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NUMERICAL INVESTIGATION ON STABILITY OF COLUMNS UNDER DYNAMIC LOADS WITH TWO FREQUENCIES

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Abstract. *Investigation of stability of structures under dynamic loads with a single frequency leads to differential equations with periodic coefficients, i.e., Mathieu-Hill equations. In many cases of civil engineering, however, the dynamic force is not periodic with a single frequency but quasi-periodic with multiple frequencies. This paper proposes a new numerical method to study the stability of columns under dynamic loads with two frequencies. The equation of motion for columns with fixed-fixed connections under parametric loads is derived and decoupled into an ordinary differential equation with variable coefficients. The first step of the numerical method is to approximate the system with two frequencies by a system with a single frequency (or period) as closely as possible. A numerical method is proposed to calculate the state transition matrix on one period through which the dynamic stability can be determined. As a verification of the proposed numerical method, a system under parametric excitation with two incommensurate frequencies is studied and compared to existing results in the literature. The efficiency and accuracy of the proposed numerical method are demonstrated. As an application example, dynamic stability of a column under parametric loads with two frequencies is investigated through instability diagrams and vibration responses. Vibration responses provide partial validation of the instability diagrams. The numerical results for instability diagrams can serve as a calibration of other approximate results. The proposed method can also be extended to multiple degrees of freedom systems under arbitrary parametric excitations with multiple frequencies.*

Keywords: Columns, Dynamic stability, Numerical simulation, Multiple frequencies, Instability diagrams.

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1 INTRODUCTION

Investigation of structural stability under dynamic loads leads to a governing partial differential equation, which is usually converted into ordinary differential equations with periodic coefficients, i.e., Mathieu-Hill equations [1]. This dynamical system is called parametrically excited system and the instability is known as parametric resonance. The traditional method for parametric resonance is from Floquet theory. Bolotin developed harmonic balance in terms of Hill infinite determinants [1] to obtain a hierarchy of instability boundaries. Mathieu-Hill equations are also solved by approximate methods such as the method of perturbation [2] and the method of averaging [3], but these methods are only applicable for small periodic excitations.

In many cases, however, the dynamic force is not periodic with a single frequency but quasi-periodic with multiple frequencies. In civil engineering, for example, parametric resonance occurs when a pile foundation is subjected to vertical earthquake excitations. If one assumes that the vertical earthquake excitation is a periodic force, the equation of motion of the file foundation turns out to be a Mathieu-Hill equation. In reality, however, earthquake excitations are not periodic; instead, they consist of different waves and contain many different frequencies. Another example is a ship sailing in longitudinal sea waves where parametric resonance often occurs because the equation of motion of the ship is a Mathieu-Hill equation [6]. However, sea waves are not periodic with a single frequency in nature; instead, sea waves contain multiple incommensurate frequencies. Unfortunately, the classical methods [1, 4] are only applicable to systems under a parametric excitation with only a single frequency. Zhang et al. [5] recently investigated dynamic stability of an axially transporting beam with two-frequency parametric excitation and internal resonance. There is no complete theory for Mathieu-Hill equations with multiple frequencies. Sharma and Sinha proposed an approximate analysis of quasi-periodic systems via Floquet theory [6]. The present paper attempts to present a numerical investigation on stability of columns under dynamic loads with two frequencies. The numerical results on dynamic stability are consolidated by vibration responses.

2 FORMULATION

The column with fixed-fixed ends in Fig. 1(a) is excited by a longitudinal dynamic force $P(t)$. Transverse vibration $v(x,t)$ occurs in the xv plane. This column may be a model of deep pile foundation under earthquake excitations. To study the dynamics of the column, an element $\mathrm{d}x$ is analyzed: the axial force P, the shear force S, the moment M, the external damping force $\beta_0 \mathrm{d}x \frac{\partial v}{\partial t}$, and the inertia force $(\rho A \mathrm{d}x)\frac{\partial v}{\partial t}$, where β_0 is the damping coefficient per unit length, ρ is the density, A is the cross-sectional area, and EI is the flexural rigidity.

Applying equations of equilibrium

$$\mathrm{d}S + (\rho A \mathrm{d}x)\ddot{v}(x,t) + \beta_0 \mathrm{d}x \dot{v}(x,t) = 0, \tag{1}$$

$$\mathrm{d}M(x,t) - S\mathrm{d}x + P(t)\mathrm{d}x v(x,t) = 0, \tag{2}$$

$$M(x,t) = EI\frac{\partial^2 v(x,t)}{\partial x^2}, \tag{3}$$

results in the equation of motion of the column

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + \rho A\frac{\partial^2 v(x,t)}{\partial x^2} + \beta_0\frac{\partial v(x,t)}{\partial x} + P(t)\frac{\partial^2 v(x,t)}{\partial x^2} = 0. \tag{4}$$

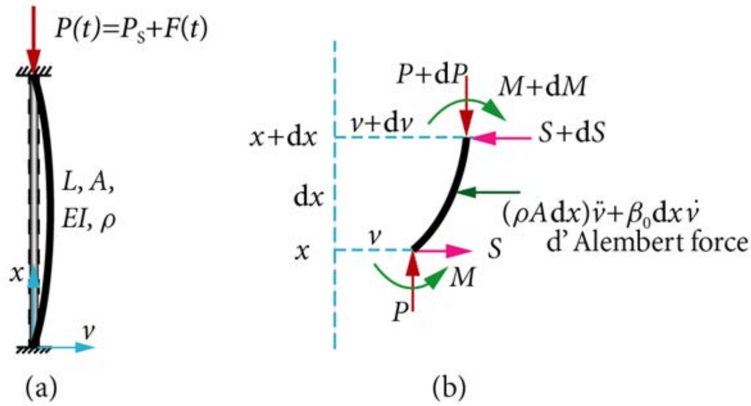


Figure 1: A column under a dynamic force (a) the column; (b) forces on the element dx.

The fixed-fixed boundary conditions are given by

$$\text{at } x=0, v(0,t)=\frac{\partial v(0,t)}{\partial x}=0;\ \text{at } x=L, v(L,t)=\frac{\partial v(L,t)}{\partial x}=0. \tag{5}$$

The method of separation of variables is used in Eq. (4)

$$v(x,t)=\sum_{n=1}^{\infty} q_n(t)\left[1-\cos\frac{2n\pi x}{L}\right], \tag{6}$$

and the equation of motion in terms of the lateral deflection $q_n(t)$ is

$$\rho A\left[1-\cos\frac{2n\pi x}{L}\right]\ddot{q}_n(t)+\beta_0\left[1-\cos\frac{2n\pi x}{L}\right]\dot{q}_n(t)+P(t)\left(\frac{2n\pi}{L}\right)^2\cos\frac{2n\pi x}{L}q_n(t)- \\ EI\left(\frac{2n\pi}{L}\right)^4\cos\frac{2n\pi x}{L}q_n(t)=0. \tag{7}$$

Performing a differentiation on Eq. (7) with respect to x and taking the first mode ($n=1$) yield

$$\ddot{q}(t)+\frac{\beta_0}{\rho A}\dot{q}(t)+\omega_0^2\left[1-\frac{P(t)}{P_0}\right]q(t)=0, \tag{8}$$

where

$$\omega_0^2=\frac{EI}{\rho A}\left(\frac{2\pi}{L}\right)^4, P_0=EI\left(\frac{2\pi}{L}\right)^2. \tag{9}$$

By assuming that the dynamic force is composed of a static component and multiple dynamic components of multiple frequencies

$$P(t)=P_s+\sum_{i=1}^{m}\left(P_{di}\cos\theta_i t\right), \theta_i=\frac{2\pi}{T_i}, \tag{10}$$

where m is the number of frequencies in the dynamic force and also the number of dynamic component forces, P_s is the static component force, P_{di} is the amplitude of the i-th dynamic component force, θ_i is the i-th angular frequency, and T_i is the i-th period.

Eq. (8) can be transformed into

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 \left[1 - \frac{P_s}{P_0} - \sum_{i=1}^{m} \left(\frac{P_{di}}{P_0} \cos\theta_i t\right)\right] q = 0, \beta = \frac{\beta_0}{2\rho A}, \tag{11}$$

$$\ddot{q} + 2\beta\dot{q} + \left[\omega_0^2 \left(1 - \frac{P_s}{P_0}\right) - \omega_0^2 \sum_{i=1}^{m} \left(\frac{P_{di}}{P_0} \cos\theta_i t\right)\right] q = 0. \tag{12}$$

By changing the variable $q(t) = e^{-\beta t}u(t)$, Eqs. (11-12) can be transformed to an equivalent undamped equation, respectively,

$$\ddot{u} + \omega_0^2 \left[1 - \frac{\beta^2}{\omega_0^2} - \frac{P_s}{P_0} - \sum_{i=1}^{m} \left(\frac{P_{di}}{P_0} \cos\theta_i t\right)\right] u = 0, \tag{13}$$

$$\ddot{u} + \left[\omega_0^2 \left(1 - \frac{P_s}{P_0}\right) - \beta^2 - \omega_0^2 \sum_{i=1}^{m} \left(\frac{P_{di}}{P_0} \cos\theta_i t\right)\right] u = 0. \tag{14}$$

3 NUMERICAL METHOD FOR DYNAMIC STABILITY

Eq. (13) or (14) is an ordinary differential equation with a variable coefficient of multiple frequencies (quasi-periodic), which is hard to solve by directly using the Bolotin's method [1]. The idea of our numerical method is to convert the quasi-periodic equation to an equation with a periodic coefficient of a single frequency, i.e., a Mathieu-Hill equation, then one can divide period T into sufficient time intervals, approximate the equation with variable coefficients by an equation with constant coefficients on each interval, and accumulate the responses on each time interval in one period T. So, the key step is to approximate the system with multiple frequencies by a system with a single frequency.

3.1 Approximation of multiple frequencies by a single frequency

The idea is to approximate the system with multiple frequencies by a system with a single frequency as closely as possible.

Consider a system with two frequencies in the dynamic force. Take $m = 2$ with angular frequency θ_1 and θ_2 as an example. The approximate single period can be determined in the following minimization search process.

$$\min_{i=1\ldots h} \min_{j=1\ldots h} \left| i\frac{2\pi}{\theta_1} - j\frac{2\pi}{\theta_2} \right|, \tag{15}$$

where h is an appropriate positive integer that is used to decide which combination of (i, j) will lead to a minimum value of $\left|i\frac{2\pi}{\theta_1} - j\frac{2\pi}{\theta_2}\right|$. In order to approximate the system with two frequencies by a system with a single frequency as closely as possible, the value of $\left|i\frac{2\pi}{\theta_1} - j\frac{2\pi}{\theta_2}\right|$ should be as small as possible, say, less than a very small specified real number such as 10^{-2}. If this small number is not satisfied during the search process, then one should increase h and search again. If θ_1 is commensurate with θ_2, i.e., their ratio θ_1/θ_2 is a rational number, the final period is accurate, $T = i\frac{2\pi}{\theta_1} = j\frac{2\pi}{\theta_2}$. A rational number is the ratio of two integers. On the contrary, if θ_1 is incommensurate with θ_2, i.e., their ratio θ_1/θ_2 is not a rational number, the final period is only approximate, $T = i\frac{2\pi}{\theta_1} \approx j\frac{2\pi}{\theta_2}$.

Take $m = 3$ with circular frequency θ_1, θ_2, and θ_3 as another example. The search process for a single period is as follows.

$$\min_{i=1\ldots h}\min_{j=1\ldots h}\min_{k=1\ldots h}\left\{\left|i\frac{2\pi}{\theta_1}-j\frac{2\pi}{\theta_2}\right|+\left|i\frac{2\pi}{\theta_1}-k\frac{2\pi}{\theta_3}\right|+\left|j\frac{2\pi}{\theta_2}-k\frac{2\pi}{\theta_3}\right|\right\}. \tag{16}$$

Locate the combination of (i, j, k) that will lead to a minimum value of $\left\{\left|i\frac{2\pi}{\theta_1}-j\frac{2\pi}{\theta_2}\right|+\left|i\frac{2\pi}{\theta_1}-k\frac{2\pi}{\theta_3}\right|+\left|j\frac{2\pi}{\theta_2}-k\frac{2\pi}{\theta_3}\right|\right\}$. If θ_1, θ_2, and θ_3 are commensurate, the final period is accurate, $T = i\frac{2\pi}{\theta_1} = j\frac{2\pi}{\theta_2} = k\frac{2\pi}{\theta_3}$. On the contrary, if θ_1, θ_2, and θ_3 are incommensurate, the final period is only approximate, $T = i\frac{2\pi}{\theta_1} \approx j\frac{2\pi}{\theta_2} \approx k\frac{2\pi}{\theta_3}$.

Consider a system with an excitation of two frequencies which is shown in Fig. 2(a): $\theta_1 = \frac{\pi}{2}(T = 4)$ and $\theta_2 = \frac{\pi}{3}(T = 6)$. Taking $h = 10$, minimization of Eq. (15) leads to $i = 3$ and $j = 2$ which make $\left|i\frac{2\pi}{\theta_1} - j\frac{2\pi}{\theta_2}\right| = 0$. Since $\theta_1 = \frac{\pi}{2}$ is commensurate with $\theta_2 = \frac{\pi}{3}$, the system with two frequencies can be accurately represented by a system with a single frequency: the single period $T = 3\frac{2\pi}{\theta_1} = 2\frac{2\pi}{\theta_2} = 12$ and the single angular frequency is $\theta = \frac{2\pi}{T} = \frac{\pi}{6}$.

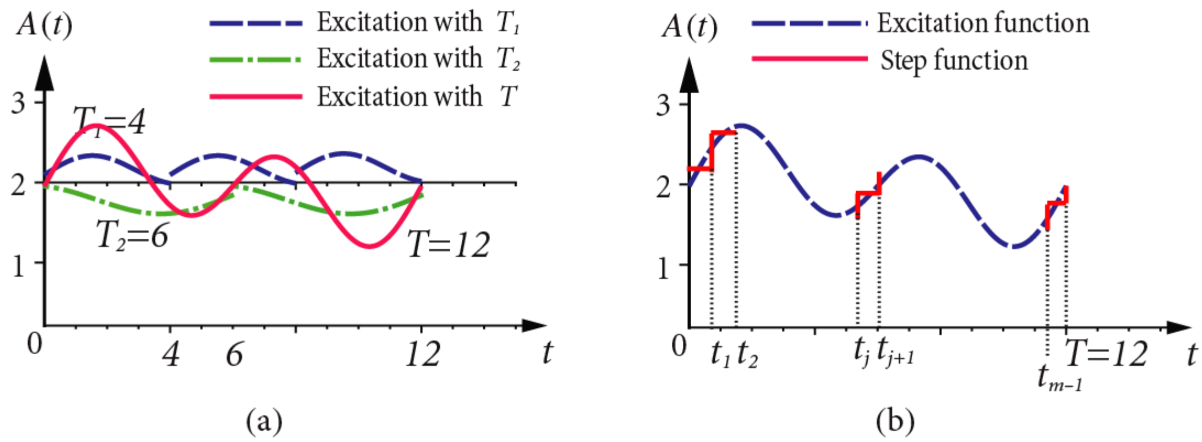


Figure 2: Approximation of two frequencies by a single frequency.

3.2 Numerical method for dynamic stability

After approximation of multiple frequencies by a single frequency, Eq. (11) is approximated by

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2[1 - S(t)]q = 0, S(t) = \frac{P_s}{P_0} + \sum_{i=1}^{m}\left(\frac{P_{di}}{P_0}\cos\theta_i t\right), S(t) \approx S(t+T) \quad 0 < t \leq T, \tag{17}$$

one can use the following numerical method to construct the instability diagram [7, 8, 9].

Divide the period T into m equal time intervals, as shown in Fig. 2(b). On interval $[t_j, t_{j+1}]$, the step function is determined by the average of the real excitation function,

$$h_j^2 = \frac{1}{t_{i+1} - t_j}\int_{t_j}^{t_{j+1}} \omega^2\left[1 - \frac{\beta^2}{\omega_0^2} - S(t)\right]dt, t_{j+1} - t_j = \frac{T}{m}, j = 0, 1, \cdots m-1. \tag{18}$$

The equation of motion in Eq. (17) is approximated by a differential equation with constant coefficients on the time interval $[t_j, t_{j+1}]$,

$$\frac{d^2q(t)}{dt^2} + {h_j}^2 q = 0, \quad t_j \le t \le t_{j+1}. \tag{19}$$

The solution of Eq. (19) is given by

$$\begin{bmatrix} q(\tau) \\ \dot{q}(\tau) \end{bmatrix} = \boldsymbol{G}(\tau) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \boldsymbol{G}(\tau) = \begin{bmatrix} \sin(h_j\tau) & \cos(h_j\tau) \\ h_j\cos(h_j\tau) & -h_j\sin(h_j\tau) \end{bmatrix}, \tau = t - t_j, \tag{20}$$

where constants C_1 and C_2 are determined by the local initial conditions, $q(t_j)$ and $\dot{q}(t_j)$,

$$\begin{bmatrix} q(t=t_j) \\ \dot{q}(t=t_j) \end{bmatrix} = \begin{bmatrix} \sin(0) & \cos(0) \\ h_j\cos(0) & -h_j\sin(0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ h_j & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \tag{21}$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \boldsymbol{G}^{-1}(0) \begin{bmatrix} q(t=t_j) \\ \dot{q}(t=t_j) \end{bmatrix} = \frac{1}{-h_j} \begin{bmatrix} 0 & -1 \\ -h_j & 0 \end{bmatrix} \begin{bmatrix} q(t=t_j) \\ \dot{q}(t=t_j) \end{bmatrix}, \tag{22}$$

$$\boldsymbol{G}^{-1}(0) = \begin{bmatrix} 0 & 1 \\ h_j & 0 \end{bmatrix}^{-1} = \frac{1}{-h_j} \begin{bmatrix} 0 & -1 \\ -h_j & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{h_j} \\ 1 & 0 \end{bmatrix}. \tag{23}$$

The solution at $t = t_{j+1}$ is determined by

$$\begin{bmatrix} q(t=t_{j+1}) \\ \dot{q}(t=t_{j+1}) \end{bmatrix} = \boldsymbol{G}(t_{j+1} - t_j)\, \boldsymbol{G}^{-1}(0) \begin{bmatrix} q(t=t_j) \\ \dot{q}(t=t_j) \end{bmatrix} = [M]_j \begin{bmatrix} q(t=t_j) \\ \dot{q}(t=t_j) \end{bmatrix}, \tag{24}$$

$$[M]_j = \boldsymbol{G}(t_{j+1} - t_j)\, \boldsymbol{G}^{-1}(0) = \boldsymbol{G}(T_0)\, \boldsymbol{G}^{-1}(0) = \begin{bmatrix} \cos(h_jT_0) & \frac{\sin(h_jT_0)}{h_j} \\ -h_j\sin(h_jT_0) & \cos(h_jT_0) \end{bmatrix}, \tag{25}$$

with $T_0 = t_{j+1} - t_j = \frac{T}{m}$.

Accumulating all responses in one period

$$\begin{bmatrix} q(t=T) \\ \dot{q}(t=T) \end{bmatrix} = [\mathcal{M}] \begin{bmatrix} q(t=t_0) \\ \dot{q}(t=t_0) \end{bmatrix}, [\mathcal{M}] = [M]_{m-1} \cdots [M]_1 [M]_0 = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix}, \tag{26}$$

where $[\mathcal{M}]$ is the state transition matrix whose eigenvalues ($\kappa_{1,2}$) can decide the stability of Eq.(17)

$$\begin{vmatrix} \mathcal{M}_{11} - \kappa & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} - \kappa \end{vmatrix} = 0, \tag{27}$$

$$\kappa^2 - (\mathcal{M}_{11} + \mathcal{M}_{22})\,\kappa + 1 = 0, \tag{28}$$

$$\kappa_{1,2} = \frac{\mathcal{M}_{11} + \mathcal{M}_{22}}{2} \pm \sqrt{\left(\frac{\mathcal{M}_{11} + M_{22}}{2}\right)^2 - 1}. \tag{29}$$

When $\left|\frac{\mathcal{M}_{11}+\mathcal{M}_{22}}{2}\right| > 1$, without loss of generality, one may suppose $\kappa_1 < 1$ then $\kappa_2 > 1$, the system is unstable if $\frac{1}{T}\log|\kappa_2| > \beta$. When $\left|\frac{\mathcal{M}_{11}+\mathcal{M}_{22}}{2}\right| < 1$, $\kappa_{1,2}$ are complex numbers, the system is stable. When $\left|\frac{\mathcal{M}_{11}+\mathcal{M}_{22}}{2}\right| = 1$, the system is in the critical state between unstable and stable, i.e., the boundaries.

4 DYNAMIC STABILITY ANALYSIS

4.1 A system with two incommensurate frequencies

Consider a system with two incommensurate frequencies:

$$\ddot{q} + d\dot{q} + [a + b_1 \cos(\pi t) + b_2 \cos(7t)]\, q = 0. \tag{30}$$

As stated in Section 3, to study the stability, the first step is to convert the system with two frequencies $\theta_1 = \pi$ and $\theta_2 = 7$ to an equivalent system with a single frequency. Four cases are listed in Table 1 with an increasing accuracy but a decreasing calculation efficiency.

Case I is for $h = 120$. The minimum value of $\left| i\frac{2\pi}{\theta_1} - j\frac{2\pi}{\theta_2} \right| = 0.012636$ is relatively large, so one can increase h such that a smaller $\left| i\frac{2\pi}{\theta_1} - j\frac{2\pi}{\theta_2} \right|$ can be achieved.

In Case II, $h = 350$ is supposed, minimization of Eq. (15) locates $i = 57$ and $j = 127$ which make $\left| i\frac{2\pi}{\theta_1} - j\frac{2\pi}{\theta_2} \right| = 0.0050665 < 10^{-2}$. So the system with two frequencies can be approximated by an equivalent system with a single frequency: the single period $T = 57\frac{2\pi}{\theta_1} \approx 127\frac{2\pi}{\theta_2} \approx 114$ and the single angular frequency is $\theta = \frac{2\pi}{T} = 0.05512$.

If more accuracy is needed, one can suppose $h = 700$ as in Case III of Table 1. Minimization of Eq. (15) leads to $i = 149$ and $j = 332$ which make $\left| i\frac{2\pi}{\theta_2} - j\frac{2\pi}{\theta_2} \right| = 0.0025033 < 10^{-2}$, then the single period $T = 149\frac{2\pi}{\theta_1} \approx 332\frac{2\pi}{\theta_2} \approx 298$ and the single angular frequency is $\theta = \frac{2\pi}{T} = 0.02108$.

If further accuracy is needed, one can suppose $h = 800$ as in Case IV of Table 1. Minimization of Eq. (15) leads to $i = 355$ and $j = 791$ which make $\left| i\frac{2\pi}{\theta_1} - j\frac{2\pi}{\theta_2} \right| = 0.00005999 < 10^{-4}$, then the single period $T = 355\frac{2\pi}{\theta_1} \approx 791\frac{2\pi}{\theta_2} \approx 710$ and the single angular frequency is $\theta = \frac{2\pi}{T} = 0.008849$.

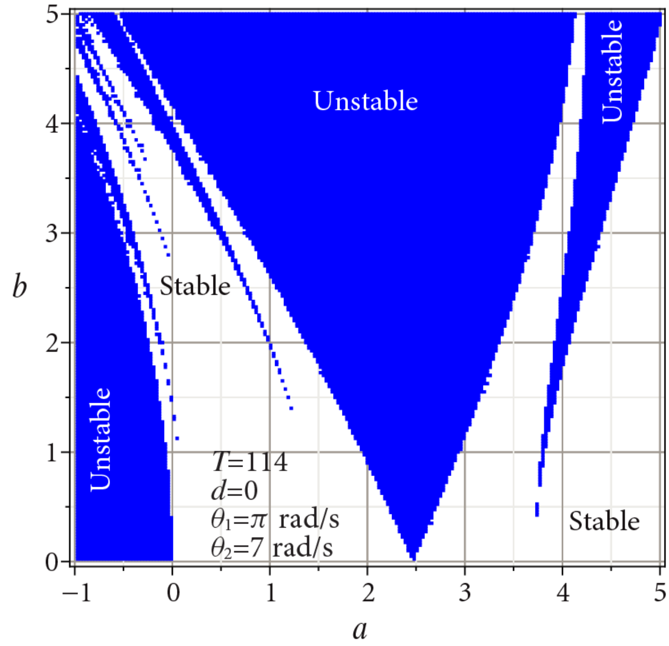
It can be seen that $T = 114$ is already accurate enough, this paper takes the single frequency/period $T = 114$. Using the numerical method in Section 3.2, the instability diagram is obtained in Fig. 3(a) for an undamped system and in Fig. 3(b) for a damped system with $d = 0.1$, where the blue shaded areas indicate stability and the blank areas represent instability. The same system has been studied by Sharma and Sinha [6] by taking $b_1 = b_2 = b$. It can be seen that the numerical results in Fig. 3(a) and Fig. 3(b) are almost identical to Fig. 3(f) and Fig. 12(i) in [6], respectively. So our numerical method is correct and accurate enough.

Case	h	i	j	$i\frac{2\pi}{\theta_1}$	$j\frac{2\pi}{\theta_2}$	T
I	120	35	78	70	70.012636	70
II	350	57	127	114	113.9949335	114
III	700	149	332	298	298.0025033	298
IV	800	355	791	710	709.9999400	710

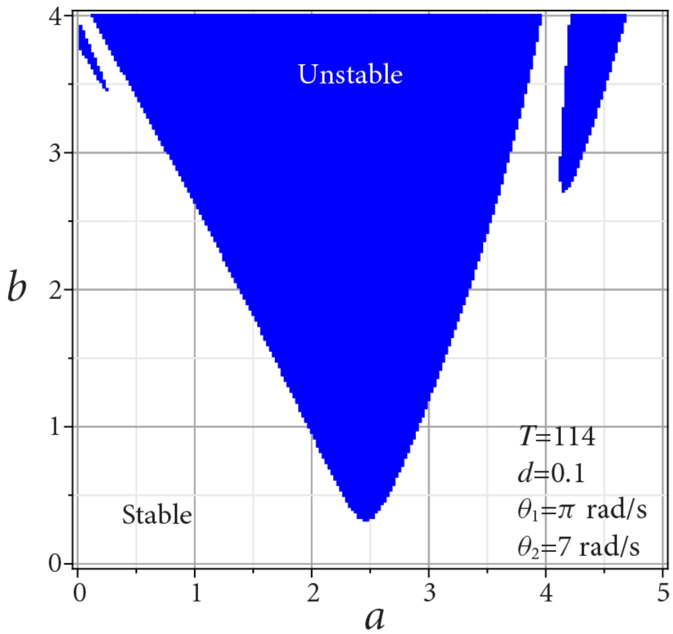
Table 1: Equivalent single period ($\theta_1 = \pi$, $\theta_2 = 7$)

4.2 A column under parametric loads with multiple frequencies

Consider a column with fixed connections: Length $L = 800$ mm, width $W = 25$ mm, thickness $t = 1$ mm, Young's modulus $E = 2 \times 10^{11}$ N/m^2, mass density $\rho = 7.8 \times 10^3$ kg/m^3. The minimum moment of inertia is $I = \frac{0.025 \times 0.001^3}{12} = 2.0833 \times 10^{-12}$ m^4.



(a)



(b)

Figure 3: Instability diagrams for a system with two incommensurate frequencies. (a) undamped; (b) damped with $d = 0.1$.

From Eq. (9), the fundamental frequency is $\omega_0 = 90.2$ rad/s or 14.4 Hz and the Euler's load is $P_0 = 102.81$ N.

$$\omega_0 = \left(\frac{2\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} = \left(\frac{2 * \pi}{0.8}\right)^2 \sqrt{\frac{2 \times 10^{11} * 2.0833 \times 10^{-12}}{7.8 \times 10^3(0.001)(0.025)}} = 90.2\frac{\text{rad}}{\text{s}} \text{ or } 14.2 \text{ Hz}, \quad (31)$$

$$P_0 = \mathrm{EI}\left(\frac{2\pi}{L}\right)^2 = 2\times 10^{11} * 2.0833\times 10^{-12} * \left(\frac{2*\pi}{0.8}\right)^2 = 25.7\ \mathrm{N}. \tag{32}$$

The equation of motion for the column is given by

$$\ddot{q} + 2\beta\dot{q} + 8130.3\left[1 - \frac{p_s}{p_0} - \sum_{i=1}^{m}\left(\frac{p_{di}}{p_0}\cos\theta_i t\right)\right] q = 0. \tag{33}$$

By assuming $\theta_1 = \frac{2\pi}{0.13} = 48.3\frac{\mathrm{rad}}{\mathrm{s}}$ [10], $\theta_2 = \frac{2\pi}{0.03} = 209.4\frac{\mathrm{rad}}{\mathrm{s}}$, and

$$P(t) = P_s + P_d(\cos 48.3t + \cos 209.4t), \lambda_s = \frac{P_s}{P_n}, \lambda_d = \frac{P_d}{P_n}. \tag{34}$$

Eq. (33) can be transformed into

$$\ddot{q} + 2\beta\dot{q} + 8130.3\left[1 - \lambda_s - \lambda_d(\cos 48.3t + \cos 209.4t)\right] q = 0. \tag{35}$$

Eq. (35) is a dynamic system under a parametric excitation with two frequencies, $\theta_1 = 48.3\frac{\mathrm{rad}}{\mathrm{s}}$ and $\theta_2 = 209.4\frac{\mathrm{rad}}{\mathrm{s}}$. This system is converted to an equivalent parametric excitation with a single frequency. Three cases are listed in Table 2 with an increasing accuracy but a decreasing calculation efficiency.

In Case I, $h = 30$ is supposed, minimization of Eq. (15) locates $i = 3$ and $j = 13$ which make $\left|i\frac{2\pi}{\theta_1} - j\frac{2\pi}{\theta_2}\right| = 0.0001863 < 10^{-3}$. So the system with two frequencies can be approximated by a system with a single frequency: the single period $T = 3\frac{2\pi}{\theta_1} \approx 13\frac{2\pi}{\theta_2} \approx 0.39$ and the single angular frequency is $\theta = \frac{2\pi}{T} = 16.1107$.

If more accuracy is needed, one can suppose $h = 700$ as in Case II of Table 2. Minimization of Eq. (15) leads to $i = 161$ and $j = 698$ which make $\left|i\frac{2\pi}{\theta_1} - j\frac{2\pi}{\theta_2}\right| = 2.9999967\times 10^{-9}$, then the single period $T = 161\frac{2\pi}{\theta_1} \approx 698\frac{2\pi}{\theta_2} \approx 0.29$ and the single angular frequency is $\theta = \frac{2\pi}{T} = 21.6662$.

Even if one supposes $h = 4000$, minimization of Eq. (15) remains unchanged as in Case II of Table 2. It can be seen that $T = 0.29$ is already accurate enough. Using the numerical method in Section 3.2, the instability diagram is obtained in Fig. 4 for a damped system with $\beta_0 = 0.1$, where the blue shaded areas indicate stability and the blank areas represent instability. From this diagrams, one can decide the stability for a specific system. For example, system A with $\lambda_s = 0.6$ and $\lambda_d = 0.3$ is stable, but system B with $\lambda_s = 0.7$ and $\lambda_d = 0.3$ is unstable, as shown in Fig. 4. This is confirmed by the bounded responses of point A and unbounded responses of point B in Fig. 5(a) and Fig. 5(b), respectively. The numerical algorithm for vibration responses is given in next section.

Case	h	i	j	$i\frac{2\pi}{\theta_1}$	$j\frac{2\pi}{\theta_2}$	T
I	30	3	13	0.3902599572	0.3900735865	0.39
II	700	161	698	20.94395104	20.943951043	0.29
III	4000	161	698	20.94395104	20.943951043	0.29

Table 2: Equivalent single period ($\theta_1 = 48.3$, $\theta_2 = 209.4395$)

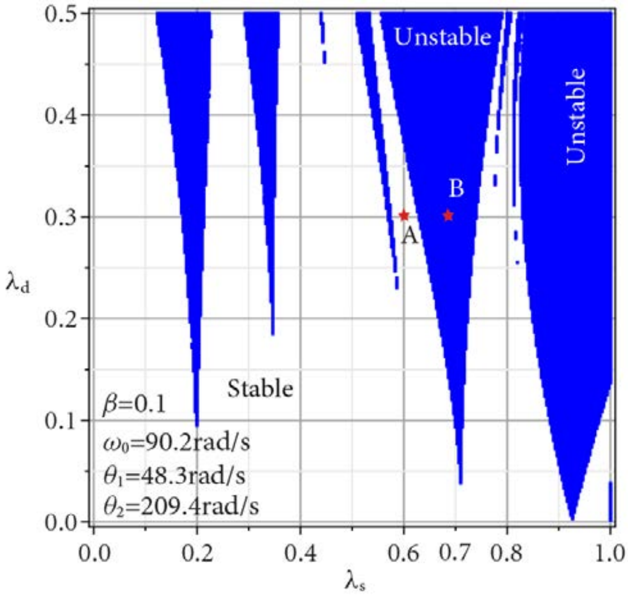


Figure 4: Instability diagrams for a column under parametric load.

4.3 Numerical solutions of vibration responses

The vibration responses in Eq. (17) can be calculated by numerical solutions of differential equations, such as the Runge-Kutta method [11]. Letting $y_1 = q$, $y_2 = \dot{q}$, one can obtained from Eq. (17)

$$\dot{y}_2 = -2\beta y_2 - \omega_0^2[1 - S(t)]y_1, \tag{36}$$

or in the matrix form,

$$\left\{ \begin{array}{c} \dot{y}_1 \\ \dot{y}_2 \end{array} \right\} = \left\{ \begin{array}{c} y_2 \\ -2\beta y_2 - \omega_0^2[1 - S(t)]y_1 \end{array} \right\} = \left\{ \begin{array}{c} f_1(t; y_1, y_2) \\ f_2(t; y_1, y_2) \end{array} \right\}. \tag{37}$$

By taking point A ($\lambda_s = 0.6, \lambda_d = 0.3$) in Fig. 4 as an example, the equation of motion is given by, from Eq. (35),

$$\ddot{q} + 0.2\dot{q} + 8130.3[0.4 - 0.3(\cos 48.3t + \cos 209.4t)]q = 0, \tag{38}$$

$$f_1(t; y_1, y_2) = y_2, f_2(t; y_1, y_2) = -0.2y_2 - 8130.3[0.4 - 0.3(\cos 48.3t + \cos 209.4t)]y_1. \tag{39}$$

Based on the initial conditions $q(t_0 = 0) = 0$ and $\dot{q}(t_0 = 0) = 0.01$ at point A, Eq. (38) can be numerically solved by the fourth-order Runge-Kutta method, which is implemented by MATLAB functions. The vibration responses at point A and point B is shown in Fig. 5(a) and Fig. 5(b), respectively. The parameters at point B are $\lambda_s = 0.7, \lambda_d = 0.3$, and the initial conditions are $q(t_0 = 0) = 0$ and $\dot{q}(t_0 = 0) = 0.00001$. It is seen that the responses at point A are bounded so the system is stable; on the contrary, the responses at point B are unbounded so the system is unstable. The vibration responses in Fig. 5 provide partial validation of the instability diagram in Fig. 4.

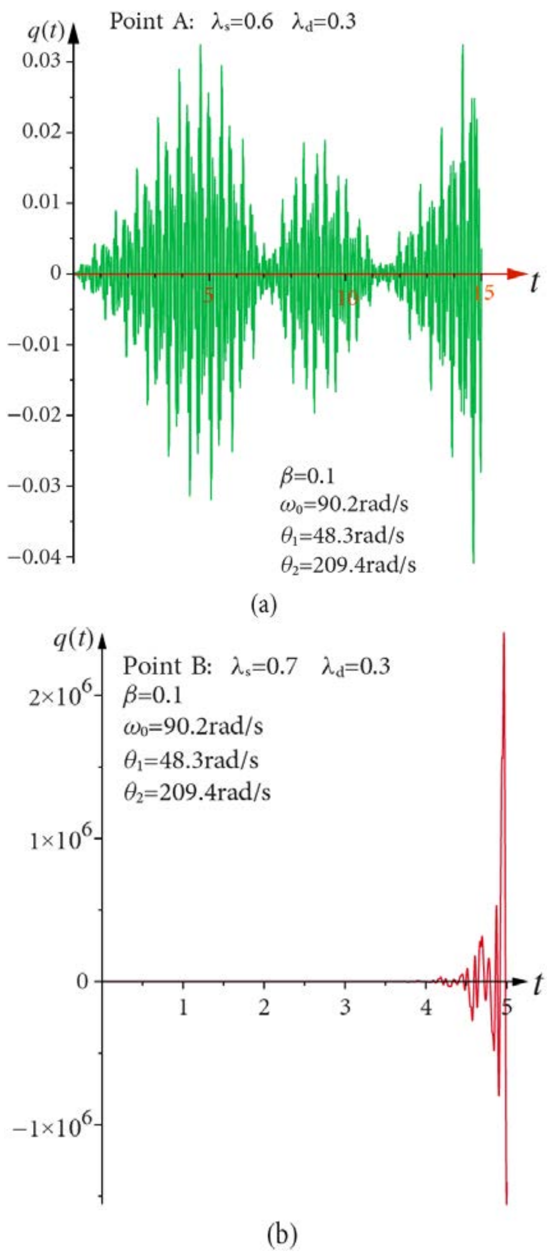


Figure 5: Responses of a column. (a) Point A; (b) Point B.

5 CONCLUSIONS

This paper proposes a new numerical method to study the stability of columns under dynamic loads with multiple frequencies. The equation of motion for columns with fixed-fixed connections under parametric load is derived and decoupled into an ordinary differential equation with variable coefficients of multiple frequencies. The first step of the numerical method is to approximate the system with multiple frequencies by an equivalent system with a single frequency (or period T) as closely as possible. Then divide the period T into enough equal time intervals. On each interval, the system is approximated by an equation of motion with constant coefficients which can be readily solved. Responses on each interval in one period T are accumulated. The stability can be investigated by the state transition matrix.

As a verification of the proposed numerical method, a system under parametric excitation with two incommensurate frequencies is studied and compared to existing results in the literature. The efficiency and accuracy of the proposed numerical method are demonstrated. As an application example, dynamic stability of a column under parametric load with two frequencies is investigated through instability diagrams and vibration responses. The vibration responses can provide partial validation of the instability diagrams.

The numerical results for instability diagrams can serve as a calibration of approximate results. The proposed method can be extended to multiple degrees of freedom systems under arbitrary parametric excitations with multiple frequencies.

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