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EXTENDED MODIFIED BRIDGE SYSTEM METHOD FOR VEHICLE-BRIDGE INTERACTION: TREATMENT OF ROTATIONAL DEGREES OF FREEDOM

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Abstract. *The extended modified bridge system (EMBS) method is a theoretically consistent approach to decouple the vehicle-bridge interaction (VBI) problem. The decoupling according to EMBS allows to solve the bridge independently of the vehicle by modifying the mechanical system of the bridge using additional (damping, stiffness, and loading) terms in the equation of motion. The EMBS method also identifies dominant coupling parameters, between the vehicle and the bridge, and their relative influence on the bridge response based on a dimensionless description of the VBI problem. The current study presents the EMBS method in the presence of rotational, in addition to translational, degrees of freedom (DOFs) in either the bridge or the vehicle subsystem. The analysis demonstrates in detail the procedure of making the multi-degree of freedom EOMs dimensionless by means of an example. Lastly, the study investigates the notion of physical similarity in VBI systems in order to highlight the significance of dimensionless parameters. Specifically, it demonstrates that two vehicle-bridge systems that possess the same dimensionless parameters will exhibit identical dimensionless response, regardless of their dimensional parameters.*

Keywords: Vehicle-bridge interaction, Railway bridges, Decoupled analysis, Rotational degrees of freedom.

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1 INTRODUCTION

The Vehicle-Bridge Interaction (VBI) phenomenon has received greater attention in recent years primarily as a result of the global expansion of high-speed railway (HSR) networks [1–7]. HSR lines are characterized by a significant number and length of bridges, and trains traverse them at speeds reaching 300 to 350 km/h [8]. In this context, VBI is important from the perspective of bridge design and improving the safety and comfort of vehicle rides.

To accurately analyze the vehicle-bridge interaction problem, it is necessary to develop robust and high-fidelity VBI simulations that can capture the dynamic behavior of the coupled system (see [9–11]). Given that modeling such a complex VBI system is challenging, it is beneficial to simplify the problem by decoupling vehicle-bridge systems. The simplest decoupling approach is the well-known moving load approximation, which considers vehicles as moving loads on the supporting bridge [12, 13]. As an improvement, Eurocode suggests combining the moving load approximation with a damping ratio, in addition to the structural damping of the bridge, to account for the favorable damping effect of VBI [14–17]. However, both the moving load and Eurocode techniques have several well-documented limitations. For example, the moving load method does not consider the effect of irregularities, and the Eurocode approach underestimates this effect [18, 19]. To overcome these limitations, Stoura and Dimitrakopoulos ([20]) proposed the Extended Modified Bridge System (EMBS) method.

The EMBS method [20] is a systematic way to decouple the vehicle-bridge interaction (VBI) problem. The EMBS approach modifies the mechanical system of the bridge with additional damping, stiffness, and loading terms, and allows to solve the bridge and vehicle subsystems independently, producing however accurate results compared to a coupled solution. The EMBS method can tackle multi-degree of freedom (MDOF) vehicle-MDOF bridge systems including, for example, complicated bridge configurations such as continuous bridges and arch bridges. Furthermore, the EMBS method identifies dominant coupling parameters and their relative influence on the bridge response based on an asymptotic expansion analysis on the dimensionless equations of motion (EOMs).

The current study presents a generalization of the EMBS approach to systems with both rotational and translational degrees of freedom (DOFs) in either the bridge or the vehicle subsystem. Specifically, it presents the governing terms in those EOMs and demonstrates the procedure of making the (MDOF) EOMs dimensionless. Additionally, the study discusses the concept of physical similarity in coupled vehicle-bridge mechanical systems in order to illustrate the importance of dimensionless parameters.

2 FORMULATION OF THE VEHICLE-BRIDGE INTERACTION PROBLEM

The EOM of a coupled MDOF vehicle-MDOF bridge system can take the form [20, 21]:

$$\mathbf{m}\ddot{\tilde{\mathbf{u}}}+\mathbf{c}\dot{\tilde{\mathbf{u}}}+\mathbf{k}\tilde{\mathbf{u}}-\mathbf{W}\tilde{\boldsymbol{\lambda}}=\mathbf{f} \tag{1}$$

where $\tilde{\mathbf{u}}(t)$ is the displacement vector of the whole (coupled) system:

$$\tilde{\mathbf{u}}^{\mathrm{T}}=\begin{bmatrix}\tilde{\mathbf{u}}_V & \tilde{\mathbf{u}}_B\end{bmatrix} \tag{2}$$

Throughout this study, an overdot indicates differentiation with respect to the dimensional time t, while prime denotes differentiation with respect to the location of the vehicle x. Superscript $()^{\mathrm{T}}$ denotes the transpose of a vector or matrix, while the tilde is used to distinguish dimensional from the corresponding dimensionless quantities, when needed (later on). $\mathbf{m}$, $\mathbf{c}$ and $\mathbf{k}$ are the corresponding mass, damping, and stiffness matrices of the entire system:

$$\mathbf{m} = \begin{bmatrix} \mathbf{m}_V & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_B \end{bmatrix}, \; \mathbf{c} = \begin{bmatrix} \mathbf{c}_V & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_B \end{bmatrix}, \; \mathbf{k} = \begin{bmatrix} \mathbf{k}_V & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_B \end{bmatrix} \tag{3}$$

The subscript $()_B$ denotes the bridge subsystem, and the subscript $()_V$ indicates the vehicle subsystem. $\tilde{\boldsymbol{\lambda}}$ is the vector of the contact forces between the vehicle and bridge subsystems, $\mathbf{W}(x)$ is the contact direction matrix, and $\mathbf{f}$ is the external force vector of the entire system:

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_V \\ -\mathbf{W}_B \end{bmatrix}, \; \mathbf{f} = \begin{bmatrix} \mathbf{f}_V \\ \mathbf{f}_B \end{bmatrix} \tag{4}$$

To estimate the contact force $\tilde{\boldsymbol{\lambda}}$ we assume "rigid contact" between the wheels and the rails, and follow the calculation procedure of [21] which leads to:

$$\mathbf{m}\ddot{\tilde{\mathbf{u}}} + \bar{\mathbf{c}}\dot{\tilde{\mathbf{u}}} + \bar{\mathbf{k}}\tilde{\mathbf{u}} = \bar{\mathbf{f}} \tag{5}$$

with:

$$\begin{aligned} \bar{\mathbf{c}} &= [\mathbf{E} - \mathbf{W}\mathbf{G}^{-1}\mathbf{W}^{\mathrm{T}}\mathbf{m}^{-1}]\mathbf{c} + 2v\mathbf{W}\mathbf{G}^{-1}\mathbf{W}'^{\mathrm{T}}, \\ \bar{\mathbf{k}} &= [\mathbf{E} - \mathbf{W}\mathbf{G}^{-1}\mathbf{W}^{\mathrm{T}}\mathbf{m}^{-1}]\mathbf{k} + v^2\mathbf{W}\mathbf{G}^{-1}\mathbf{W}''^{\mathrm{T}}, \\ \bar{\mathbf{f}} &= [\mathbf{E} - \mathbf{W}\mathbf{G}^{-1}\mathbf{W}^{\mathrm{T}}\mathbf{m}^{-1}]\mathbf{f} + v^2\mathbf{W}\mathbf{G}^{-1}\mathbf{r}_c'' \end{aligned} \tag{6}$$

where $\mathbf{E}$ is the identity matrix, $\mathbf{G}^{-1}$ is the mass participating in the contact interaction between the wheels and the bridge where its inverse $\mathbf{G} = \mathbf{W}^{\mathrm{T}}\mathbf{m}^{-1}\mathbf{W}$, $\mathbf{r}_c(x)$ is the irregularities vector, and v indicates the vehicle speed.

The EMBS method (Stoura and Dimitrakopoulos [20]) proceeds from Eq. (5) by partitioning Eq. (5) into the bridge and vehicle subsystems and further distinguishing the vehicle subsystem into the upper and wheel parts. Ultimately, the decoupling according to the EMBS method hinges on a dimensionless description of the vehicle-bridge EOM (Eq. (5)).

However, the first version of the EMBS method ([20]), did not distinguish between translational DOFs (or equivalently force equilibrium equations) and rotational DOFs (or equivalently moment equilibrium equations). Strictly speaking, the generic EOM (Eq. (5)) was dimensionalized only for translational DOFs ([20]). Here we extend the applicability of the EMBS method by explicitly considering rotational DOFs, in addition to translational DOFs, and dimensionalizing accordingly the different DOFs.

In particular, without loss of generality, we start from Eq. (5) and partition the original DOF vector $\tilde{\mathbf{u}}(t)$ into translational DOFs $\tilde{\mathbf{x}}$ and rotational DOFs $\tilde{\boldsymbol{\phi}}$:

$$\tilde{\mathbf{u}}^{\mathrm{T}} = \begin{bmatrix} \tilde{\mathbf{x}} & \tilde{\boldsymbol{\phi}} \end{bmatrix} \tag{7}$$

The EOM (Eq. (5)) of the whole (coupled) system becomes:

$$\begin{bmatrix} \mathbf{m}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_\phi \end{bmatrix} \begin{bmatrix} \ddot{\tilde{\mathbf{x}}} \\ \ddot{\tilde{\boldsymbol{\phi}}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{c}}_x & \bar{\mathbf{c}}_{x,\phi} \\ \bar{\mathbf{c}}_{x,\phi}^{\mathrm{T}} & \bar{\mathbf{c}}_\phi \end{bmatrix} \begin{bmatrix} \dot{\tilde{\mathbf{x}}} \\ \dot{\tilde{\boldsymbol{\phi}}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{k}}_x & \bar{\mathbf{k}}_{x,\phi} \\ \bar{\mathbf{k}}_{x,\phi}^{\mathrm{T}} & \bar{\mathbf{k}}_\phi \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}} \\ \tilde{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}}_x \\ \bar{\boldsymbol{f}}_\phi \end{bmatrix} \tag{8}$$

We dimensionalize the first of the matrix equations of Eq. (8) by dividing with a reference force $\mathscr{F}_{ref} = m_{ref} l_{ref} t_{ref}^{-2}$ and the second of Eq. (8) by dividing with a reference moment $\mathscr{M}_{ref} =$

$m_{ref}l_{ref}^{2}t_{ref}^{-2}$ respectively:

$$\begin{aligned}\frac{1}{\mathscr{F}_{ref}}\left(\mathbf{m}_x\ddot{\tilde{\mathbf{x}}}+\bar{\mathbf{c}}_x\dot{\tilde{\mathbf{x}}}+\bar{\mathbf{c}}_{x,\phi}\dot{\tilde{\boldsymbol{\phi}}}+\bar{\mathbf{k}}_x\tilde{\mathbf{x}}+\bar{\mathbf{k}}_{x,\phi}\tilde{\boldsymbol{\phi}}\right)&=\frac{1}{\mathscr{F}_{ref}}\left(\bar{\mathbf{f}}_x\right)\\ \frac{1}{\mathscr{M}_{ref}}\left(\mathbf{m}_\phi\ddot{\tilde{\boldsymbol{\phi}}}+\bar{\mathbf{c}}_\phi\dot{\tilde{\boldsymbol{\phi}}}+\bar{\mathbf{c}}_{x,\phi}^{\mathrm{T}}\dot{\tilde{\mathbf{x}}}+\bar{\mathbf{k}}_\phi\tilde{\boldsymbol{\phi}}+\bar{\mathbf{k}}_{x,\phi}^{\mathrm{T}}\tilde{\mathbf{x}}\right)&=\frac{1}{\mathscr{M}_{ref}}\left(\bar{\mathbf{f}}_\phi\right)\end{aligned} \tag{9}$$

The specific expressions of the reference force $\mathscr{F}_{ref}$ and the reference moment $\mathscr{M}_{ref}$ depend on the problem at hand. In general, there are multiple eligible and meaningful ways to dimensionalize a physical problem depending on the perspective and the goal of the analysis. For instance, if the interest is on the bridge subsystem, we can adopt as reference mass, the generalized mass m_B of the first mode of the bridge $m_{ref}=m_B$, as reference length the span of the bridge $l_{ref}=L_B$, and as reference time the first eigenfrequency ω_B of the bridge $t_{ref}=\omega_B^{-1}$ (Stoura and Dimitrakopoulos [18]). Another option is to define the dimensionless time with respect to the loading frequency $t_{ref}=(v/L_B)^{-1}$ (Dimitrakopoulos and Zeng [21]).

Regardless of the specific a choice of parameters m_{ref}, l_{ref}, and t_{ref}, it holds:

$$\begin{aligned}&\left(\frac{1}{m_{ref}}\mathbf{m}_x\right)\left(\frac{1}{l_{ref}t_{ref}^{-2}}\ddot{\tilde{\mathbf{x}}}\right)+\left(\frac{1}{m_{ref}t_{ref}^{-1}}\bar{\mathbf{c}}_x\right)\left(\frac{1}{l_{ref}t_{ref}^{-1}}\dot{\tilde{\mathbf{x}}}\right)\\&+\left(\frac{1}{m_{ref}l_{ref}t_{ref}^{-1}}\bar{\mathbf{c}}_{x,\phi}\right)\left(\frac{1}{t_{ref}^{-1}}\dot{\tilde{\boldsymbol{\phi}}}\right)+\left(\frac{1}{m_{ref}t_{ref}^{-2}}\bar{\mathbf{k}}_x\right)\left(\frac{1}{l_{ref}}\tilde{\mathbf{x}}\right)\\&+\left(\frac{1}{m_{ref}l_{ref}t_{ref}^{-2}}\bar{\mathbf{k}}_{x,\phi}\right)\left(\tilde{\boldsymbol{\phi}}\right)=\frac{1}{m_{ref}l_{ref}t_{ref}^{-2}}\left(\bar{\mathbf{f}}_x\right)\end{aligned} \tag{10}$$

$$\begin{aligned}&\left(\frac{1}{m_{ref}l_{ref}^{2}}\mathbf{m}_\phi\right)\left(\frac{1}{t_{ref}^{-2}}\ddot{\tilde{\boldsymbol{\phi}}}\right)+\left(\frac{1}{m_{ref}l_{ref}^{2}t_{ref}^{-1}}\bar{\mathbf{c}}_\phi\right)\left(\frac{1}{t_{ref}^{-1}}\dot{\tilde{\boldsymbol{\phi}}}\right)\\&+\left(\frac{1}{m_{ref}l_{ref}t_{ref}^{-1}}\bar{\mathbf{c}}_{x,\phi}^{\mathrm{T}}\right)\left(\frac{1}{l_{ref}t_{ref}^{-1}}\dot{\tilde{\mathbf{x}}}\right)+\left(\frac{1}{m_{ref}l_{ref}^{2}t_{ref}^{-2}}\bar{\mathbf{k}}_\phi\right)\left(\tilde{\boldsymbol{\phi}}\right)\\&+\left(\frac{1}{m_{ref}l_{ref}t_{ref}^{-2}}\bar{\mathbf{k}}_{x,\phi}^{\mathrm{T}}\right)\left(\frac{1}{l_{ref}}\tilde{\mathbf{x}}\right)=\frac{1}{m_{ref}l_{ref}^{2}t_{ref}^{-2}}\left(\bar{\mathbf{f}}_\phi\right)\end{aligned} \tag{11}$$

where each parenthesis represents accordingly a dimensionless response quantity or a dimensionless system parameter.

In matrix terms the dimensionless EOMs (Eqs. (10) and (11)) are:

$$\begin{aligned}\mathbf{M}_x\ddot{\mathbf{x}}+\bar{\mathbf{C}}_x\dot{\mathbf{x}}+\bar{\mathbf{C}}_{x,\phi}\dot{\boldsymbol{\phi}}+\bar{\mathbf{K}}_x\mathbf{x}+\bar{\mathbf{K}}_{x,\phi}\boldsymbol{\phi}&=\bar{\mathbf{F}}_x\\ \mathbf{M}_\phi\ddot{\boldsymbol{\phi}}+\bar{\mathbf{C}}_\phi\dot{\boldsymbol{\phi}}+\bar{\mathbf{C}}_{x,\phi}^{\mathrm{T}}\dot{\mathbf{x}}+\bar{\mathbf{K}}_\phi\boldsymbol{\phi}+\bar{\mathbf{K}}_{x,\phi}^{\mathrm{T}}\mathbf{x}&=\bar{\mathbf{F}}_\phi\end{aligned} \tag{12}$$

We can state the dimensionless EOM of the whole (coupled) system (Eq. 12) in the matrix form as:

$$\begin{bmatrix}\mathbf{M}_x & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_\phi\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{x}}\\ \ddot{\boldsymbol{\phi}}\end{bmatrix}+\begin{bmatrix}\bar{\mathbf{C}}_x & \bar{\mathbf{C}}_{x,\phi}\\ \bar{\mathbf{C}}_{x,\phi}^{\mathrm{T}} & \bar{\mathbf{C}}_\phi\end{bmatrix}\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\boldsymbol{\phi}}\end{bmatrix}+\begin{bmatrix}\bar{\mathbf{K}}_x & \bar{\mathbf{K}}_{x,\phi}\\ \bar{\mathbf{K}}_{x,\phi}^{\mathrm{T}} & \bar{\mathbf{K}}_\phi\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \boldsymbol{\phi}\end{bmatrix}=\begin{bmatrix}\bar{\mathbf{F}}_x\\ \bar{\mathbf{F}}_\phi\end{bmatrix} \tag{13}$$

or equivalently:

$$\mathbf{M}\ddot{\mathbf{u}}+\bar{\mathbf{C}}\dot{\mathbf{u}}+\bar{\mathbf{K}}\mathbf{u}=\bar{\mathbf{F}} \tag{14}$$

where $\mathbf{M} = \begin{bmatrix} \mathbf{M}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_\phi \end{bmatrix}$, $\bar{\mathbf{C}} = \begin{bmatrix} \bar{\mathbf{C}}_x & \bar{\mathbf{C}}_{x,\phi} \\ \bar{\mathbf{C}}^{\mathrm{T}}_{x,\phi} & \bar{\mathbf{C}}_\phi \end{bmatrix}$ and $\bar{\mathbf{K}} = \begin{bmatrix} \bar{\mathbf{K}}_x & \bar{\mathbf{K}}_{x,\phi} \\ \bar{\mathbf{K}}^{\mathrm{T}}_{x,\phi} & \bar{\mathbf{K}}_\phi \end{bmatrix}$ are the corresponding dimensionless mass, damping, and stiffness matrices of the entire system and $\mathbf{u}^{\mathrm{T}}(t) = \begin{bmatrix} \mathbf{x} & \boldsymbol{\phi} \end{bmatrix}$ denotes the dimensionless displacement vector of the (vehicle-bridge) system, with dimensionless translational DOFs:

$$\ddot{\mathbf{x}} = \frac{1}{L_B \omega_B^2} \ddot{\tilde{\mathbf{x}}};\ \dot{\mathbf{x}} = \frac{1}{L_B \omega_B} \dot{\tilde{\mathbf{x}}};\ \mathbf{x} = \frac{1}{L_B} \tilde{\mathbf{x}} \tag{15}$$

and dimensionless rotational DOFs:

$$\ddot{\boldsymbol{\phi}} = \frac{1}{\omega_B^2} \ddot{\tilde{\boldsymbol{\phi}}};\ \dot{\boldsymbol{\phi}} = \frac{1}{\omega_B} \dot{\tilde{\boldsymbol{\phi}}};\ \boldsymbol{\phi} = \tilde{\boldsymbol{\phi}} \tag{16}$$

Note that the time-derivatives of the dimensionless DOFs x and ϕ emerge naturally for differentiation with respect to the dimensionless time $\tau = \omega_B t$.

The pertinent dimensionless system submatrices (partitioned according to the nature of the DOFs: translation vs rotation) are:

$$\begin{gathered}
\mathbf{M}_x = \frac{1}{m_B} \mathbf{m}_x;\ \mathbf{M}_\phi = \frac{1}{m_B L_B^2} \mathbf{m}_\phi \\
\bar{\mathbf{C}}_x = \frac{1}{m_B \omega_B} \bar{\mathbf{c}}_x;\ \bar{\mathbf{C}}_{x,\phi} = \frac{1}{m_B L_B \omega_B} \bar{\mathbf{c}}_{x,\phi};\ \bar{\mathbf{C}}_\phi = \frac{1}{m_B L_B^2 \omega_B} \bar{\mathbf{c}}_\phi \\
\bar{\mathbf{K}}_x = \frac{1}{m_B \omega_B^2} \bar{\mathbf{k}}_x;\ \bar{\mathbf{K}}_{x,\phi} = \frac{1}{m_B L_B \omega_B^2} \bar{\mathbf{k}}_{x,\phi};\ \bar{\mathbf{K}}_\phi = \frac{1}{m_B L_B^2 \omega_B^2} \bar{\mathbf{k}}_\phi \\
\bar{\mathbf{F}}_x = \frac{1}{m_B L_B \omega_B^2} \bar{\mathbf{f}}_x;\ \bar{\mathbf{F}}_\phi = \frac{1}{m_B L_B^2 \omega_B^2} \bar{\mathbf{f}}_\phi
\end{gathered} \tag{17}$$

Consequently, to dimensionalize the generic EOM (Eq. (5)) we need to partition degrees of freedom into translational and rotational DOFs and generate the diagonal matrix $\boldsymbol{\Delta}_{ref}$ accordingly:

$$\boldsymbol{\Delta}_{ref} = \begin{bmatrix} \frac{1}{\mathscr{F}_{ref}} \mathbf{E}_{x,x} & \mathbf{0}_{x,\phi} \\ \mathbf{0}_{\phi,x} & \frac{1}{\mathscr{M}_{ref}} \mathbf{E}_{\phi,\phi} \end{bmatrix} = \begin{bmatrix} \frac{1}{m_{ref} l_{ref} t_{ref}^{-2}} \mathbf{E}_{x,x} & \mathbf{0}_{x,\phi} \\ \mathbf{0}_{\phi,x} & \frac{1}{m_{ref} l_{ref}^2 t_{ref}^{-2}} \mathbf{E}_{\phi,\phi} \end{bmatrix} \tag{18}$$

where $\mathbf{E}$ is the identity matrix, subscript x is the number of translational DOFs and subscript ϕ is the number of rotation DOFs. Multiplying both sides of the EOM (Eq. (5)) with the matrix $\boldsymbol{\Delta}_{ref}$ produces the dimensionless form of the EOMs pertinent to both translational and rotation DOFs:

$$\boldsymbol{\Delta}_{ref} \left(\mathbf{m}\ddot{\tilde{\mathbf{u}}} + \bar{\mathbf{c}}\dot{\tilde{\mathbf{u}}} + \bar{\mathbf{k}}\tilde{\mathbf{u}} \right) = \boldsymbol{\Delta}_{ref} \left(\bar{\mathbf{f}} \right) \tag{19}$$

Note that the proposed procedure for making the generic EOM (Eq. (5)) dimensionless is applicable to any equation of motion in matrix form for the VBI system.

3 ILLUSTRATION OF THE PROPOSED DIMENSIONLESS DESCRIPTION

This section illustrates the proposed procedure of making the vehicle-bridge EOMs dimensionless with reference to a generic bridge model and a generic vehicle model. To this end, consider the vehicle-bridge system of Figure 1, where the vehicle has 4 DOFs. Partition the DOFs of the vehicle into the upper part (i.e., non-contact) DOFs:

$$\tilde{\mathbf{u}}_u^{\mathrm{T}} = \begin{bmatrix} \tilde{z}_c & \tilde{\theta}_c \end{bmatrix} \tag{20}$$

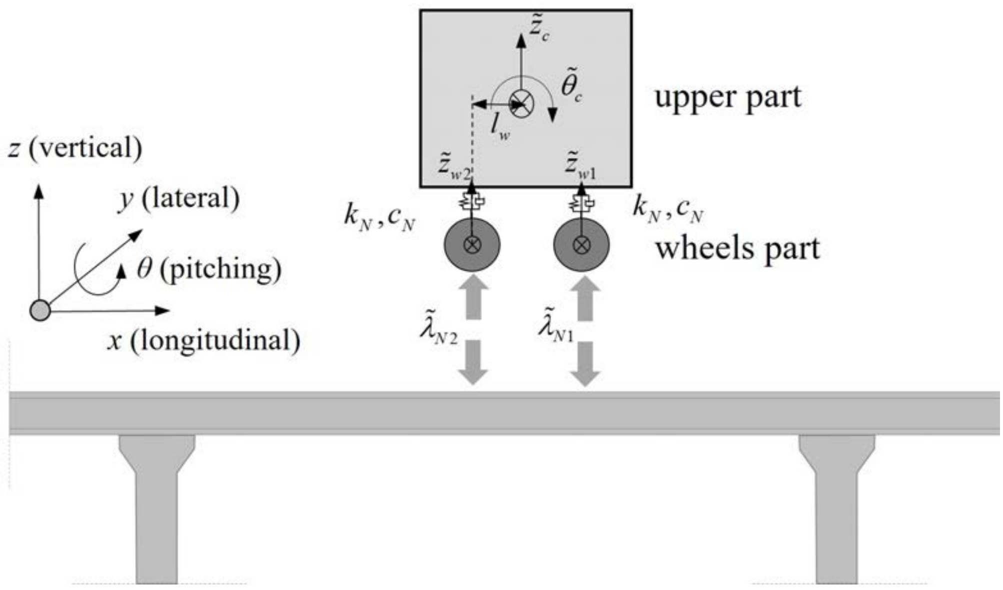

Figure 1: The vehicle-bridge interacting model

and the wheel part (i.e., contact) DOFs:

$$\tilde{\mathbf{u}}_w^{\mathrm{T}} = \begin{bmatrix} \tilde{z}_{w1} & \tilde{z}_{w2} \end{bmatrix} \tag{21}$$

Accordingly, we partition the mass, stiffness, and damping matrices of the vehicle as:

$$\mathbf{m}_V = \begin{bmatrix} \mathbf{m}_u & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_w \end{bmatrix},\ \mathbf{k}_V = \begin{bmatrix} \mathbf{k}_u & \mathbf{k}_{u,w} \\ \mathbf{k}_{w,u} & \mathbf{k}_w \end{bmatrix},\ \mathbf{c}_V = \begin{bmatrix} \mathbf{c}_u & \mathbf{c}_{u,w} \\ \mathbf{c}_{w,u} & \mathbf{c}_w \end{bmatrix} \tag{22}$$

The pertinent mass submatrices for the vehicle system shown in Fig. 1 are:

$$\mathbf{m}_u = diag[m_c, I_{c\theta}],\ \mathbf{m}_w = diag[m_w] = m_w\mathbf{E} \tag{23}$$

The pertinent stiffness submatrices are:

$$\mathbf{k}_u = \begin{bmatrix} 4k_N & 0 \\ 0 & 4k_N l_w^2 \end{bmatrix},\ \mathbf{k}_w = \begin{bmatrix} 2k_N & 0 \\ 0 & 2k_N \end{bmatrix},\ \mathbf{k}_{u,w} = \mathbf{k}_{w,u}^{\mathrm{T}} = \begin{bmatrix} -2k_N & -2k_N \\ -2k_N l_w & -2k_N l_w \end{bmatrix} \tag{24}$$

The damping matrix and submatrices are similar to the stiffness ones if one replaces the stiffness constant with the damping constant:

$$\mathbf{c}_u = \begin{bmatrix} 4c_N & 0 \\ 0 & 4c_N l_w^2 \end{bmatrix},\ \mathbf{c}_w = \begin{bmatrix} 2c_N & 0 \\ 0 & 2c_N \end{bmatrix},\ \mathbf{c}_{u,w} = \mathbf{c}_{w,u}^{\mathrm{T}} = \begin{bmatrix} -2c_N & -2c_N \\ -2c_N l_w & -2c_N l_w \end{bmatrix} \tag{25}$$

The contact force vector and contact direction matrix are respectively:

$$\tilde{\boldsymbol{\lambda}} = \begin{bmatrix} \tilde{\lambda}_{\mathrm{N}1} \\ \tilde{\lambda}_{\mathrm{N}2} \end{bmatrix},\ \mathbf{W}_w = \mathbf{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{26}$$

If we model the bridge using shape functions (e.g., the fundamental mode):

$$\mathbf{W}_B = \left[\psi^*_{\text{BN}}(x_1) \quad \psi^*_{\text{BN}}(x_2)\right] \tag{27}$$

where $\psi^*_{\text{BN}}(x_i) = \psi_{\text{BN}}(x_i)h(x_i)$ and $h(x_i) = H(x_i/v) - H((x_i - L)/v)$ in which $H(x)$ is the Heaviside step function, L is length of the bridge, and x_i indicates location of each contact point on the bridge. The function $h(x_i)$ activates the operation of each wheel on the bridge when it enters the bridge, and deactivates it when the wheel exits the bridge. $\psi_B(x)$ denotes the mode shape of the fundamental mode of the bridge. For example, $\psi_B(x)$ for a simply supported bridge is:

$$\psi_B(x) = sin\left(\frac{\pi x}{L}\right), \;\; 0 \leq x \leq L \tag{28}$$

The matrix $\mathbf{G}$ is given by:

$$\begin{aligned}\mathbf{G} = \mathbf{W}^{\text{T}}\mathbf{m}^{-1}\mathbf{W} &= \begin{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{W}_w\end{bmatrix}\\ -\mathbf{W}_B\end{bmatrix}^{\text{T}} \begin{bmatrix}\begin{bmatrix}\mathbf{m}_u & \mathbf{0}\\ \mathbf{0} & \mathbf{m}_w\end{bmatrix} & \mathbf{0}\\ \mathbf{0} & \mathbf{m}_B\end{bmatrix}^{-1} \begin{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{W}_w\end{bmatrix}\\ -\mathbf{W}_B\end{bmatrix}\\ &= \mathbf{W}_w^{\text{T}}\mathbf{m}_w^{-1}\mathbf{W}_w + \mathbf{W}_B^{\text{T}}\mathbf{m}_B^{-1}\mathbf{W}_B = \begin{bmatrix}\frac{1}{m_w} + \frac{\left(\psi^*_{\text{BN}}(x_1)\right)^2}{M_{\text{BN}}} & \frac{\left(\psi^*_{\text{BN}}(x_1)\psi^*_{\text{BN}}(x_2)\right)}{M_{\text{BN}}}\\ \frac{\left(\psi^*_{\text{BN}}(x_1)\psi^*_{\text{BN}}(x_2)\right)}{M_{\text{BN}}} & \frac{1}{m_w} + \frac{\left(\psi^*_{\text{BN}}(x_2)\right)^2}{M_{\text{BN}}}\end{bmatrix}\end{aligned} \tag{29}$$

Stoura and Dimitrakopoulos [20] presented the EOM of the upper part of the vehicle as:

$$\mathbf{m}_u\ddot{\tilde{\mathbf{u}}}_u + \mathbf{c}_u\dot{\tilde{\mathbf{u}}}_u + \mathbf{k}_u\tilde{\mathbf{u}}_u = -\mathbf{c}_{u,w}\mathbf{W}_w^{\text{-T}}\left(v\mathbf{W'}^{\text{T}}_B\tilde{\mathbf{u}}_B + \mathbf{W}_B^{\text{T}}\dot{\tilde{\mathbf{u}}}_B + v\mathbf{r}'_c\right) - \mathbf{k}_{u,w}\mathbf{W}_w^{\text{-T}}\left(\mathbf{W}_B^{\text{T}}\tilde{\mathbf{u}}_B + \mathbf{r}_c\right) \tag{30}$$

Importantly, for such two-dimensional VBI problems $\mathbf{W}_w = \mathbf{E}$ and thus its inverse exists and is equal to $\mathbf{W}_w^{\text{-T}} = \mathbf{E}$.

Substituting the mass (Eq. (23)), stiffness (Eq. (24)), and damping (Eq. (25)) matrices and degrees of freedom vector of the vehicle (Eq. (20)) into the EOM of the upper part of the vehicle (Eq. (30)) and considering the first mode of the bridge (Eq. (27)) gives:

$$\begin{aligned}&\begin{bmatrix}m_c & 0\\ 0 & I_{c\theta}\end{bmatrix}\begin{bmatrix}\ddot{\tilde{z}}_c\\ \ddot{\tilde{\theta}}_c\end{bmatrix} + \begin{bmatrix}4c_N & 0\\ 0 & 4c_Nl_w^2\end{bmatrix}\begin{bmatrix}\dot{\tilde{z}}_c\\ \dot{\tilde{\theta}}_c\end{bmatrix} + \begin{bmatrix}4k_N & 0\\ 0 & 4k_Nl_w^2\end{bmatrix}\begin{bmatrix}\tilde{z}_c\\ \tilde{\theta}_c\end{bmatrix} =\\ &-\begin{bmatrix}-2c_N & -2c_N\\ -2c_Nl_w & -2c_Nl_w\end{bmatrix}\left(v\begin{bmatrix}\psi^{*\prime}_{\text{BN}}(x_1)\\ \psi^{*\prime}_{\text{BN}}(x_2)\end{bmatrix}\tilde{\mathbf{u}}_B + \begin{bmatrix}\psi^*_{\text{BN}}(x_1)\\ \psi^*_{\text{BN}}(x_2)\end{bmatrix}\dot{\tilde{\mathbf{u}}}_B + v\begin{bmatrix}r'_c(x_1)\\ r'_c(x_2)\end{bmatrix}\right)\\ &-\begin{bmatrix}-2k_N & -2k_N\\ -2k_Nl_w & -2k_Nl_w\end{bmatrix}\left(\begin{bmatrix}\psi^*_{\text{BN}}(x_1)\\ \psi^*_{\text{BN}}(x_2)\end{bmatrix}\tilde{\mathbf{u}}_B + \begin{bmatrix}r_c(x_1)\\ r_c(x_2)\end{bmatrix}\right)\end{aligned} \tag{31}$$

In other words, the EOM of the translational DOF of the vehicle is:

$$\begin{aligned}&m_c\ddot{\tilde{z}}_c + 4c_N\dot{\tilde{z}}_c + 4k_N\tilde{z}_c =\\ &2\left(c_Nv\sum_{j=1}^{2}\psi^{*\prime}_{\text{BN}}(x_j) + k_N\sum_{j=1}^{2}\psi^*_{\text{BN}}(x_j)\right)\tilde{\mathbf{u}}_B + 2c_N\sum_{j=1}^{2}\psi^*_{\text{BN}}(x_j)\,\dot{\tilde{\mathbf{u}}}_B +\\ &2\left(c_Nv\sum_{j=1}^{2}r'_{\text{c}}(x_j) + k_N\sum_{j=1}^{2}r_{\text{c}}(x_j)\right)\end{aligned} \tag{32}$$

and the EOM of the rotational (pitching) DOF of the vehicle is:

$$
\begin{gathered}
I_{c\theta}\ddot{\tilde{\theta}}_c+4c_N l_w^2\dot{\tilde{\theta}}_c+4k_N l_w^2\tilde{\theta}_c= \\
2\left(c_N v l_w\sum_{j=1}^{2}\psi_{\mathrm{BN}}^{*\prime}\left(x_j\right)+k_N l_w\sum_{j=1}^{2}\psi_{\mathrm{BN}}^{*}\left(x_j\right)\right)\tilde{\mathbf{u}}_B+2c_N l_w\sum_{j=1}^{2}\psi_{\mathrm{BN}}^{*}\left(x_j\right)\dot{\tilde{\mathbf{u}}}_B+ \\
2\left(c_N v l_w\sum_{j=1}^{2}r_{\mathrm{c}}'\left(x_j\right)+k_N l_w\sum_{j=1}^{2}r_{\mathrm{c}}\left(x_j\right)\right)
\end{gathered}
\tag{33}
$$

As explained in section 2, to convert an equation of motion in matrix form into a dimensionless form, we need to partition the degrees of freedom into translational and rotational DOFs and create the matrix $\mathbf{\Delta}_{ref}$ for the system. Then, multiplying both sides of the equation with the matrix $\mathbf{\Delta}_{ref}$ produces dimensionless EOM. Given that the vehicle system in this example has one translational and one rotational DOF, the matrix $\mathbf{\Delta}_{ref}$ for this system is:

$$
\mathbf{\Delta}_{ref}=\begin{bmatrix}\frac{1}{\mathscr{F}_{ref}}\mathbf{E}_{x,x} & \mathbf{0}_{x,\phi}\\ \mathbf{0}_{\phi,x} & \frac{1}{\mathscr{M}_{ref}}\mathbf{E}_{\phi,\phi}\end{bmatrix}=\begin{bmatrix}\frac{1}{\mathscr{F}_{ref}} & 0\\ 0 & \frac{1}{\mathscr{M}_{ref}}\end{bmatrix}
\tag{34}
$$

Therefore, we need to multiply both sides of Eq. (31) with the matrix $\mathbf{\Delta}_{ref}$ (Eq. 34) to make it dimensionless. This leads to dividing both sides of the EOM corresponding to the translational DOF (Eq. (32)) with the reference force e.g., $\mathscr{F}_{ref}=m_B L_B\omega_B^2$, and the EOM corresponding to the rotational DOF (Eq. (33)) with the reference moment e.g., $\mathscr{M}_{ref}=m_B L_B^2\omega_B^2$. Hence, dividing both sides of Eq. (32) with $\mathscr{F}_{ref}=m_B L_B\omega_B^2$ and grouping the dimensionless terms leads:

$$
\begin{gathered}
\frac{m_c}{m_B}\frac{\ddot{\tilde{z}}_c}{L_B\omega_B^2}+4\frac{c_N}{m_B\omega_B}\frac{\dot{\tilde{z}}_c}{L_B\omega_B}+4\frac{k_N}{m_B\omega_B^2}\frac{\tilde{z}_c}{L_B}= \\
2\left(\frac{c_N}{m_B\omega_B}\frac{v}{L_B\omega_B}\sum_{j=1}^{2}L_B\psi_{\mathrm{BN}}^{*\prime}\left(x_j\right)+\frac{k_N}{m_B\omega_B^2}\sum_{j=1}^{2}\psi_{\mathrm{BN}}^{*}\left(x_j\right)\right)\frac{\tilde{\mathbf{u}}_B}{L_B}+2\frac{c_N}{m_B\omega_B}\sum_{j=1}^{2}\psi_{\mathrm{BN}}^{*}\left(x_j\right)\frac{\dot{\tilde{\mathbf{u}}}_B}{L_B\omega_B}+ \\
2\left(\frac{c_N}{m_B\omega_B}\frac{v}{L_B\omega_B}\sum_{j=1}^{2}r'_{\mathrm{c}}\left(x_j\right)+\frac{k_N}{m_B\omega_B^2}\sum_{j=1}^{2}\frac{r_{\mathrm{c}}\left(x_j\right)}{L_B}\right)
\end{gathered}
\tag{35}
$$

Similarly, dividing both sides of Eq. (33) with $\mathscr{M}_{ref}=m_B L_B^2\omega_B^2$ and grouping the dimensionless terms yields:

$$
\begin{gathered}
\frac{I_{c\theta}}{m_B L_B^2}\frac{\ddot{\tilde{\theta}}_c}{\omega_B^2}+4\frac{c_N l_w^2}{m_B\omega_B L_B^2}\frac{\dot{\tilde{\theta}}_c}{\omega_B}+4\frac{k_N l_w^2}{m_B L_B^2\omega_B^2}\tilde{\theta}_c= \\
2\left(\frac{c_N l_w}{m_B\omega_B L_B}\frac{v}{L_B\omega_B}\sum_{j=1}^{2}L_B\psi_{\mathrm{BN}}^{*\prime}\left(x_j\right)+\frac{k_N l_w}{m_B\omega_B^2 L_B}\sum_{j=1}^{2}\psi_{\mathrm{BN}}^{*}\left(x_j\right)\right)\frac{\tilde{\mathbf{u}}_B}{L_B}+2\frac{c_N l_w}{m_B\omega_B L_B}\sum_{j=1}^{2}\psi_{\mathrm{BN}}^{*}\left(x_j\right)\frac{\dot{\tilde{\mathbf{u}}}_B}{L_B\omega_B} \\
+2\left(\frac{c_N l_w}{m_B\omega_B L_B}\frac{v}{L_B\omega_B}\sum_{j=1}^{2}r'_{\mathrm{c}}\left(x_j\right)+\frac{k_N l_w}{m_B\omega_B^2 L_B}\sum_{j=1}^{2}\frac{r_{\mathrm{c}}\left(x_j\right)}{L_B}\right)
\end{gathered}
\tag{36}
$$

Substituting dimensionless parameters from Eqs. (15), (16), and (17) and also using dimensionless speed parameter $S_v=v/\left(L_B\omega_B\right)$, scaled slope of irregularities vector $R'_{\mathrm{c}}=r'_{\mathrm{c}}$, and scaled irregularities vector $R_c\left(x_j\right)=r_{\mathrm{c}}\left(x_j\right)/L_B$ in Eqs. (35) and (36) yields:

$$\begin{gathered}M_x\ddot{z}_c+4C_x\dot{z}_c+4K_xz_c=\\ 2\left(C_xS_v\sum_{j=1}^{2}L_B\psi_{\mathrm{BN}}^{*\prime}\left(x_j\right)+K_x\sum_{j=1}^{2}\psi_{\mathrm{BN}}^{*}\left(x_j\right)\right)\mathbf{u}_B+2C_x\sum_{j=1}^{2}\psi_{\mathrm{BN}}^{*}\left(x_j\right)\dot{\mathbf{u}}_B+\\ 2\left(C_xS_v\sum_{j=1}^{2}R_c'\left(x_j\right)+K_x\sum_{j=1}^{2}R_c\left(x_j\right)\right)\end{gathered}\tag{37}$$

and:

$$\begin{gathered}M_\phi\ddot{\theta}_c+4C_\phi\dot{\theta}_c+4K_\phi\theta_c=\\ 2\left(C_{x,\phi}S_v\sum_{j=1}^{2}L_B\psi_{\mathrm{BN}}^{*\prime}\left(x_j\right)+K_{x,\phi}\sum_{j=1}^{2}\psi_{\mathrm{BN}}^{*}\left(x_j\right)\right)\mathbf{u}_B+2C_{x,\phi}\sum_{j=1}^{2}\psi_{\mathrm{BN}}^{*}\left(x_j\right)\dot{\mathbf{u}}_B+\\ 2\left(C_{x,\phi}S_v\sum_{j=1}^{2}R'{}_c\left(x_j\right)+K_{x,\phi}\sum_{j=1}^{2}R_c\left(x_j\right)\right)\end{gathered}\tag{38}$$

where all variables and parameters in Eqs. (37) and (38) are dimensionless.

3.1 FEM description of the bridge

In this section, we discuss the differences in the EOMs that arise from using the finite element method instead of considering the first mode of the bridge. To this end, assume we model the bridge using the FEM, and consider the DOF vector of a typical bridge-deck element (two-node Euler-Bernoulli beam element with two DOFs at each node); for example:

$$\tilde{\mathbf{u}}_B^{\mathrm{T}}=\begin{bmatrix}\tilde{w}_1 & \tilde{\theta}_{y1} & \tilde{w}_2 & \tilde{\theta}_{y2}\end{bmatrix}\tag{39}$$

The corresponding direction matrix would be:

$$\mathbf{W}_B\left(x\right)=\begin{bmatrix}\psi_{\mathrm{BN1}}^{*}\left(x_1\right) & \psi_{\mathrm{BN1}}^{*}\left(x_2\right)\\ \psi_{\mathrm{BN2}}^{*}\left(x_1\right) & \psi_{\mathrm{BN2}}^{*}\left(x_2\right)\\ \psi_{\mathrm{BN3}}^{*}\left(x_1\right) & \psi_{\mathrm{BN3}}^{*}\left(x_2\right)\\ \psi_{\mathrm{BN4}}^{*}\left(x_1\right) & \psi_{\mathrm{BN4}}^{*}\left(x_2\right)\end{bmatrix}\tag{40}$$

where again, $\psi_{\mathrm{BN}}^{*}(x_i)=\psi_{\mathrm{BN}}(x_i)h(x_i)$ and $h(x_i)=H(x_i/v)-H((x_i-L_e)/v)$, but now L_e is the length of the element. Function $h(x_i)$ here activates and deactivates the action of each wheel of the vehicle on the pertinent element of the bridge, when it enters and leaves the element, respectively. We can state shape functions $\psi_{\mathrm{BN}}(x)$ for a two-node Euler-Bernoulli beam element with two DOFs at each node as follows:

$$\begin{gathered}\psi_{\mathrm{BN1}}\left(x\right)=1-3\left(\frac{x}{L_e}\right)^2+2\left(\frac{x}{L_e}\right)^3\\ \psi_{\mathrm{BN2}}\left(x\right)=L_e\left(\frac{x}{L_e}\right)-2L_e\left(\frac{x}{L_e}\right)^2+L_e\left(\frac{x}{L_e}\right)^3\\ \psi_{\mathrm{BN3}}\left(x\right)=3\left(\frac{x}{L_e}\right)^2-2\left(\frac{x}{L_e}\right)^3\\ \psi_{\mathrm{BN4}}\left(x\right)=-L_e\left(\frac{x}{L_e}\right)^2+L_e\left(\frac{x}{L_e}\right)^3\end{gathered}\tag{41}$$

The matrix $\mathbf{G}$ becomes:

$$\mathbf{G} = \mathbf{W}^T \mathbf{m}^{-1} \mathbf{W} = \mathbf{W}_w^{\mathrm{T}} \mathbf{m}_w^{-1} \mathbf{W}_w + \mathbf{W}_B^{\mathrm{T}} \mathbf{m}_B^{-1} \mathbf{W}_B = \begin{bmatrix} \frac{1}{m_w} & 0 \\ 0 & \frac{1}{m_w} \end{bmatrix} + \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} \tag{42}$$

with:

$$\begin{aligned} g_{11} &= \frac{\left(\psi^*_{\mathrm{BN1}}(x_1)\right)^2}{m_{\tilde{w}_1}} + \frac{\left(\psi^*_{\mathrm{BN2}}(x_1)\right)^2}{m_{\tilde{\theta}_{y1}}} + \frac{\left(\psi^*_{\mathrm{BN3}}(x_1)\right)^2}{m_{\tilde{w}_2}} + \frac{\left(\psi^*_{\mathrm{BN4}}(x_1)\right)^2}{m_{\tilde{\theta}_{y2}}} \\ g_{12} &= \frac{\psi^*_{\mathrm{BN1}}(x_1)\,\psi^*_{\mathrm{BN1}}(x_2)}{m_{\tilde{w}_1}} + \frac{\psi^*_{\mathrm{BN2}}(x_1)\,\psi^*_{\mathrm{BN2}}(x_2)}{m_{\tilde{\theta}_{y1}}} + \frac{\psi^*_{\mathrm{BN3}}(x_1)\,\psi^*_{\mathrm{BN3}}(x_2)}{m_{\tilde{w}_2}} + \frac{\psi^*_{\mathrm{BN4}}(x_1)\,\psi^*_{\mathrm{BN4}}(x_2)}{m_{\tilde{\theta}_{y2}}} \\ g_{22} &= \frac{\left(\psi^*_{\mathrm{BN1}}(x_2)\right)^2}{m_{\tilde{w}_1}} + \frac{\left(\psi^*_{\mathrm{BN2}}(x_2)\right)^2}{m_{\tilde{\theta}_{y1}}} + \frac{\left(\psi^*_{\mathrm{BN3}}(x_2)\right)^2}{m_{\tilde{w}_2}} + \frac{\left(\psi^*_{\mathrm{BN4}}(x_2)\right)^2}{m_{\tilde{\theta}_{y2}}} \end{aligned} \tag{43}$$

where $m_{\tilde{w}_1}$, $m_{\tilde{\theta}_{y1}}$, $m_{\tilde{w}_2}$, and $m_{\tilde{\theta}_{y2}}$ are the bridge element masses corresponding to the element DOFs (Eq. 39). When we model the bridge with FEM, both translational and rotational DOFs are present in the bridge subsystem. Therefore, to convert the equations of motion of the bridge into a dimensionless form, we need to partition EOMs into translational and rotational DOFs and create the matrix $\boldsymbol{\Delta}_{ref}$ (Eq. (18)) for the system. By multiplying the equations of motion with the appropriate $\boldsymbol{\Delta}_{ref}$ matrix, we can obtain the dimensionless equations of motion for both translational and rotational degrees of freedom. Specifically, we can express $\boldsymbol{\Delta}_{ref}$ for the EOMs of a bridge when we model it with a two-node Euler-Bernoulli beam element and its DOF vector is given in Eq. (39) as:

$$\boldsymbol{\Delta}_{\mathbf{ref}} = \begin{bmatrix} \frac{1}{\mathscr{F}_{ref}} & 0 & 0 & 0 \\ 0 & \frac{1}{\mathscr{M}_{ref}} & 0 & 0 \\ 0 & 0 & \frac{1}{\mathscr{F}_{ref}} & 0 \\ 0 & 0 & 0 & \frac{1}{\mathscr{M}_{ref}} \end{bmatrix} = \begin{bmatrix} \frac{1}{m_B L_B \omega_B^2} & 0 & 0 & 0 \\ 0 & \frac{1}{m_B L_B^2 \omega_B^2} & 0 & 0 \\ 0 & 0 & \frac{1}{m_B L_B \omega_B^2} & 0 \\ 0 & 0 & 0 & \frac{1}{m_B L_B^2 \omega_B^2} \end{bmatrix} \tag{44}$$

4 PHYSICAL SIMILARITY

This section investigates the concept of physical similarity to highlight the importance of dimensionless parameters in the vehicle-bridge interaction problem. The underlying notion behind physical similarity in VBI systems is that if the dimensionless terms of two mechanical systems are the same, regardless of whether the dimensional terms are different, the dimensionless response for two systems will be the same and the dimensional response will be proportional. To illustrate, we consider two (coupled) vehicle-bridge systems (A and B). Table 1 presents the dimensional and dimensionless properties of both systems. System A corresponds the Pioneer train and Skidträsk bridge (Pioneer - Skidträsk bridge) [7, 18]. Although the dimensional parameters of systems A and B differ, their dimensionless parameters are identical (Table 1). Hence, the two systems are physically similar.

Figure 2 depicts the dimensional and dimensionless displacement time history at the midspan of the bridge under the passage of a ten-vehicle train, as well as the dimensional and dimensionless acceleration of the pitching DOF of the first vehicle. Note that for obtaining the numerical results we model the bridge with the finite element method (FEM) and the vehicle with 10 degrees of freedom (DOFs). As expected according to the concept of physical similarity, the

Table 1: Dimensional and dimensionless properties of two systems

Dimensional parameters		
Description (unit)	System A	System B
Bridge length L_B (m)	36	54
Mass of the bridge m_B (kg)	306,000	459,000
Moment of inertia I (m^4)	0.82	2.77
Young's modulus E (GPa)	210	210
Fundamental frequency f_B (Hz)	3.86	3.15
Damping ratio ζ_B (%)	0.5	0.5
Stiffness of the bridge k_B (N/m)	$179.76 \cdot 10^6$	$179.76 \cdot 10^6$
Mass of the train m_V (kg)	58,000	87,000
Mass moment of inertia of the train $I_{c\theta}$ ($kg \cdot m^2$)	1,064,400	3,592,350
Stiffness of the train $k_{V,i}$ (kN/m)	$2.08 \cdot 10^3$	$2.08 \cdot 10^3$
Damping of the train $c_{V,i}$ (kNs/m)	60	73.8
Length of the train $\tilde{d}$ (m)	25	37.5
Speed of the vehicle v (km/h)	250	307.5
Dimensionless parameters		
Description	Systems A and B	
Scaled distance $d_i = \tilde{d}/L_B$	0.69	
Speed parameter $S_v = v/(\omega_B L_B)$	0.08	
Mass ratio $M_x = m_V/m_B$	0.19	
Mass ratio $M_\phi = I_{c\theta}/(m_B L_B^2)$	0.0027	
Stiffness ratio $K_i = k_{V,i}/k_B$	0.012	
Impedance ratio $C_i = c_{V,i}/(m_B \omega_B)$	0.008	
Normalized vehicle weight $F_{B,i} = f_{B,i}/(k_B L_B)$	$2.2 \cdot 10^{-5}$	

dimensional response (displacement and acceleration) for systems A and B, as shown in Figures 2a, 2b, 2d, and 2e, are different since the dimensional parameters of these two systems are different. More importantly though, Figs. 2c and 2f indicate that the dimensionless response for both systems is the same, since the dimensionless parameters for both systems are the same. This demonstrates that two VBI systems that are physically similar produce the identical dimensionless response, confirming the physical similarity concept.

5 CONCLUSIONS

The extended modified bridge system (EMBS) method offers a consistent way to decouple a coupled vehicle-bridge system. Ultimately, it allows to solve the bridge and vehicle subsystems independently, providing results that are comparable to a coupled solution. The present study focuses on the vertical plane of vehicle-bridge interaction (VBI), and considers a generic train-bridge system with multi-degree of freedom (MDOF), which can be applied to a wide range of train-bridge systems. The aim is to demonstrate the application of the EMBS method on systems that have both translational and rotational degrees of freedom. In this context, the study first provides a comprehensive description of the governing terms in the equations of motion (EOMs). Then, it illustrates the procedure of converting the multi-degree of freedom EOMs into a dimensionless form. In this context, the analysis explores the differences in the EOMs that arise from simulating the bridge using the finite element method, as opposed to using mode superposition (or merely the first mode of the bridge). Furthermore, the study

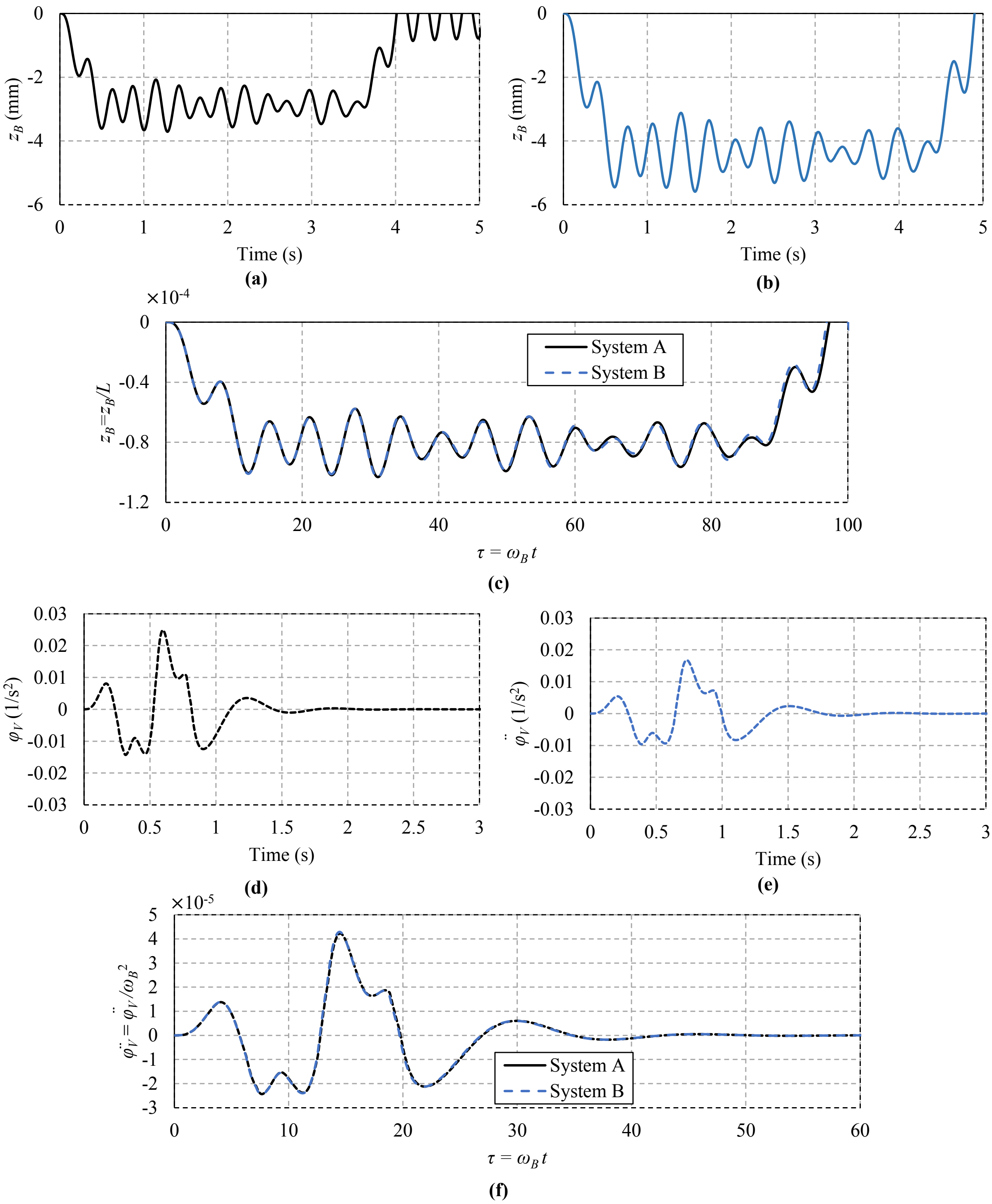

Figure 2: Time history of the displacement of the bridge midpoint (a) dimensional for system A (b) dimensional for system B (c) dimensionless for both systems A and B Time history of the acceleration of the pitching DOF of the vehicle (d) Dimensional for system A (e) dimensional for system B (f) dimensionless for both systems A and B

highlights the importance of a dimensionless description of the VBI problem by demonstrating the principle of physical similarity: the dimensionless response of two vehicle-bridge systems will be the same as long as their dimensionless terms are identical, even if two systems have varying dimensional parameters.

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