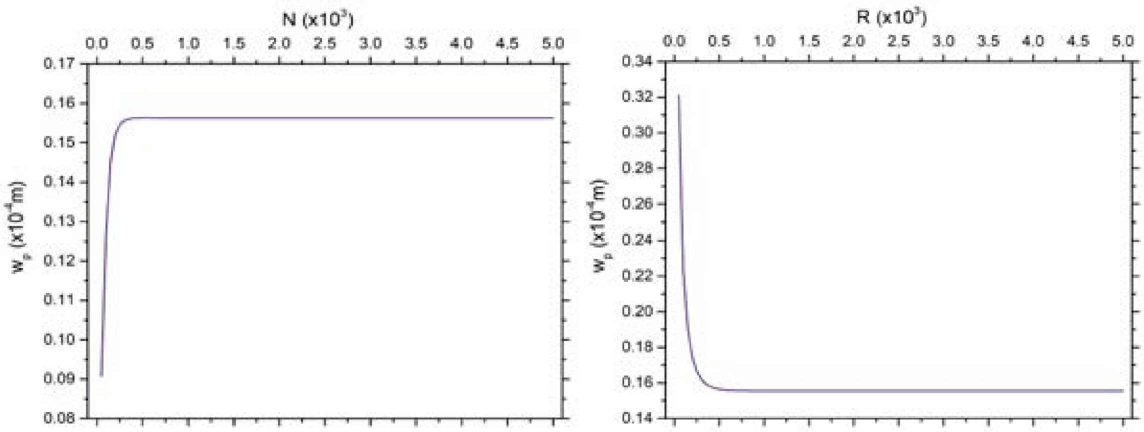
For the second verification, the problem of a load moving on the surface of a homogeneous cross-anisotropic half-space medium is solved by the proposed method and compared with the solution of Ba et al [27]. Thus, assuming the same values with Ba et al [27], i.e., E=5 GPa, E'=2E, v=v'=0.25, $G = E/2(1+v)$, G'=0.3 E' and ρ=2000 kg/m^3, one can obtain the normalized vertical displacement $\overline{w}=G_1 z/P$ at a depth z=10 m as a function of the normalized load speed $\overline{V}= V/\sqrt{G'/\rho}$ as shown in Fig. 3b. The concentrated load P was modeled by the proposed method assuming a very small rectangular area of $l_1 = l_2 = 0.015$ m, while the absence of the plate is approximated by the proposed method by assigning a very small value to h_p, e.g., $h_p = 10^{-4}$ m. Finally, the solution is obtained assuming L_1=L_2=75 m, $N = M = 512$, and $r = 5$. One can observe in Fig. 3b (a) that the response obtained by the proposed method coincides with the solution of Ba et al [24].

Next, a convergence study with respect to the required values of N and R in Eqs. (43a) and (55), respectively, or N, M, and r in Eqs. (43b) and (58), respectively, is presented, where the maximum vertical displacement of the plate at the center of the load versus N, M, and r is shown, respectively. One can observe from Figs. 4 and 5 that values of N, M >400 are sufficient to obtain solutions with high accuracy. As far as the required maximum value of r is concerned, it should be noted here that this value depends on the non-homogeneity index Ξ. Thus, for higher values of Ξ, higher values of r are required. It has been observed that for very small values of Ξ, e.g., homogeneous soil, a value of r ≥ 5 is sufficient. In order to determine the highest possible required value of r in this convergence study, a very high value of non-homogeneity index is assumed, i.e., $\Xi = 0.99$ and as it is shown in Figs. 4b and 5c, values of r ≥ 250 are sufficient to obtain results of high accuracy.



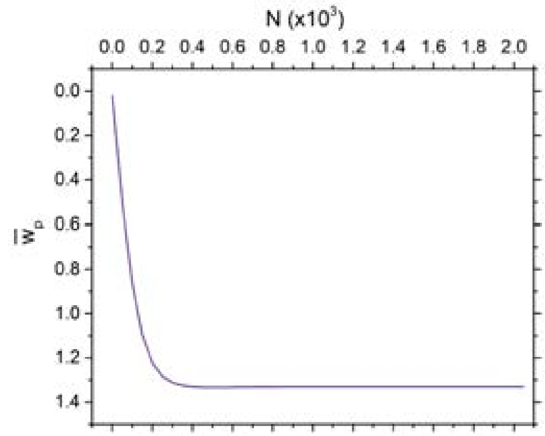
(a) (b)

Figure 3: Use of the present solution for the special cases of (a) a nonhomogeneous isotropic half-plane without plate of [34] and (b) the homogeneous and cross-anisotropic soil of Ba et al [27].

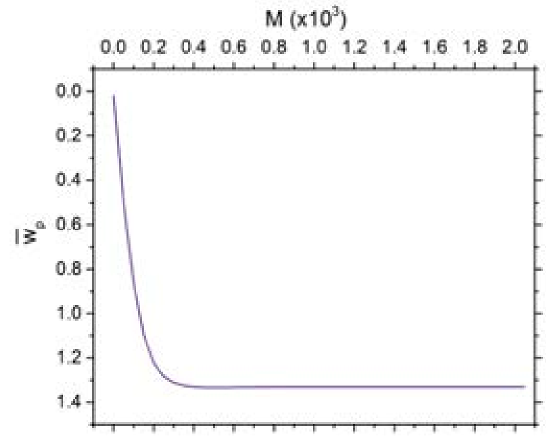


(a) (b)

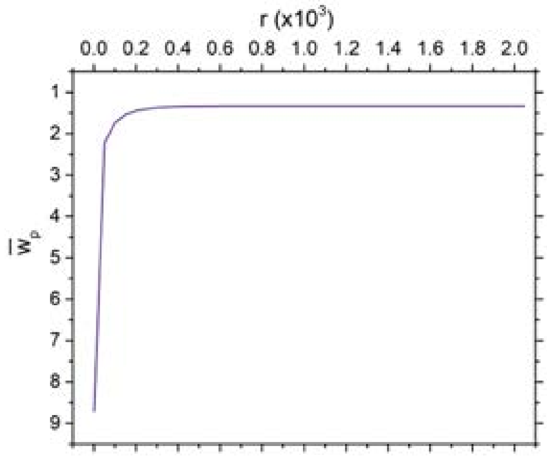
Figure 4: Convergence analysis: Maximum plate vertical displacement w_p versus (a) the number N in Eq. (43a) and (b) the number R in Eq. (55).



(a)



(b)



(c)

Figure 5: Convergence analysis: Maximum plate vertical displacement w_p versus (a) the number N, (b) the number M in Eq. (43b) and (c) the number r in Eq. (58).

Finally, a comparison of the results produced by the half-plane and half-space models is presented. Both models share the same material properties for the soil (E=0.2 GPa, E′=E, v=v′=0.35, G′=0.30 E′, ρ=2100 kg/m^3 and Ξ=0.01) and the plate (E_p=30 GPa, ρ_p=2300 kg/m^3, v_p=0.2 and h_p=0.2 m) and are associated with the same data for the moving load (P=80 kN, l_1=l_2=0.15 m and V=80 m/s). One can observe in Fig. 6 (a) that the half-space model with the plate results in much smaller maximum vertical displacement at the surface (z=0 m) compared to that of the half-plane model with the plate. The same above problem is solved again assuming very small values for the plate parameters in order to model the absence of the plate and the response is shown in Fig. 6 (b). One can observe that in this case, both models provide similar maximum vertical displacement with the half-plane model resulting in a slightly higher displacement. Furthermore, it is interesting to observe that in both cases the half-plane model results in a much wider displacement-time response shape than the half-space one and clearly provides higher displacement values than those from the half-space model away from the area of the load application.

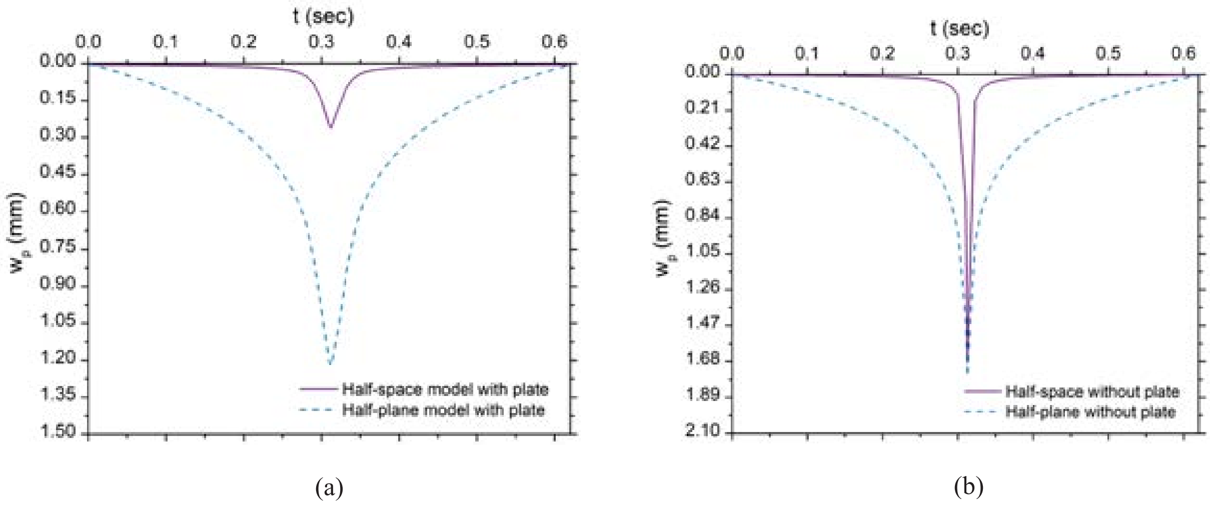


Figure 6: Comparison of the present half-space with the half-plane (a) considering the plate and (b) without the plate.

5 CONCLUSIONS

1. The problem of a distributed load moving with constant speed on the surface of an elastic plate resting on a non-homogeneous and cross-anisotropic viscoelastic half-plane or half-space has been analytically treated by using the complex Fourier series method.
2. The proposed solutions were verified for the two special cases of a load acting on a homogeneous cross-anisotropic elastic half-plane or half-plane.
3. A comparison of the proposed half-space and half-plane models has shown that both models result in a similar maximum vertical displacement only for the case of the half-plane model without the plate and under the load where the half-plane displacement has a slightly higher value. However, the half-plane model without the plate clearly shows higher displacements than those of the half-space far from the area of the load application. On the other hand, in case there is a plate, the half-space model results in a much lower vertical response compared to that of the half-plane one.

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