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# WAVE FINITE ELEMENT METHOD FOR THE DYNAMICS OF STRUCTURES WITH CYCLIC SYMMETRY

**T. Hoang**[1], **D. Duhamel**[1], **G. Foret**[1], **and L-H. Tran**[2]

[1]Laboratoire Navier, Ecole des Ponts, CNRS, Uni Gustave Eiffel
Champs-sur-Marne, France
e-mail: {tien.hoang,denis.duhamel,gilles.foret}@enpc.fr

[2]Faculty of Civil Engineering - VNU Hanoi
University of Engineering and Technology, Hanoi, Vietnam
e-mail: hungtl@vnu.edu.vn

**Abstract.** *The finite element method for axisymmetric and cyclic symmetric structures has been developed and integrated in numerous commercial FEM codes. Based on the periodicity of the geometry and loads, the dynamics of a 3D structure can be solved by considering only its section or one period, which permits to reduce the calculation time. However, this method cannot be used easily when the boundary conditions and loads are not symmetric. This article presents a new approach of the wave finite element method (WFE) to calculate the dynamic responses of such a structure in a general case: non-symmetric loads and boundary conditions. Based on the WFE for periodic structures, we can determine the wave decomposition of a substructure's responses in the cylindrical coordinate system. The loads on the substructure can be represented as waves added to the response. Then, we apply the wave decomposition to all substructures until getting the first one and this results a relation between the wave amplitudes and the external loads. This relation is simple and can be applied for arbitrary boundary conditions via the reaction forces. The numerical results show the advantage of this methods in the calculation time in comparing with FEM.*

**Keywords:** Wave finite element, dynamics, periodic structure, axisymmetry, cyclic symmetry, reduced model.

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## 1 INTRODUCTION

Many engineering structures are axisymmetric or cyclic symmetric. Some techniques of the finite element method can deal with this type of structure without any difficulty when the loads and boundary condition are also symmetric [1]. However, when the ”perfect” symmetry is not respected, the classical FEM and some reduced models can be applied but they have sometimes disadvantage because of the heavy calculation due to the great number degree of freedom or the model construction cost [2, 3]. This article presents a new technique of the wave finite element method (WFE) for such a structure in general case: non-symmetric loads and boundary conditions.

The wave finite element method was developed originally for the wave propagation in periodic structure [4]. This method permits to calculate the dynamic responses of a period structure by using the wavemodes of one period of the structure with different approaches [5, 6, 7, 8, 9, 10, 11, 12, 13]. Recently, this method has been applied for cyclic symmetric structures in several researches. Mencik [14] has used the force response and dynamic flexibility modes of cyclic symmetric structures. This technique is also applied to compute the free and forced vibration of a tire [15]. Reno et al. [16] proposed a the wave decomposition in angular and axis directions for calculating the forced response of cylinder and cylindrical shells. However, these methods can not applied easily when the periodic structure presents boundary conditions.

This article proposes a new and simple expression (see equation 21) to compute the response of a cyclic symmetric structures with WFE. By rewriting the dynamic equation of one period of the structure in the frequency domain, we can obtain the wavemodes which propagate in the two circumstantial directions of the structure. The loads on a period appear in this relation as an additive term. On the other side, the boundary of the last substructure is also a boundary of the first one. This characteristic leads to a simple expression of the wave amplitudes of the responses according to the external loads on the structure. When the structure has some non-symmetric boundary conditions, this method can be applied by calculating via the reaction forces. For applications, the first example of a 2D gear subjected to forces at a teeth and a fix boundary condition shows a time reduction of 90.4%. The second example is a cooling tower which is represented by a shell model and the calculation shows that the new technique allows a reduction of of 30% computing time compared to the FEM.

## 2 BASIC FRAMEWORKS

We consider a cyclic symmetric structure which is composed of $N$ substructures. Each substructure is subjected to external loads which can be different for each substructure. By using the finite element method, the dynamic equation of a substructure can be written as follows

$$\tilde{\mathbf{D}}\mathbf{q} = \mathbf{F} \tag{1}$$

where $\mathbf{q}$ and $\mathbf{F}$ are the DOF and nodal loads; $\tilde{\mathbf{D}}$ is the dynamic stiffness matrix of the cell.

*Change of the coordinate system:* In order to take into account of the geometric symmetry, equation(1) needs to be established in the cylindrical coordinate system. If it is in a Cartesian coordinate, we can change the system by using the rotation matrix. For each node of the substructure, we have the element rotation matrix for a vector $q^T = [q_x \quad q_y \quad q_z]$ given by

$$\begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_r \\ q_\theta \\ q_z \end{bmatrix} \tag{2}$$

where $\theta$ is the angle coordinate of the node. Then, the rotation matrix $\mathbf{R}$ is the diagonal block matrix of all element matrix. Finally, the dynamic stiffness matrix in the cylindrical system is given by $\mathbf{R}^T\tilde{\mathbf{D}}\mathbf{R}$.

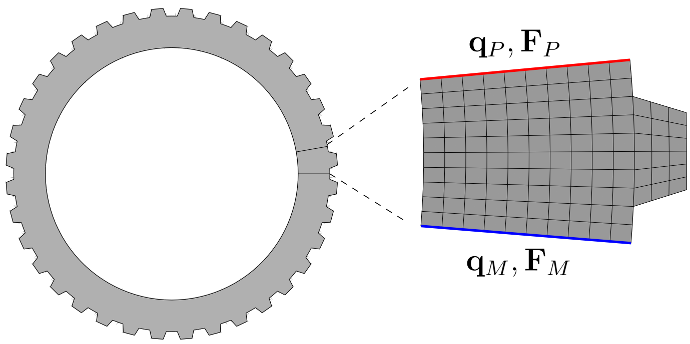


Figure 1: A cyclic symmetric structure presented by a substructure

Each substructure connects to the other ones by its boundaries in counter-clockwise (P) and clockwise (M) directions (see figure 1). Thus, we can rewrite equation (1) as follows

$$\begin{bmatrix} \tilde{\mathbf{D}}_{II} & \tilde{\mathbf{D}}_{IM} & \tilde{\mathbf{D}}_{IP} \\ \tilde{\mathbf{D}}_{MI} & \tilde{\mathbf{D}}_{MM} & \tilde{\mathbf{D}}_{MP} \\ \tilde{\mathbf{D}}_{PI} & \tilde{\mathbf{D}}_{PM} & \tilde{\mathbf{D}}_{PP} \end{bmatrix} \begin{bmatrix} \mathbf{q}_I \\ \mathbf{q}_M \\ \mathbf{q}_P \end{bmatrix} = \begin{bmatrix} \mathbf{F}_I \\ \mathbf{F}_M \\ \mathbf{F}_P \end{bmatrix} \tag{3}$$

where $I$ denote for the inner nodes. We can reduce the inner nodes $\mathbf{q}_I$ from the first row of equation (3) as follows

$$\mathbf{q}_I = \tilde{\mathbf{D}}_{II}^{-1}\left[\mathbf{F}_I - \tilde{\mathbf{D}}_{IM}\,\mathbf{q}_M - \tilde{\mathbf{D}}_{IP}\mathbf{q}_P\right] \tag{4}$$

Then, by substituting the aforementioned equation into equation (3), we obtain

$$\begin{bmatrix} \mathbf{D}_{MI}\mathbf{F}_I \\ \mathbf{D}_{PI}\mathbf{F}_I \end{bmatrix} + \begin{bmatrix} \mathbf{D}_{MM} & \mathbf{D}_{MP} \\ \mathbf{D}_{PM} & \mathbf{D}_{PP} \end{bmatrix} \begin{bmatrix} \mathbf{q}_M \\ \mathbf{q}_P \end{bmatrix} = \begin{bmatrix} \mathbf{F}_M \\ \mathbf{F}_P \end{bmatrix} \tag{5}$$

where

$$\begin{array}{ll} \mathbf{D}_{MI} = \tilde{\mathbf{D}}_{MI}\tilde{\mathbf{D}}_{II}^{-1} & \mathbf{D}_{PI} = \tilde{\mathbf{D}}_{PI}\tilde{\mathbf{D}}_{II}^{-1} \\ \mathbf{D}_{MM} = \tilde{\mathbf{D}}_{MM} - \tilde{\mathbf{D}}_{MI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IM} & \mathbf{D}_{PP} = \tilde{\mathbf{D}}_{PP} - \tilde{\mathbf{D}}_{PI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IP} \\ \mathbf{D}_{MP} = \tilde{\mathbf{D}}_{MP} - \tilde{\mathbf{D}}_{MI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IP} & \mathbf{D}_{PM} = \tilde{\mathbf{D}}_{PM} - \tilde{\mathbf{D}}_{PI}\tilde{\mathbf{D}}_{II}^{-1}\tilde{\mathbf{D}}_{IM} \end{array} \tag{6}$$

We see that equation (5) presents a relation between the nodal loads and DOF at the counterclockwise and clockwise boundaries of the substructure. For the two consecutive substruc-

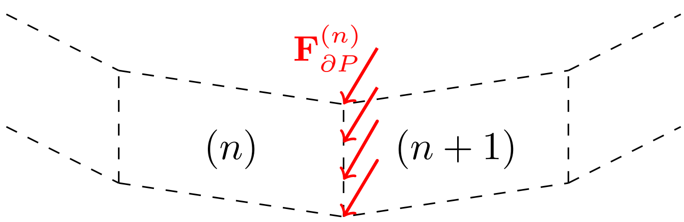


Figure 2: Two consecutive substructures

tures as shown in Figure 2, the counterclockwise boundary of $(n)$ is the clockwise boundary of

$(n+1)$. Therefore, we have

$$\begin{aligned} \mathbf{q}_P^{(n)} &= \mathbf{q}_M^{(n+1)} \\ \mathbf{F}_P^{(n)} + \mathbf{F}_M^{(n+1)} &= -\mathbf{F}_{\partial P}^{(n)} \end{aligned} \tag{7}$$

where $\mathbf{F}_{\partial P}^{(n)}$ are the external nodal loads at the counterclockwise boundary $P$ of the cell $(n)$ as shown in Figure 2. By combining equations (5) and (7), we obtain

$$\begin{bmatrix} \mathbf{q}_M^{(n+1)} \\ -\mathbf{F}_M^{(n+1)} \end{bmatrix} = \begin{bmatrix} -\mathbf{D}_{MP}^{-1}\mathbf{D}_{MI}\mathbf{F}_I^{(n)} \\ \left(\mathbf{D}_{PI} - \mathbf{D}_{PP}\mathbf{D}_{MP}^{-1}\mathbf{D}_{MI}\right)\mathbf{F}_I^{(n)} + \mathbf{F}_{\partial P}^{(n)} \end{bmatrix} + \mathbf{S}\begin{bmatrix} \mathbf{q}_M^{(n)} \\ -\mathbf{F}_M^{(n)} \end{bmatrix} \tag{8}$$

where

$$\mathbf{S} = \begin{bmatrix} -\mathbf{D}_{MP}^{-1}\mathbf{D}_{MM} & -\mathbf{D}_{MP}^{-1} \\ \mathbf{D}_{PM} - \mathbf{D}_{PP}\mathbf{D}_{MP}^{-1}\mathbf{D}_{MM} & -\mathbf{D}_{PP}\mathbf{D}_{MP}^{-1} \end{bmatrix}, \tag{9}$$

We can also rewrite equation (6) as follows

$$\mathbf{u}^{(n+1)} = \mathbf{S}\mathbf{u}^{(n)} + \mathbf{b}^{(n)} \tag{10}$$

where

$$\mathbf{u}^{(n)} = \begin{bmatrix} \mathbf{q}_M^{(n)} \\ -\mathbf{F}_M^{(n)} \end{bmatrix}, \quad \mathbf{b}^{(n)} = \begin{bmatrix} -\mathbf{D}_{MP}^{-1}\mathbf{D}_{MI}\mathbf{F}_I^{(n)} \\ \left(\mathbf{D}_{PI} - \mathbf{D}_{PP}\mathbf{D}_{MP}^{-1}\mathbf{D}_{MI}\right)\mathbf{F}_I^{(n)} + \mathbf{F}_{\partial P}^{(n)} \end{bmatrix} \tag{11}$$

Equation (10) presents a relation between the response (DOF and nodal loads) of the substructures $(n)$ and $(n+1)$. Here $\mathbf{b}^{(n)}$ presents the external loads on the substructure $(n)$ (when the cell is free, $\mathbf{b}^{(n)} = 0$). In the periodic structure of $N$ substructures, this equation presents a recurrent relation with regard to $n$, which can be reduced to the following results

$$\mathbf{u}^{(n)} = \mathbf{S}^n\mathbf{u}^{(0)} + \sum_{k=1}^{n} \mathbf{S}^{n-k}\mathbf{b}^{(k-1)} \tag{12}$$

Equation (11) gives the relations between the responses at the substructre $(n)$ and $\mathbf{u}^{(0)}$. We note that $\mathbf{u}^{(0)}$ can be chosen at any substructure because the structure is periodic.

## 3 WAVE DECOMPOSITION

The matrix $\mathbf{S}$ given by equation (9) is a linear transformation which yields complex eigenvalues and eigenvectors $\{\mu_j, \phi_j\}_j$ defined by $\mathbf{S}\phi_j = \mu_j\phi_j$. Due to the symplectic nature of the matrix $\mathbf{S}$ (see [5]), we can put the eigenvalues in pair $(\mu_j, \mu_j^\star)$ with $\mu_j^\star = 1/\mu_j$ which correspond to the pair of eigenvectors $(\phi_j, \phi_j^\star)$. We call $\phi_i$ the wave from the left to right with eigenvalue $\|\mu_j\| \leq 1$ and $\phi_i^\star$ the wave from the right to left with eigenvalue $\|\mu_j^\star\| \geq 1$.

The set of $\phi_j$ and $\phi_j^\star$ forms a base of the vectorial space of $\mathbf{u}^{(n)}$ and $\mathbf{b}^{(n)}$ defined by equation (10). This base is symplectic orthogonal in the meaning of $\phi_j^T\mathbf{J}\phi_i = \phi_j^{\star T}\mathbf{J}\phi_i^\star = 0$ $(\forall i, j)$ and $\phi_j^{\star T}\mathbf{J}\phi_i = \phi_j^T\mathbf{J}\phi_i^\star = 0$ $(\forall i \neq j)$(see [8]). We can normalize this wave base in the meaning of $\phi_i^{\star T}\mathbf{J}\phi_i = -\phi_i^T\mathbf{J}\phi_i^\star = 1$.

If we note $\mathbf{\Phi} = [\phi_1 \cdots \phi_n]$ and $\mathbf{\Phi}^\star = [\phi_1^\star \cdots \phi_n^\star]$, we can decompose each vector of equation (9) in this wave base as follows

$$\begin{aligned} \mathbf{u}^{(n)} &= \mathbf{\Phi}\mathbf{Q}^{(n)} - \mathbf{\Phi}^\star\mathbf{Q}^{\star(n)} \\ \mathbf{b}^{(n)} &= \mathbf{\Phi}\mathbf{Q}_E^{(n)} - \mathbf{\Phi}^\star\mathbf{Q}_E^{\star(n)} \end{aligned} \tag{13}$$

where $\mathbf{Q}^{(n)}, \mathbf{Q}^{\star(n)}$ are the wave amplitudes of $\mathbf{u}^{(n)}$ and $\mathbf{Q}_E^{(k)}, \mathbf{Q}_E^{\star(k)}$ are the wave amplitudes of the external loads on the intermediate substructures $\mathbf{b}^{(n)}$.

The wave decomposition in equation (13) is different from usual expression for WFE by the minus sign on the right to left waves. The advantage of this expression is that we can calculate directly the wave amplitudes by using the symplectic orthogonality of the wave basis as the following

$$\begin{aligned} \mathbf{Q}^{(n)} &= \mathbf{\Phi}^{\star T}\mathbf{J}\mathbf{u}^{(n)}, \quad \mathbf{Q}^{\star(n)} = \mathbf{\Phi}^{T}\mathbf{J}\mathbf{u}^{(n)} \\ \mathbf{Q}_E^{(n)} &= \mathbf{\Phi}^{\star T}\mathbf{J}\mathbf{b}^{(n)}, \quad \mathbf{Q}_E^{\star(n)} = \mathbf{\Phi}^{T}\mathbf{J}\mathbf{b}^{(n)} \end{aligned} \tag{14}$$

By substituting equation (11) into equation (14), we obtain (see A)

$$\begin{aligned} \mathbf{Q}_E^{(k)} &= \left(\boldsymbol{\mu}\mathbf{\Phi}_q^{\star T}\mathbf{D}_{MI} + \mathbf{\Phi}_q^{\star T}\mathbf{D}_{PI}\right)\mathbf{F}_I^{(k)} + \mathbf{\Phi}_q^{\star T}\mathbf{F}_{\partial P}^{(k)} \\ \mathbf{Q}_E^{\star(k)} &= \left(\boldsymbol{\mu}^{\star}\mathbf{\Phi}_q^{T}\mathbf{D}_{MI} + \mathbf{\Phi}_q^{T}\mathbf{D}_{PI}\right)\mathbf{F}_I^{(k)} + \mathbf{\Phi}_q^{T}\mathbf{F}_{\partial P}^{(k)} \end{aligned} \tag{15}$$

Equation (15) shows that the loads on each substructure creates two waves which propagate to the counterclockwise and clockwise directions of the structures.

Now we will calculate the responses of a substructure in function of the wave amplitudes. By replacing equation (13) with $n = 0$ into equation (12), we obtain

$$\mathbf{u}^{(n)} = \mathbf{S}^n\left(\mathbf{\Phi}\mathbf{Q} - \mathbf{\Phi}^{\star}\mathbf{Q}^{\star}\right) + \sum_{k=1}^{n}\mathbf{S}^{n-k}\left(\mathbf{\Phi}\mathbf{Q}_E^{(k-1)} - \mathbf{\Phi}^{\star}\mathbf{Q}_E^{\star(k-1)}\right) \tag{16}$$

where $\mathbf{Q}, \mathbf{Q}^{\star}$ are the wave amplitudes of $\mathbf{u}^{(0)}$. In addition, we have by definition of the wave bases

$$\mathbf{S}^n\mathbf{\Phi} = \mathbf{\Phi}\boldsymbol{\mu}^n, \quad \mathbf{S}^n\mathbf{\Phi}^{\star} = \mathbf{\Phi}^{\star}\boldsymbol{\mu}^{\star n} \tag{17}$$

Thus, we can rewrite equation (16) as follows

$$\mathbf{u}^{(n)} = \mathbf{\Phi}\boldsymbol{\mu}^n\mathbf{Q} - \mathbf{\Phi}^{\star}\boldsymbol{\mu}^{\star n}\mathbf{Q}^{\star} + \sum_{k=1}^{n}\mathbf{\Phi}\boldsymbol{\mu}^{n-k}\mathbf{Q}_E^{(k-1)} - \mathbf{\Phi}^{\star}\boldsymbol{\mu}^{\star n-k}\mathbf{Q}_E^{\star(k-1)} \tag{18}$$

Because $\boldsymbol{\mu}^{\star} = \boldsymbol{\mu}^{-1}$, we have

$$\mathbf{u}^{(n)} = \mathbf{\Phi}\boldsymbol{\mu}^n\left(\mathbf{Q} + \sum_{k=1}^{n}\boldsymbol{\mu}^{\star k}\mathbf{Q}_E^{(k-1)}\right) - \mathbf{\Phi}^{\star}\boldsymbol{\mu}^{\star n}\left(\mathbf{Q}^{\star} + \sum_{k=1}^{n}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{\star(k-1)}\right) \tag{19}$$

On the other side, the counterclockwise boundary of the substructure $N$ is also the clockwise boundary of the substructure $0$, we have $\mathbf{u}^{(N)} = \mathbf{u}^{(0)}$. In addition, the wave decomposition of a boundary in the wave base $\{\mathbf{\Phi}, \mathbf{\Phi}^{\star}\}$ is unique. Thus, we have

$$\mathbf{Q} = \left[\mathbf{I} - \boldsymbol{\mu}^N\right]^{-1}\sum_{k=1}^{N}\boldsymbol{\mu}^{N-k}\mathbf{Q}_E^{(k)}, \quad \mathbf{Q}^{\star} = \left[\boldsymbol{\mu}^N - \mathbf{I}\right]^{-1}\sum_{k=1}^{N}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{\star(k)} \tag{20}$$

Finally, we obtain the expression of the response by substituting the aforementioned equation into equation (13)

$$\mathbf{u}^{(0)} = \mathbf{\Phi}\left[\mathbf{I} - \boldsymbol{\mu}^N\right]^{-1}\sum_{k=1}^{N}\boldsymbol{\mu}^{N-k}\mathbf{Q}_E^{(k)} + \mathbf{\Phi}^{\star}\left[\mathbf{I} - \boldsymbol{\mu}^N\right]^{-1}\sum_{k=1}^{N}\boldsymbol{\mu}^{k}\mathbf{Q}_E^{\star(k)} \tag{21}$$

Equation (21) presents the response at the boundary of each substructure and the external loads on the structure. For the inner nodes of the substructures, we can use equation (4) to compute the response. We note that this method needs to calculate only once the wave amplitudes of the external loads on each substructure and we don't need to inverse any matrix as in the classical FEM.

**Remarks**

*Non-symmetric boundary conditions*: We see that the structure's responses can be calculated from the external loads via their wave amplitudes given in equation (15). However, if the boundary condition is given instead of the forces, we can use use this method by calculating via the reaction force. Firstly, we write the relation between the displacement of the boundary condition in function of the reaction forces and external loads. Then, we can write the boundary condition to obtain an equation on the reaction forces. Once the reaction forces are calculated, we can compute the responses of the structure.

*Symmetric loads*: When the loads on the structure is symmetric, we have $\mathbf{Q}_E^{(k)} = \mathbf{Q}_E$ and $\mathbf{Q}_E^{\star(k)} = \mathbf{Q}_E^{\star}$ for all substructures. We obtain the following result from equation (21)

$$\mathbf{u}^{(0)} = \mathbf{\Phi} \left[\mathbf{I} - \boldsymbol{\mu}\right]^{-1} \mathbf{Q}_E + \mathbf{\Phi}^{\star} \left[\mathbf{I} - \boldsymbol{\mu}\right]^{-1} \mathbf{Q}_E^{\star} \tag{22}$$

It is remarkable that this result does not depend on the number of substructures.

## 4 NUMERICAL APPLICATIONS

Let's consider a 2D gear with 36 teeth as shown in Figure 1.The structure is subjected to a load at the tooth of the substructure number 1 and a fixed constrain at the radius of the substructure number 10. We use the finite element method with the element type S4 in Abaqus to obtain the dynamic stiffness matrix of a substructure. Then, the calculation is performed with FEM and WFE methods in Matlab. Figure 3 shows the results obtained by the two methods. The results agrees well and the calculation times are 17.7s and 1.7s for FEM and WFE respectively, which means 90.4% of time reduction.

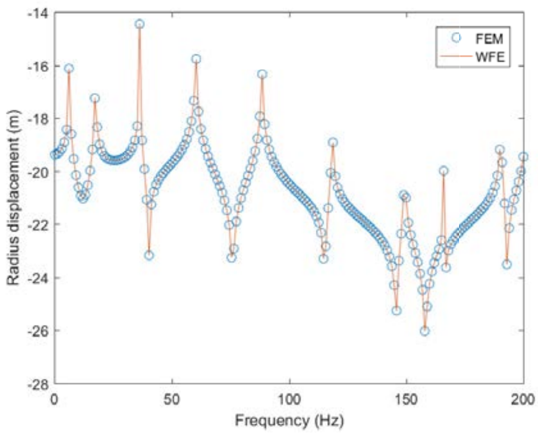


Figure 3: Result example 1

We take another example of a hyperbolic cooling tower in a nuclear center. The dimensions of the tower are given by [17] with height of 138m, the fixed base is of 108m in diameter. The tower is subjected to a pressure in a rectangular domain of 10x5m with the center at the height of 100m. This tower has an axisymmetric geometry but the load is not symmetric. Therefore, we can not use the axisymmetric element in FEM code. We use the shell element S4 to compute the dynamic stiffness of the whole tower and its substructure which composes only 1 element in the tangent direction, corresponding to an arc of 5°. Thus, the number of substructures is $\frac{360}{5} = 72$. It is remarkable that each node contains 6 degrees of freedom corresponding to the displacements and rotations in 3 directions. Therefore, the rotation matrix of one node contains not only the matrix $3 \times 3$ in equation (2) but a $6 \times 6$ matrix with 2 diagonal blocks of this $3 \times 3$ matrix. Figure 5 shows the results obtained by FEM and the new method. The two results agree

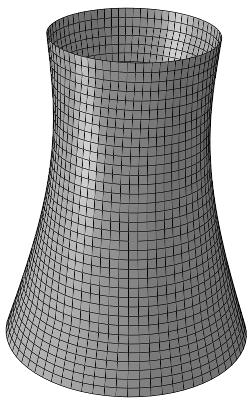

Figure 4: Example of a cooling tower

well and the calculation times are 102.2s for FEM and 70.7 for WFE.

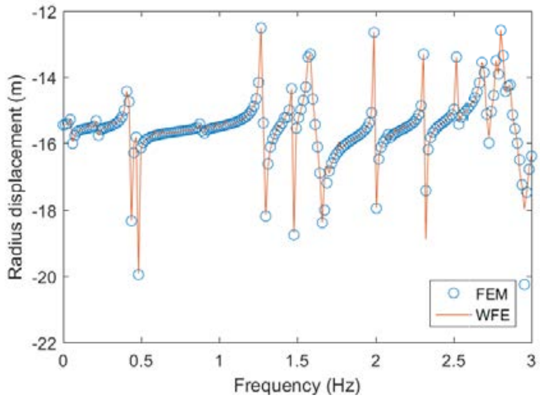


Figure 5: Example of a cooling tower

## 5 CONCLUSIONS

By using the wave finite element method, we demonstrate that the dynamic response of a cyclic symmetric structure is a sum of all waves generated by external loads in each substructure. This method permits to reduce the DOF of the structure to only one substructure and therefore reduce the calculation times.

## A CALCULATION OF WAVE AMPLITUDES OF EXTERNAL LOADS

Because the vector $\mathbf{u}^{(n)}$ contains two components corresponding to the displacements and forces, we can separate the components of the wave base as follows

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_q \\ \mathbf{\Phi}_F \end{bmatrix} \quad \mathbf{\Phi}^\star = \begin{bmatrix} \mathbf{\Phi}_q^\star \\ \mathbf{\Phi}_F^\star \end{bmatrix} \tag{23}$$

By substituting equation (11) into equation (14), we obtain

$$\begin{aligned}
\mathbf{Q}_E^{(n)} &= \mathbf{\Phi}_q^{\star T} \left( \mathbf{F}_{\partial P}^{(n)} + \left( \mathbf{D}_{PI} - \mathbf{D}_{PP}\mathbf{D}_{MP}^{-1}\mathbf{D}_{MI} \right) \mathbf{F}_I^{(n)} \right) + \mathbf{\Phi}_F^{\star T}\mathbf{D}_{MP}^{-1}\mathbf{D}_{MI}\mathbf{F}_I^{(n)} \\
&= \mathbf{\Phi}_q^{\star T}\mathbf{F}_{\partial P}^{(n)} + \left( \left( \mathbf{\Phi}_F^{\star T} - \mathbf{\Phi}_q^{\star T}\mathbf{D}_{PP} \right) \mathbf{D}_{MP}^{-1}\mathbf{D}_{MI} + \mathbf{\Phi}_q^{\star T}\mathbf{D}_{PI} \right) \mathbf{F}_I^{(n)} \\
\mathbf{Q}_E^{\star(n)} &= \mathbf{\Phi}_q^{T} \left( \mathbf{F}_{\partial P}^{(n)} + \left( \mathbf{D}_{PI} - \mathbf{D}_{PP}\mathbf{D}_{MP}^{-1}\mathbf{D}_{MI} \right) \mathbf{F}_I^{(n)} \right) + \mathbf{\Phi}_F^{T}\mathbf{D}_{MP}^{-1}\mathbf{D}_{MI}\mathbf{F}_I^{(n)} \\
&= \mathbf{\Phi}_q^{T}\mathbf{F}_{\partial P}^{(n)} + \left( \left( \mathbf{\Phi}_F^{T} - \mathbf{\Phi}_q^{T}\mathbf{D}_{PP} \right) \mathbf{D}_{MP}^{-1}\mathbf{D}_{MI} + \mathbf{\Phi}_q^{T}\mathbf{D}_{PI} \right) \mathbf{F}_I^{(n)}
\end{aligned} \tag{24}$$

In addition, we have the relation between the $\mathbf{\Phi}_q$ and $\mathbf{\Phi}_F$ as follows (see [8])

$$\begin{aligned}
\mathbf{\Phi}_F &= \mathbf{D}_{PP}\mathbf{\Phi}_q + \mathbf{D}_{PM}\mathbf{\Phi}_q\boldsymbol{\mu}^\star \\
\mathbf{\Phi}_F^\star &= \mathbf{D}_{PP}\mathbf{\Phi}_q^\star + \mathbf{D}_{PM}\mathbf{\Phi}_q^\star\boldsymbol{\mu}
\end{aligned} \tag{25}$$

By substituting equation (25) into (24), we obtain

$$\begin{aligned}
\mathbf{Q}_E^{(k)} &= \mathbf{\Phi}_q^{\star T}\mathbf{F}_{\partial P}^{(k)} + \left( \boldsymbol{\mu}\mathbf{\Phi}_q^{\star T}\mathbf{D}_{MI} + \mathbf{\Phi}_q^{\star T}\mathbf{D}_{PI} \right) \mathbf{F}_I^{(k)} \\
\mathbf{Q}_E^{\star(k)} &= \mathbf{\Phi}_q^{T}\mathbf{F}_{\partial P}^{(k)} + \left( \boldsymbol{\mu}^\star\mathbf{\Phi}_q^{T}\mathbf{D}_{MI} + \mathbf{\Phi}_q^{T}\mathbf{D}_{PI} \right) \mathbf{F}_I^{(k)}
\end{aligned} \tag{26}$$

If we note

$$\mathbf{F}_E^{(k)} = \begin{bmatrix} \mathbf{F}_I^{(k)} \\ \mathbf{F}_{\partial P}^{(k)} \end{bmatrix}, \qquad \begin{aligned} \mathbf{\Phi}_E &= \begin{bmatrix} \boldsymbol{\mu}\mathbf{\Phi}_q^{\star T}\mathbf{D}_{MI} + \mathbf{\Phi}_q^{\star T}\mathbf{D}_{PI} & \mathbf{\Phi}_q^{\star T} \end{bmatrix} \\ \mathbf{\Phi}_E^\star &= \begin{bmatrix} \boldsymbol{\mu}^\star\mathbf{\Phi}_q^{T}\mathbf{D}_{MI} + \mathbf{\Phi}_q^{T}\mathbf{D}_{PI} & \mathbf{\Phi}_q^{T} \end{bmatrix} \end{aligned} \tag{27}$$

Equation (26) becomes

$$\mathbf{Q}_E^{(k)} = \mathbf{\Phi}_E\mathbf{F}_E^{(k)} \quad \text{and} \quad \mathbf{Q}_E^{\star(k)} = \mathbf{\Phi}_E^\star\mathbf{F}_E^{(k)} \tag{28}$$

On the other side, $\mathbf{F}_E^{(k)}$ is the vector of the external loads applying on the substructure $k$. Equation (28) presents a simple relation between the external loads and its wave amplitudes.

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