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A NEW PHYSICS-GUIDED DATA ASSIMILATION FRAMEWORK FOR ONLINE STRUCTURAL MONITORING: APPLICATION TO SHAKING-TABLE TESTS

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Abstract. *The modified Constitutive Relation Error (mCRE) is a model updating functional whose physics-based construction avoids the explicit dependency into a priori user's expertise. Its robustness to measurement noise and remarkable convexity properties make it a credible alternative to classical model updating methods. In recent works, the authors developed an automated mCRE-based model updating framework for the offline update of finite element models of structures submitted to ground motion inputs, before deriving it to data assimilation with the development of the Modified Dual Kalman Filter (MDKF). In the latter, we addressed sequential data assimilation by integrating the mCRE as new observer within a dual Kalman filter. Contrary to classical Kalman filters, the comparison between measurements and model predictions is achieved through the mCRE functional itself, in which the data-to-model distance is enriched with a model error term with strong mechanical content (the CRE).*

In this contribution, we focus on MDKF numerical improvements to achieve real-time data assimilation for structural health monitoring applications where an extensive number of parameters may need to be tracked with a few amount of data, which makes the identification process ill-posed and prone to instabilities. Using (i) parallelized mCRE computation and sigma-points propagation, (ii) the Scaled Spherical Simplex Kalman filter structure and (iii) a CRE-based clustering pre-step, CPU requirements are reduced by partial state update without loss of accuracy. The MDKF methodology and the novel key ingredients for efficient (near-optimal) data assimilation are illustrated throughout (simulated) earthquake engineering examples of growing complexity.

Keywords: Data assimilation, Kalman filtering, modified Constitutive Relation Error, Shaking-table experiments, Earthquake Engineering.

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1 Introduction

1.1 Earthquake engineering context

Earthquake engineering problems are no exception to the need of using experimental data to build, validate and operate robust numerical models, in particular when it comes to monitor and predict the structural state of civil engineering structures [5]. Such a data-to-model interaction is commonly referred to as digital twin [25] or Dynamic Data-Driven Application System (DDDAS) [8] as the updated model somehow impacts the response or the behavior of experimental device. The problem that motivates the present contribution is the control of hydraulic actuators of the CEA/TAMARIS shaking-tables when performing experiments on damaging specimens. If the control of electro-hydraulic shaking-tables for seismic replication is a well-identified issue [23], most of the proposed strategies do not consider the fact that sudden damage occurence in the tested specimen may lead to unstable experiments. Actually, because the actuators, the shaking-table and the tested specimen are physically strongly coupled, the controllability of the full device particularly depends on the modal signature of the specimen. When damage propagates, the modal signature suddenly changes, which may turn the control strategy inappropriate, the test unstable, and put the safety of the facility at risk. To avoid such situations, an alternative lies in the above-mentioned DDDAS paradigm in which a close dialog between the experimental device and a digital twin is performed in real-time (see Fig. 1).

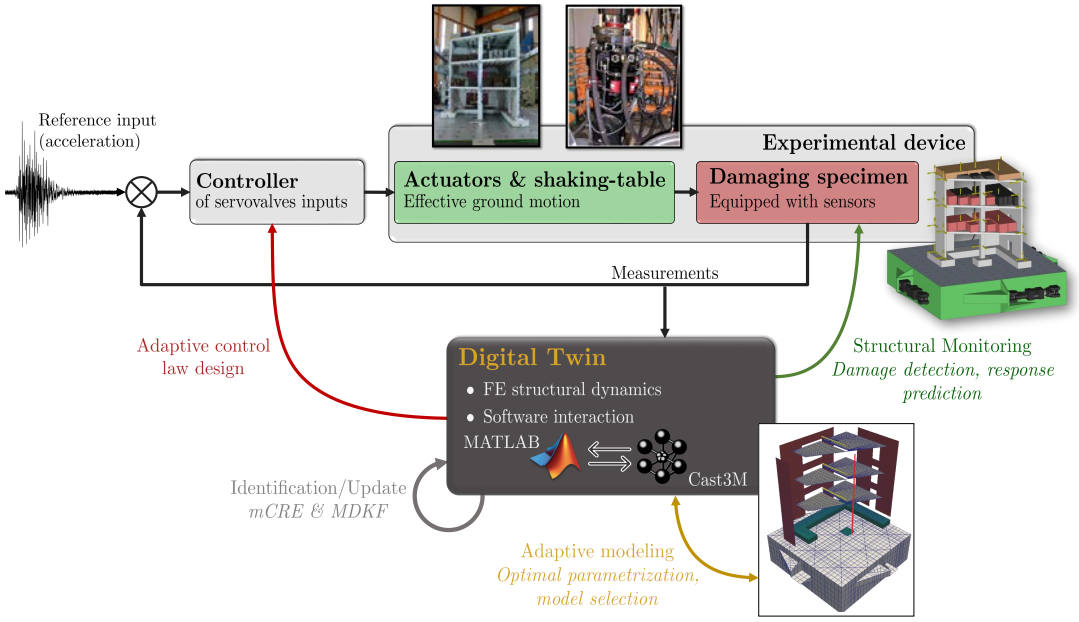


Figure 1: DDDAS framework for enhanced monitoring and control of shaking-table experiments.

1.2 Data assimilation and Kalman filtering

The feasibility of Dynamic Data Driven Application Systems (DDDAS) lies (among others) in efficient data assimilation schemes able to update model predictions from acquired data. In particular, such algorithms must be able to operate in real-time whereas the computational burden carried by recursive calls to model predictions can be prohibitive. With the extended availability of data and the difficulties of validating complex nonlinear full order models, last

decades have seen the emergence of hybrid approaches, often described as *data-driven* techniques. They systematically involve a reduced order model and the correction of model predictions (due to mismodeling or unanticipated phenomena) by a data-based enrichment step. Data assimilation techniques then differ according to the (reduced order) model structure and the way data are integrated and exploited to update model predictions.

For the sake of conciseness, we will mostly focus on sequential data assimilation by Kalman filtering (KF) in this contribution. Initially presented as an optimal linear unbiased quadratic estimator for linear dynamic systems under Gaussian assumptions [15], the KF algorithm is structured as a recursive prediction-correction scheme, where model predictions are sequentially corrected based on newly assimilated measurements. The KF has been massively derived and extended to process nonlinear systems. Among them, the *Extended Kalman Filter* (EKF) is based on the principle of operator linearization [14], whereas the *Unscented Kalman Filter* (UKF) [21] and the *Scaled Spherical Simplex Filter* (S3F) [22] are alternative algorithms based on statistical regularization: these techniques are based on the fact that sampling points (called σ-points) transformed by nonlinear operators allow for a better approximation of state statistics when nonlinearities are hardly linearizable. The *Ensemble Kalman Filter* (EnKF) [13] and the *Particle Filter* (PF) [6] use the same principle with much more propagated sampling-points as they aim to reconstruct the full probability density of the estimated state.

In addition, parameter estimation techniques using the Dual or Joint extensions of nonlinear KFs have also been developed to allow the simultaneous identification of mechanical state and structural parameters.

Of course, all the performance of these different KF have been extensively compared [2, 17, 20], highlighting that UKF is often more efficient than EnKF and PF in terms of CPU time, and is more accurate than EKF for an equivalent computational cost. Finally, it is worth noticing that the recent S3F outperforms UKF: S3F can be used in all applications where UKF can be involved, achieving the exact same numerical performance in terms of accuracy. However, by sampling $n+2$ sigma-points instead of $2n+1$ (n being the state space dimension), S3F preserves all the important features of UKF with the minimal amount of sampling points to propagate and significant CPU time savings (see Fig. 2).

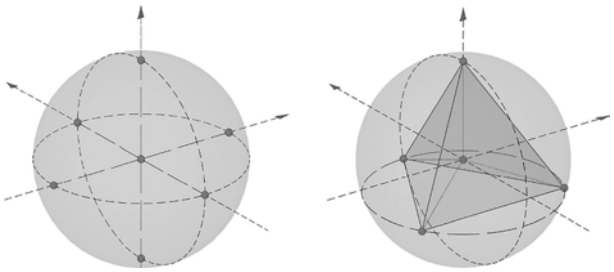


Figure 2: Sigma-points locations around the current mean state estimate ($n = 3$) for UKF (spread over a hypersphere - left) and S3F (corners of a simplex - right) - from [22].

1.3 The modified Constitutive Relation Error

Initially proposed for model updating in dynamics [7, 9] by Ladevèze and co-workers, an alternative to standard deterministic/stochastic model updating techniques consists in using the concept of *modified Constitutive Relation Error* (mCRE). The mCRE functional is defined as a quadratic model-to-measurements distance enriched with a term based on the concept of *Constitutive Relation Error* (CRE) [16]. The idea behind this enrichment is to improve the ellipticity

properties of the functional by adding a term which has a strong mechanical content and relaxes the most unreliable parts of the model description. This energy-based residual offers interesting advantages. First, it improves local convexity properties compared to a classical least-square functional [1]. Besides, the CRE part of the residual, computed over the whole structure, allows to select the most erroneous areas in order to restrain the updating process to a few parameters [11]. The numerous applications of mCRE for model updating during the last two decades have proven its robustness and efficiency throughout a wide range of applications. Recently, a mCRE-based model updating algorithm has been fully automated in low-frequency dynamics, with enhanced robustness with respect to measurement noise and a dedicated formulation when dealing with random inputs [10, 11].

Although dedicated to model updating, this functional has been combined to model order reduction for real-time data assimilation, leading to the Modified Kalman Filter in quasi-statics [18, 19] and to the Modified Dual Kalman Filter (MDKF) in low-frequency dynamics [12].

1.4 Main contributions

In this paper, following the work initiated in [12], we will essentially focus on the Modified Dual Kalman Filter algorithm and enhance its numerical performance. First, we compare and assess several Kalman filters that integrate the mCRE as observation operator in order to exhibit which KF structure is most appropriate for real-time data assimilation, with effort paid to optimize the numerical performance of the MDKF. Despite a well-chosen algorithmic structure, it cannot still be enough to reach real-time constraints when the parameter space under study is of high dimension. It is particularly the case when it comes to detect damage from linear FE models [11]. A solution may lie in using at best the information stored in the Constitutive Relation Error term. We thus propose to add a clustering step in the recursive KF-loop to generate and propagate an optimal number of σ-points at each data assimilation step (in the CRE sense), with corrections focused on parameters that are identified as erroneous.

The remainder of the paper is structured as follows: Section 2 recalls the fundamentals of mCRE and MDKF for FE stiffness model updating in low-frequency dynamics. Improvements for enhanced numerical performance are presented in Section 3 before being compared on an academic earthquake-engineering-inspired example in Section 4. Conclusions and prospects are finally drawn in Section 5.

2 The Modified Dual Kalman Filter for FE stiffness updating in low-frequency dynamics

This section intends to briefly recall the fundamentals of the mCRE-based model updating framework for low-frequency dynamics, starting from a linear finite element (FE) problem written in the frequency domain. The interested reader is respectively referred to [9, 10] and [12] for additional information about mCRE and MDKF.

2.1 FE framework, measurements and stiffness parametrization

Let us consider the general case of an elastic structure Ω spatially discretized in E (non-overlapping) finite elements such that $\Omega = \cup_{e=1}^{E} \Omega_e$ subjected to a given dynamical excitation F. We denote by K, D, M the stiffness, damping and mass FE matrices, respectively. F_ω and U_ω are the frequency counterparts of nodal loading conditions and displacement field. With these notations, the dynamical equilibrium written in the frequency domain at a given angular frequency ω reads:

$$\left[-\omega^2 M + i\omega D + K\right] U_\omega = F_\omega \tag{1}$$

In addition to the dynamic equilibrium, a set of sensors is used to measure the magnitude of certain kinematic quantities (displacement, velocity and/or accelerations). In the frequency domain, assuming measurements are perfect, such information can be written without loss of generality as:

$$\Pi U_\omega = Y_\omega \tag{2}$$

where Y_ω refers to the frequency counterpart of measurements at angular frequency ω and Π denotes a projection matrix that allows to extract the components of U_ω that are measured.

As one can interpret damage as local stiffness loss, a convenient manner to parametrize a linear FE problem for damage detection is to parametrize the stiffness matrix. Doing so, in this contribution, the n_θ parameters to update $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$ only affect the stiffness matrix K. More precisely, the stiffness FE matrix is decomposed in n_θ non-overlapping subdomains and parametrized as follows:

$$K(\theta) = \sum_{i=1}^{n_\theta} \frac{\theta_i}{\theta_{0,i}} K_{0,i} \quad \text{with} \quad K(\theta_0) = \sum_{i=1}^{n_\theta} K_{0,i} \tag{3}$$

Note that the subdomains can perfectly match with finite elements or gather some of them to reduce the number n_θ of parameters to identify.

2.2 The modified CRE: a physics-regularized approach for inverse problems

Originally intended to perform model validation [16], the key idea for the construction of the CRE residual lies into the distinction between reliable and unreliable information on the reference mechanical problem. The mCRE concept extends this distinction to the experimental data as well. Indeed, the redundant information gathered in equations (1) and (2) means all equations cannot be exactly verified. Some of them must be considered less reliable, and thus relaxed as they will be only verified at best by minimizing the so-called mCRE functional.

At a given angular frequency ω, the mCRE for stiffness identification is defined as:

$$\begin{aligned} e_\omega^2(s_\omega, \theta;\, Y_\omega) \;&\triangleq\; \zeta_\omega^2(s_\omega, \theta) \;+\; \frac{\alpha}{2}\left\|\Pi U_\omega - Y_\omega\right\|_G^2 \\ \text{with } \zeta_\omega^2(s_\omega, \theta) = \frac{1}{2}(U_\omega - V_\omega)^H K(\theta)(U_\omega - V_\omega) &= \frac{1}{2}\left\|U_\omega - V_\omega\right\|_{K(\theta)}^2 \text{ and } s_\omega = (U_\omega, V_\omega) \end{aligned} \tag{4}$$

- The CRE $\zeta_\omega^2(s_\omega, \theta)$ measures the closeness of the mechanical displacement fields (U_ω, V_ω) in the sense of the stiffness matrix. U_ω verifies the reliable kinematic equations of the problem (boundary conditions, closeness to measurements as involved in the data-to-model distance) and V_ω is constrained to satisfy the dynamic equilibrium of the problem. In practice, s_ω is obtained by solving an optimization problem for all ω (for a given parameter set θ):

$$s_\omega(\theta;\, Y_\omega) \;=\; \arg\min \left[e_\omega^2(s_\omega, \theta;\, Y_\omega)\right] \text{ s.t. } [-\omega^2 M + i\omega D]U_\omega + K(\theta)V_\omega = F_\omega \tag{5}$$

  which is equivalent to solve the following matrix system:

$$AX = b \text{ with } \begin{cases} A &= \begin{bmatrix} [K(\theta) + i\omega D - \omega^2 M]^H & \frac{r}{1-r}\Pi^H G \Pi \\ -K^H(\theta) & [K(\theta) + i\omega D - \omega^2 M] \end{bmatrix} \\ X &= \begin{Bmatrix} U_\omega - V_\omega & U_\omega \end{Bmatrix}^T \\ b &= \begin{Bmatrix} \frac{r}{1-r}\Pi^H G Y_\omega & F_\omega \end{Bmatrix}^T \end{cases} \tag{6}$$

- The data-to-model distance is a Hermitian norm of the gap between model predictions U_ω with measurements Y_ω. G is a symmetric positive-definite matrix guaranteeing that $\|\square\|_G$ is homogeneous to ζ_ω^2 and equivalent in level. The choice of G is not critical: herein, G is taken proportional to the identity matrix and weighted by the strain energy stored in the first (involved) eigenmodes. The tuning factor $\alpha \in \mathbb{R}^+$ enables one to give more or less confidence to the measurements; large values can be specified when measurements are considered reliable whereas close-to-zero values are better suited to corrupted or noisy recordings. The choice of α is therefore crucial for providing relevant parameter estimates. Its optimal calibration has been deeply studied in [10, 11], with attention paid to its dependency into the measurement noise level if known.

As the model updating procedure is conducted on a given frequency bandwidth D_ω which contains the essential information about the response of the structure, the mCRE functional $\mathcal{J}$ to be minimized is thus obtained by direct integration over D_ω:

$$\mathcal{J}(\theta,\, Y) = \int_{D_\omega} z(\omega) e_\omega^2(s(\theta,\, Y_\omega), \theta,\, Y_\omega)\, \mathrm{d}\omega \tag{7}$$

where $z(\omega)$ is a frequency weighting function such that $\int_{D_\omega} z(\omega)\, \mathrm{d}\omega = 1$ allowing to modulate the importance of specific frequencies of D_ω. Indeed, a frequency weighting function derived from the experimental frequency content enables to automatically favor the vicinity of the experimental eigenfrequencies; this enhances the robustness of the functional to measurement noise in low-frequency dynamics [10].

2.3 The MDKF: integration of the mCRE as a nonlinear observer

As previously mentioned in the introduction, nonlinear KFs are relevant tools to update on-the-fly FE models of dynamical (and evolutive) systems. They permit to track and identify accurately structural parameters from sparse and noisy measurements. The starting point of the KF theory is the definition of a dynamical system under state-space discrete form. Considering the evolution with time of structural parameters alongside the physical state of the system, the full dynamical system reads:

$$\begin{cases} \theta_{k+1} &= \theta_k + w_{\theta,k} \\ x_{k+1} &= \mathcal{M}_k(x_k, \theta_k, e_k) + w_{x,k} \\ y_{k+1} &= \mathcal{H}_{k+1}(x_{k+1}, e_{k+1}) + v_{k+1} \end{cases} \tag{8}$$

In terms of notations, $x_k \in \mathbb{R}^{n_x}$ is the state vector at time point t_k. y_k is the vector of observations. $w_x \sim \mathcal{N}(0, Q_x)$ and $v \sim \mathcal{N}(0, R)$ are zero-mean Gaussian white-noise accounting for additive model and measurement errors respectively. The dynamical system is based on two (possibly nonlinear) operators, namely the model operator $\mathcal{M}$ and the observation operator $\mathcal{H}$. The latter classically extracts partial information regarding the predicted state x_k or inputs e_k. Without any *a priori* knowledge on parameters $\theta_k \in \mathbb{R}^{n_\theta}$, one can formulate a simple stationary evolution law. As it remains a restrictive hypothesis (*e.g.* damage is not), the stationarity assumption is then relaxed with the addition of a last zero-mean Gaussian white-noise $w_{\theta,k} \sim \mathcal{N}(0, Q_\theta)$ so as to model uncertainty on parameters and enable their evolution during the data assimilation process.

Although the common definition of a projection matrix as observer seems rather intuitive since sensors directly collect measurements to be compared to model predictions, the choice

of the observation metric (and thus the way measurements are processed) can be reconsidered for enhanced robustness with respect to measurement noise. From a dual KF viewpoint, one can choose to replace the dual observation operator (classically being a state prediction Kalman filter composed of the last two equations of (8)) with another functional able to quantify the closeness between model predictions and assimilated measurements.

The Modified Dual Kalman Filter (MDKF) derives the mCRE as new observation operator in a dual Kalman filter framework [12]. Practically, in a similar manner as one would compute optimal parameters minimizing the functional $\mathcal{J}$ from (7), the observation equation of the MDKF will reinvest the mCRE gradient $\nabla_\theta \mathcal{J}$ in order to guarantee the stationarity of the cost function. Please note that no additional numerical error is made with the call to the mCRE gradient as its analytical expression can be explicitly derived from the constrained minimization problem (6) (see *e.g.* [4] for complementary details). The MDKF dynamical system thus reads:

$$\begin{cases} \theta_{k+1} &= \theta_k + w_{\theta,k} \\ 0 &= \nabla_\theta \mathcal{J}(\theta_k,\, Y_k) + v_k \end{cases} \tag{9}$$

This new framework thus differs from classical KFs as measurements are indirectly compared with model predictions through the mCRE functional. Parameter estimates are then sought as minimizers of the mCRE (according to current measurements). The doubt put on the mCRE gradient (with the observation noise v) then quantifies the authorized proximity of estimates to the optimal set of parameters that minimizes the mCRE at each time step.

2.3.1 Algorithmic subtleties about MDKF

The use of MDKF raises several questions, in particular regarding its robustness with respect to measurement noise, the calibration of internal parameters and the achievable numerical performance.

It is worth noticing that the coupling between mCRE and dual Kalman filtering avoids the calibration of process and measurement covariance matrices for state estimation. However, mCRE internal parameters should be well-tuned if one intends to obtain relevant data assimilation results. It can be realistically assumed that the mCRE functional can be calibrated using the rules given in [11] after a first training stage involving a low-level non-damaging random input before performing data assimilation. Besides, the selection of filtering parameters, *i.e.* noise covariance matrices, is made easier by several user-friendly guidelines that have been proposed and validated to calibrate Q_θ and R in [12].

If complete explanations are available in [12], let us finally recall that MDKF implies time-frequency domains nested interaction when mixing sequential data assimilation (in time) and mCRE (written in the frequency domain). Compared to classical KF approaches, updating parameters for all new data points does not seem relevant as the observer operates on data in the frequency domain. A sliding window technique is thus proposed for handling the time-frequency nested interaction. The overlapping rate between two consecutive windows defines the amount of new assimilated data at each time step. Note that it also fully conditions the real-time prospects of the MDKF algorithm to the extent that one considers data is assimilated in real-time if the necessary CPU time per iteration is lower than the time between two consecutive data assimilation time steps.

3 MDKF improvements for real-time data assimilation

To prove the relevance of MDKF to perform data assimilation efficiently, the last point to address concerns the numerical performance, in particular when the parameter set θ and/or the number of degrees-of-freedom of the FE problem become important. This section is dedicated to present several improvements that allowed to enhance numerical performance of MDKF.

3.1 Efficient mCRE computation

The most consuming operations of the MDKF algorithm are the frequency domain data preprocessing and the computation of the mCRE for each set of σ-points and frequency of D_ω. Even if the call to Fast Fourier Transforms is unavoidable, several actions can be made to reduce the computational burden due to mCRE computations:

- The computation of the mCRE value for all σ-points per iteration can be drastically shortened using reduced basis to decrease the size of the system (6) [9, 10].
- The gradient of the mCRE functional $\mathcal{J}$ with respect to parameters can be expressed analytically and obtained after a low-cost post-processing operation once system (6) is solved. It can be shown that:

$$\forall k \in [\![1; n_\theta]\!],\ \frac{\mathrm{d}e_\omega^2}{\mathrm{d}\theta_k} = \frac{1}{2}\left[U_r^T K_{0,k} U_r + U_i^T K_{0,k} U_i - V_r^T K_{0,k} V_r - V_i^T K_{0,k} V_i\right] \tag{10}$$

  with $U_r = \Re(U_\omega), U_i = \Im(U_\omega), V_r = \Re(V_\omega)$, and $V_i = \Im(V_\omega)$.
- Parallelization techniques can be used to propagate σ-points so as to overcome the computational burden of sampling-based KF, whose CPU time significantly increases with the dimension of the parameter vector to update. In particular, the solution of system (6) for all frequencies in D_ω can be parallelized easily using new in-core page-wise left matrix divide function `pagemldivide` (available since MATLAB© 2022a release).

3.2 Nonlinear Kalman filter structure

The system (9) is not constrained to be used in an UKF framework as originally presented in [12]. As mentioned earlier, the S3F is an alternative sigma-point Kalman Filter that has proved to have the same numerical performance than UKF with almost 50% CPU time reduction. Indeed, the latter samples only $(n_\theta + 2)$ σ-points whereas UKF needs $(2n_\theta + 1)$. Fundamentally, the S3F and UKF algorithms do only differ in the way σ-points (and associated weights) are declared. The algorithmic modifications from UKF to S3F are therefore limited to a few code lines.

Besides, UKF was preferred to EKF in the initial MDKF due to its better accuracy in higher order statistical moments. However, from the computational viewpoint, EKF avoids the (costly) propagation of σ-points, which implies it should be quicker without being much inaccurate as the analytical expression of the mCRE gradient is available.

MDKF based on dual EKF, dual UKF and dual S3F structures will be compared in terms of accuracy and CPU time in the next section of this contribution.

3.3 CRE-based clustering for partial state-update

The CPU performance of sampling-based KFs remains directly proportional to n_θ. Therefore, the processing of data in real-time may not be possible in cases where $n_\theta \gg 1$, which can be the case in realistic monitoring applications on large-scale structures. A possible way

to improve numerical performance without loss of accuracy is to use the full potential of the mCRE within the MDKF. The CRE term provides a direct insight regarding the validity of the model itself, making it a relevant tool for identifying erroneous parts of the model as all finite element contributions to CRE can be computed independently. If stiffness parameters affect group of elements (*i.e.* subdomains of the FE model), the CRE per subdomain S_i then reads:

$$\forall\, i \in [\![1; n_\theta]\!], \quad \zeta^2_{\omega,i}(s_\omega, \theta) = \sum_{e \subset S_i} \frac{1}{2} \left\| U_\omega - V_\omega \right\|^2_{K_e(\theta)} \tag{11}$$

Thus, if the CRE can be a convenient local model indicator, one can imagine exploiting it so that only most erroneous parts of the model should be corrected. To do so in an automated way, a CRE-based clustering technique can be used. Briefly, clustering techniques allow to partition a space in subsets (the so-called clusters) in the sense of a given distance: within a cluster, the distance between objects of a cluster remains small, whereas a distance between objects of different clusters is larger (see [24] for further details). Among the wide range of clustering techniques, a fuzzy k-means clustering algorithm developed in [3] is reinvested in this work. Originally intended for separating the physical modes from the spurious modes produced by a growing model-order identification algorithm in modal analysis, its robustness with respect to non-convex, poorly dissociated and heterogeneous clusters makes it a relevant tool to define parameter distributions from the CRE map. Starting from the initial model, the key idea is to identify a cluster of most erroneous elements through the computation of the model error (CRE) map per element which naturally emphasizes on the damaged area if the initial parameter guess is well determined. In other words, we perform a two-cluster distinction from

$$X = \left\{ \int_{D_\omega} z(\omega) \zeta^2_{\omega,i} \mathrm{d}\omega \right\}_{i=1}^{n_\theta} \tag{12}$$

and forthcoming correcting actions can be focused on the cluster of most erroneous elements in the CRE sense. Note that this is not a computationally expensive procedure as it is a (vectorizable) post-processing operation once mechanical fields solution of (6) has been obtained. Note that in an offline model updating context, it was shown that the clustering step works as a form of regularization (in the Tikhonov sense) of the mCRE functional. An illustration of the CRE-based clustering potentialities to update a large number of parameters from a few amount of acceleration data is given in Fig. 3.

As a complement, the pseudo-code of the MDKF coupled to the CRE-based clustering is given in Alg. 1.

4 Application to stiffness parameter tracking from sparse accelerometer data

4.1 Plane frame submitted to random ground motion

In this section, we intend to validate and compare MDKF structures (EKF, UKF, S3F) and the benefits of the CRE-based clustering approach with a typical earthquake engineering academic example. Let us consider the plane frame structure shown in Fig. 4, whose stiffness distribution is assumed unknown. This structure is clamped to a rigid moving support (*e.g.* a shaking-table) and submitted to a 120 s random bi-axial ground motion input. The reference stiffness field presents a defect that suddenly appears in the wall W10 while the initial guess does not perfectly correspond to the healthy reference configuration (10% underestimation). The FE model is made of elastic beam elements and built with the CEA FE simulation software

Algorithm 1: Clustered Modified Dual Kalman Filter (for a sampling-based KF structure)

Initialization:

- Sliding window, data assimilation time steps $\{t_k\}$
- mCRE tuning parameters: $D_\omega, \alpha, z(\omega)$ and reduced basis
- Initial statistics on parameters: mean $\widehat{\theta}_0$ and covariance P_0^θ
- Noise process and measurements covariance matrices: Q_θ, R

for $k = 0 : \infty$ **do**

0) **CRE-based clustering**

Computes the CRE map at the current estimate: $\zeta_i^2(\widehat{\theta}_k)\ \forall\, i \in [\![1; n_\theta]\!]$
Identification of a restricted subset in need of correction $\theta_{R,k},\ R \subset [\![1; n_\theta]\!]$

1) **Generation of samples**

Computation of σ-points for the restricted parameter set $\theta_{R,k}$

2) **Prediction step**

Computation of *a priori* mean and covariance for $\theta_{R,k}$

3) **Processing new data in the frequency domain**

Extraction of the new data block with the sliding window: $y_k(t)$
Fast Fourier transform for mCRE analysis: $Y_k(\omega)\ \forall \omega \in D_\omega$

4) **Correction step**

Propagation of the σ-points through the mCRE functional and gradient computation
Correction of predictions to compute *a posteriori* statistics of θ_R
The statistics of $\theta_{R^C,k+1}$ remain unchanged compared to $\theta_{R^C,k}$.

end

Result: Successive estimates of the parameter vector statistics $(\widehat{\theta}_k, P_k^\theta)$

CASTEM. Following (3), the updated stiffness FE matrix is decomposed in $n_\theta = 6$ or $n_\theta = 12$ non-overlapping subdomains. They follow an intuitive wall/slab decomposition (see Fig. 4).

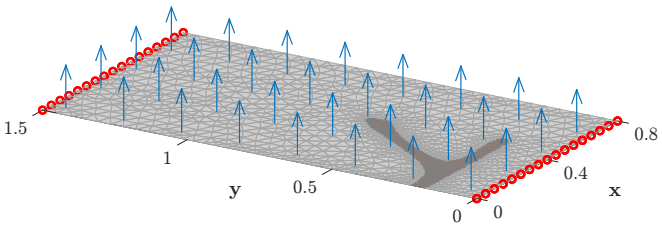
In order to assess the robustness of the data assimilation algorithms with respect to measurement noise, a white noise of known standard deviation is added to the synthetic measurements. The noise level δ (in $\%$) is then defined according to the magnitude of the input ground acceleration $\ddot{U}_d$ such that noisy synthetic acceleration data is obtained as follows:

$$\ddot{y}_{noisy}(t) = \ddot{y}(t) + \delta\ \text{std}\left(\ddot{U}_d(t)\right)\eta(t) \tag{13}$$

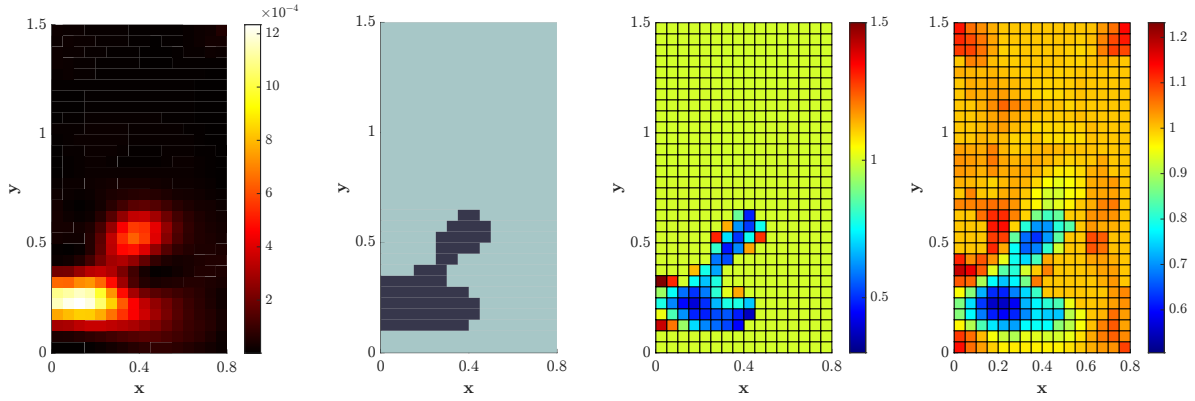
where $\eta(t)$ is a random Gaussian vector of zero mean and unitary standard deviation.

4.2 Comparative study between mCRE-based Kalman filters - procedure, results and discussion

The first objective of this study is to assess which of nonlinear KF structures (UKF, EKF, S3F) is the most appropriate to perform stiffness parameter tracking with MDKF accurately in real-time from discrete acceleration measurements. Various levels of noise are considered $\delta = \{0; 5; 15\}\%$ to assess the robustness to measurement noise. For the sake of conciseness and clarity, we will only present in the forthcoming figures the probability density function

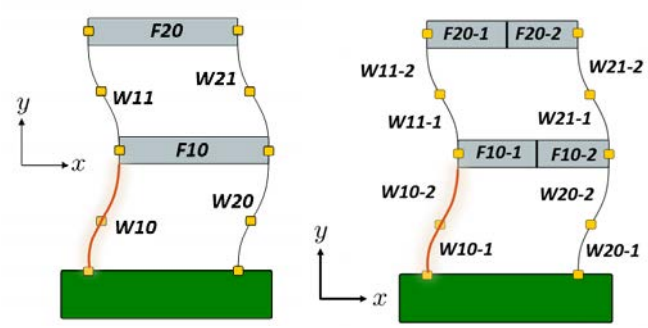


(a) Plane wall with damage defect to identify from sparse acceleration measurements.

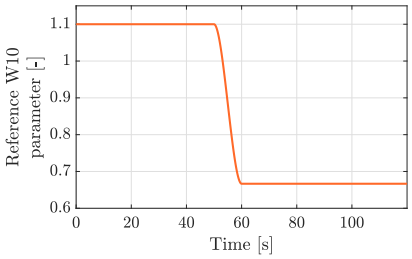


(b) Initial CRE map. (c) CRE-based clustering. (d) Parameter estimate. (e) Parameter estimate without clustering.

Figure 3: Regularizing effect of CRE-based clustering for stiffness identification of a non-convex defect on a plane wall submitted to random ground motion. Acceleration noisy measurements are processed, and CRE-based clustering allows for better parameter estimation with 75% of unchanged parameters - from [11].



(a) Subdomains labelling and sensors location.



(b) W10 reference stiffness parameter evolution.

Figure 4: Two-story plane frame problem with sensors location (yellow dots) and subdomains.

of subdomain W10, even if all parameters are updated alltogether. A L_2-norm misfit error on parameters allows to properly assess the different KFs in terms of accuracy. It is denoted ϵ_θ is defined as, :

$$\epsilon_\theta = \frac{\int \left(\widehat{\theta}(t) - \theta^*(t)\right)^2 \mathrm{d}t}{\int \theta^*(t)^2 \mathrm{d}t} \tag{14}$$

MDKF framework	**Reference value**		
Data sampling frequency	f_s	$=$	1000 Hz
Overlapping rate for MDKF windows	α_{ovlp}	$=$	95%
mCRE frequency interval	D_ω	$=$	$[1\text{ Hz}; 20\text{ Hz}]$
mCRE frequency sampling step	Δf	$=$	0.1 Hz
Amount of new data per window	$(1-\alpha_{\text{ovlp}})/\Delta f$	$=$	0.1 s
Covariance on parameter state	Q_θ	$=$	$10^{-3}I$
Covariance on mCRE	R	$=$	10^{-8}
Initial covariance on parameters	P_0^θ	$=$	$0.01I$

Table 1: Reference setting parameters of MDKF.

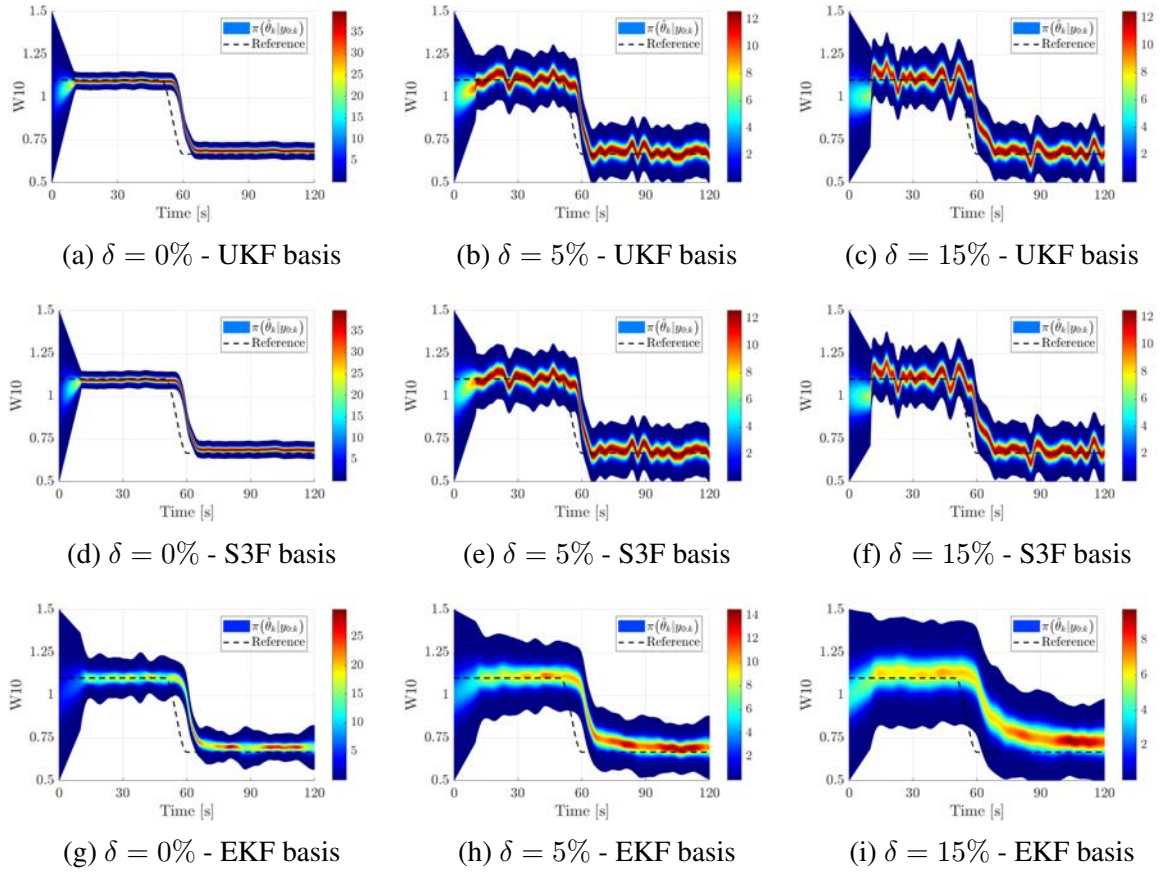
with θ^* the reference parameter value and $\widehat{\theta}$ the mean estimate provided by KFs. The key results of this study are all stored in Tab. 2, and illustrated in Figure 5. For the reader information, they have been obtained on a standard personal laptop (Intel i5-10310, 8Go RAM).

	$\delta = 0\%$ (perfect data)			$\delta = 5\%$ (noisy)			$\delta = 15\%$ (highly-noisy)		
KF basis	UKF	S3F	EKF	UKF	S3F	EKF	UKF	S3F	EKF
ϵ_{W10}	0.0052	0.0052	0.0083	0.0054	0.0054	0.0111	0.0069	0.0070	0.0198
ϵ_{W11}	0.0001	0.0001	0.0001	0.0005	0.0005	0.0004	0.0017	0.0017	0.0011
ϵ_{W20}	0.0002	0.0002	0.0002	0.0004	0.0004	0.0005	0.0004	0.0004	0.0005
ϵ_{W21}	0.0001	0.0001	0.0001	0.0009	0.0009	0.0015	0.0009	0.0009	0.0015
ϵ_{F10}	0.0004	0.0004	0.0003	0.0007	0.0007	0.0006	0.0007	0.0007	0.0006
ϵ_{F20}	0.0001	0.0001	0.0000	0.0003	0.0003	0.0004	0.0003	0.0003	0.0004
Average CPU time per step [s]	0.22303	0.16379	0.13590	0.21300	0.16154	0.13137	0.21424	0.16375	0.12753

Table 2: Comparison of mCRE-based KFs for stiffness parameter tracking from several datasets. All filters perform in real-time as new assimilated data per iteration last 0.5 seconds.

Several remarks can be made from these results:

- Whatever the KF structure in which the mCRE has been integrated, the tracking of parameters is successfully performed in real-time, with always more than 50% of data assimilation step remaining time to perform other operations online. Even for highly-noisy measurements, the mCRE-based data assimilation framework provides a correct parameter tracking, particularly the well captured fall of W10 stiffness.
- From the error indicator values $\epsilon_\bullet$ displayed in Tab. 2, one can consider the identification of correct quality. The fact that ϵ_{W10} is one order of magnitude higher is due to the slight delay at the identification of the sudden change in W10. This delay is not abnormal as it is a direct consequence of the sliding window technique [12].
- Owing to the fact that UKF and S3F produce the exact same estimates [22], Fig. 5a-5c and Fig. 5d-5f are identical. As almost half sigma-points are required for S3F compared to UKF, the analysis of CPU times in Tab. 2 confirms that using S3F σ-points allows a significant gain on this academical example (30% faster, 8 σ-points instead of 13 in that case). There is no doubt that this CPU gain will be higher in problems where the parameter space dimension n_θ is larger. The author thus strongly recommend the systematic use of S3F instead of UKF, as it is as accurate for a lower computational cost.



(a) $\delta = 0\%$ - UKF basis (b) $\delta = 5\%$ - UKF basis (c) $\delta = 15\%$ - UKF basis

(d) $\delta = 0\%$ - S3F basis (e) $\delta = 5\%$ - S3F basis (f) $\delta = 15\%$ - S3F basis

(g) $\delta = 0\%$ - EKF basis (h) $\delta = 5\%$ - EKF basis (i) $\delta = 15\%$ - EKF basis

Figure 5: On-the-fly update of W10 stiffness parameter using several mCRE-based KFs.

- EKF-based MDKF benefits from the simplicity of EKF and the parallel computation of mCRE computations to process data faster than σ-points-based MDKF. However, it also naturally carries the limitations of EKF in terms of accuracy, with the loss of tracking reactivity that appears at high measurement noise levels. Therefore, if EKF-based MDKF is faster, it is also less accurate than mCRE-based sigma-points KFs.

In the end, it appears that S3F-based MDKF is the best compromise between computational cost and accurate estimates. It will thus be chosen as reference mCRE-based KF algorithm.

4.3 A proof of concept for the relevance of the clustered MDKF

In this last paragraph, we intend to complete the case study with the validation of the clustered MDKF when trying to update the frame parametrization described by $n_\theta = 12$ non-overlapping subdomains. Having "too numerous" subdomains (as it is the case for damage detection problems) will allow to present a proof of concept of the relevance of the clustered MDKF for large dimensional parameter tracking problems. More specifically, the objective of this study is to assess the ability to recover the expected parameters from simulated acceleration measurements acquired by discrete sensors scattered over the structure (yellow dots in Fig. 4), with particular attention paid to accuracy and CPU time savings compared to standard MDKF. The internal parameters of MDKF used to obtain the forthcoming results are still the ones specified in Tab. 1. In order to avoid falling into local minimas as the initial guess does not match correctly with the expected parameters (10% gaps), the first 10 iterations (over 110) are forced

to update the full set of parameters. Afterwards, the CRE-based clustering automatically drives which parameters have to be updated.

The clustered MDKF is applied to assimilate the same data that has been processed to obtain the results displayed in Fig. 6. Parameter identification results are shown in Fig. 7 and the comparison between CPU times is summarized in Tab. 3.

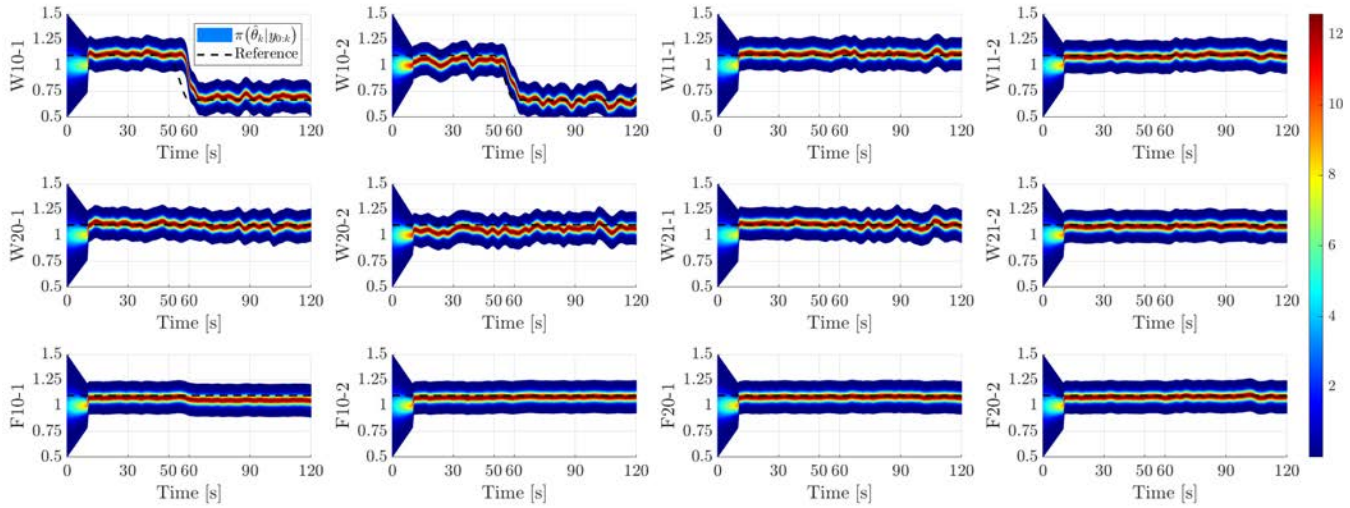


Figure 6: MDKF results of the frame model from noisy measurements ($\delta = 2\%$). Probability density functions are plotted for each of the $n_\theta = 12$ parameters with expected values.

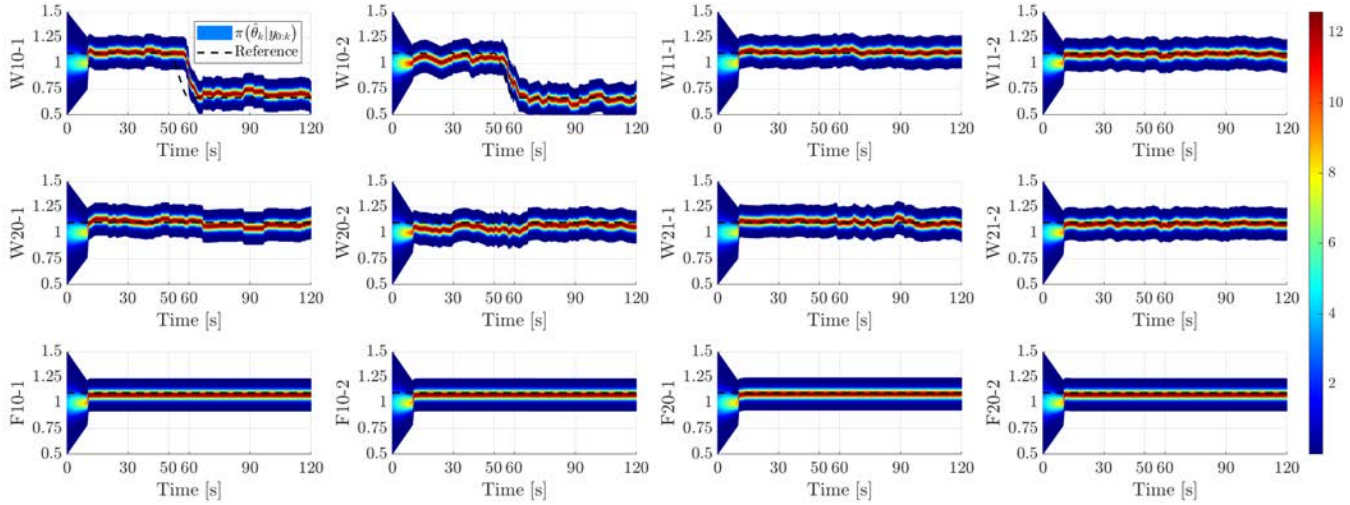


Figure 7: Clustered MDKF results of the frame model from noisy measurements ($\delta = 2\%$). Probability density functions are plotted for each of the $n_\theta = 12$ parameters with expected values.

The analysis of these results allow to conclude that clustered MDKF does not have significant effects on the identification quality of the results by respectively comparing Fig.6-7. Indeed, the noise level and MDKF internal parameters still condition accuracy results, whether the clustering step is present or not.

Algorithm	n_θ	Average CPU time per iteration
UKF-based MDKF	6	0.33 s
S3F-based MDKF	6	0.16 s
UKF-based MDKF	12	0.76 s
S3F-based MDKF	12	0.44 s
Clustered UKF-based MDKF	12	0.48 s (including 0.02 s for the clustering step)
Clustered S3F-based MDKF	12	0.35 s (including 0.02 s for the clustering step)

Table 3: Comparison of CPU time per iteration between MDKF and clustered MDKF to assimilate the same amount of data.

In terms of computational times, it is undeniable that the clustering step is not a time-consuming procedure: only 0.02 s on the 0.5 s allowed to process data in real-time. However, it provides a significant speed-up according to the results given in Tab. 3: in most time steps only half of the clusters (the ones located at the bottom of the frame) are updated. The remaining saved time can therefore be spent for DDDAS purposes instead of trying to update low-sensitive or already well-identified parameters (with respect to the CRE). Besides, one should note that only 12 parameters were updated here and that the value of n_θ does not explictly affect much the required CPU time for clustering. The speedup one can expect when $n_\theta \gg 1$ may thus be even larger than the one observed in this case study.

As a last remark, CPU time comparisons between UKF-based MDKF and S3F-based MDKF have also been done in Tab. 3 in order to show that when combining (i) a KF in which the number of sigma-points is optimized and (ii) an adaptive parametrization of the problem, considerable gains can be made for substantially equivalent precision.

5 Conclusion and prospects

In this paper, a new sequential data assimilation method, using the mCRE as an alternative metrics for observations, has been presented and several improvements has been proposed to improve its performance. We discussed and fairly compared possible variants of the MDKF according to the KF structure in which the mCRE functional is embedded, with the conclusion that the mCRE-based S3F is the best compromise between computational cost and estimation accuracy. Finally, an extension of MDKF for partial state update without loss of accuracy was presented. An additional (low-cost) CRE-based clustering step was included in the data assimilation scheme so as to exploit at best the information carried by the model error term of the mCRE observer. The clustering step permits to restrain the amount of updated parameters per iteration, which allows to achieve real-time constraints without losing accuracy, and to save extra-time for post-processing operations in a DDDAS context. A proof of concept from an academic earthquake engineering example allowed to validate the relevance of the proposed MDKF extension and to draw first limitations that are (unsurprisingly) inherent to the mCRE ones.

The MDKF algorithm will be soon reinvested to perform adaptive model-based control in the context illustrated in Fig. 1 to enhance the control of time-evolving systems (*e.g.* damaging structures). The clustered MDKF would be particularly relevant when considering rich/full-field measurements and damage detection problems such as the ones that can occur in structural health monitoring.

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