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A NOVEL FINITE ELEMENT FORMULATION FOR THE DYNAMIC ANALYSIS OF DAMPED EULER-BERNOULLI BEAMS UNDER MOVING LOADS

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Abstract. *A novel finite element formulation for the dynamic analysis of linear-elastic damped Euler-Bernoulli beams subjected to moving loads is proposed. The formulation is consistent with a complementary form of the well known Hamilton's variational principle, and will be used to address some numerical tests in both frequency and time domains. The effectiveness and accuracy of the formulation will be assessed and discussed by comparison between the obtained results and those rendered by the standard displacement-based finite element formulation. As it will be demonstrated, the proposed formulation not only renders continuous bending moment and shear-force distributions, a desired feature in the structural design field, but also has a superior accuracy than that provided by the displacement formulation that uses the same number of nodal degrees-of-freedom.*

Keywords: Structural Dynamics, Euler-Bernoulli Beams, Moving Loads, Finite Element Analysis.

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1 INTRODUCTION

Moving loads are commonly found in practical civil, mechanical, and aerospace engineering problems. For example, moving train wheels over a railway track, a moving vehicle on a bridge, a roller coaster moving on its track, an overhead crane used either for manufacturing or maintenance applications, a sliding robot manipulator, a span morphing aircraft wing, etc. Early studies highlighted the fact that moving loads on structures may induce significantly higher dynamic deflections and stresses than those observed in the static case. Therefore, it is crucial to conceive structural models in which the dynamic effects are taken into account and, hence, to develop effective and accurate methods that can be used to analyse such models.

In fact, the dynamics of structures under moving loads has been an active research area, particularly in the past decades, with the focus on the numerical analysis and modelling [1], mainly due to the rapidly increasing computer performance which has allowed the development of highly robust and sophisticated computational and numerical methods.

In moving load type problems, the applied load changes its position on the structure in a time-dependent way. The numerical discretization of time-dependent problems involves the following two stages: (i) spatial discretization and (ii) temporal discretization. Several methods have been employed for the spatial discretization, such as the finite difference, finite element, boundary element, and differential quadrature methods. Among these, the finite element method has been the most adopted one, due to its versatility, capability to handle complex geometries, etc. The application of the finite element method results in a set of ordinary differential equations in time for the finite element variables. The resulting system of equations needs, thus, to be further discretized in time, often using finite difference schemes for the time derivatives, such as Newmark's, Wilson's or Runge-Kutta schemes.

Among the various formulations of the finite element method, its displacement-based formulation has been the most adopted one. In this formulation, the displacements are taken as the fundamental unknowns, selected such that only those patterns which satisfy compatibility requirements are to be admitted as possible solutions. Examples of this formulation applied to the dynamics of beams can be found in [2, 3, 4]. However, this formulation leads to bending moment and shear-force distributions that are discontinuous between elements and that do not satisfy, in general, the equilibrium boundary conditions. As a result, and since for many structural engineering purposes the 'stress' distribution is, very often, the paramount information needed, applications of this formulation usually involve an averaging procedure to obtain continuous 'stress' distributions for design calculations. Formulations that avoid the need for these averaging procedures are the so-called equilibrium-based formulations, which are capable of producing equilibrated solutions, *i.e.*, solutions satisfying in strong form the equilibrium differential equations, as well as the equilibrium boundary conditions. Examples of these formulations applied to one-dimensional beam problems were proposed in, *e.g.*, [5, 6, 7, 8], but all of them are limited to statics.

The standard displacement-based finite element formulations for structural dynamics may be derived from the well known Hamilton's variational principle. However, attempts to formulate dual, or complementary, forms of the Hamilton's principle are very scarce in the literature. A few exceptions in the framework of the free vibration analysis of beams can be found in [9, 10].

To the best of the author's knowledge, equilibrium-based finite element formulations for the dynamic analysis of beams under moving loads have not yet been proposed in the literature. It is the goal of the present work to introduce a novel finite element formulation, which relies on a complementary variational principle, in conjunction with a Newmark's time discretization scheme, for the dynamic analysis of linear-elastic damped Euler-Bernoulli beams subjected to moving loads. The formulation will be used to address some numerical tests in both frequency and time domains, and its effectiveness and accuracy will be assessed and discussed by comparison between the obtained results and those rendered by the standard two-node displacement-based finite element formulation. As it will be shown, the proposed formulation not only can produce continuous bending moment and shear-force distributions, a desired feature in the field of structural design, but also has a superior accuracy than that given by the displacement-based formulation.

2 DYNAMICS OF EULER-BERNOULLI BEAMS UNDER MOVING LOADS: BVP

Let us consider a beam traversed by a moving load, assumed as a concentrated load with constant speed v and amplitude $g(t)$. The following assumptions will be adopted in this study: (1) The beam is homogeneous and of constant cross-section; (2) plane cross-sections remain plane after deformation; (3) the Euler-Bernoulli hypothesis holds; (4) only a single moving load is considered to travel on the beam at a time; (5) the damping of the beam is of the viscous (external) type; and (6) the beam is initially at rest before the load moves in.

The moving load may be represented in terms of a distribution over the beam domain as

$$f(x,t) = g(t)\delta(x - vt) \tag{1}$$

where δ represents the Dirac delta function, with vt the load position at time t.

We shall use $w(x,t)$ to denote the beam deflection along the y axis at position x and time t. L is the length of the beam, m the mass of the beam per unit length, c the viscous damping coefficient per unit length, E the Young's modulus, and I the moment of inertia of the beam. Based on the aforementioned assumptions, the governing equation of motion of the beam can be written as

$$EIw'''' + m\ddot{w} + c\dot{w} = f(x,t) \tag{2}$$

where primes ($'$) and dots ($\dot{}$) denote differentiation with respect to coordinate x and time t, respectively.

The response of the beam is said to be a forced response for

$$0 \leq \frac{vt}{L} \leq 1 \tag{3}$$

and a free response for

$$\frac{vt}{L} > 1 \tag{4}$$

The boundary conditions depend on the beam supports. Considering, for instance, a clamped-hinged beam, as illustrated in Figure 1, the following conditions must hold

$$w(0,t) = 0 \tag{5a}$$
$$w(L,t) = 0 \tag{5b}$$
$$w'(0,t) = 0 \tag{5c}$$
$$EIw''(L,t) = 0 \tag{5d}$$

Equations (5a-5c) represent kinematical boundary conditions, whereas equation (5d) stands for an equilibrium boundary condition.

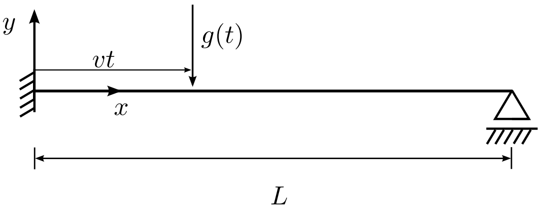

Figure 1: Clamped-Hinged Beam.

The initial conditions are

$$w(x,0) = 0 \tag{6a}$$
$$\dot{w}(x,0) = 0 \tag{6b}$$

as the beam is assumed to be at rest prior to the arrival of the moving load.

Noting that both the impulse per unit length, p, and the bending moment, M, can be related with the beam deflection, w, as

$$p = m\dot{w} \tag{7}$$

and

$$\frac{M}{EI} - w'' = 0 \tag{8}$$

the governing equation (2) can be rewritten in the following form

$$M'' + \dot{p} + \frac{c}{m}p = f \tag{9}$$

which is known as the equilibrium differential equation of the beam.

The shear-force acting on the beam, Q, is related with the bending moment, M, as follows

$$M' - Q = 0 \tag{10}$$

3 VARIATIONAL SETTING

3.1 Primal (or Hamilton's type) Variational Principle

Based on the classical variational approach, the Lagrangian of an undamped ($c = 0$) beam is defined as the following one-field energy functional

$$L(w, \dot{w}) = T(\dot{w}) - V(w) \tag{11}$$

with T and V the kinetic and potential energies of a beam element, respectively, given by

$$T(w) = \frac{1}{2}\int_0^L m\dot{w}^2\,dx \tag{12}$$

and

$$V(w) = \frac{1}{2}\int_0^L EIw''^2\,dx - \int_0^L fw\,dx \tag{13}$$

The Hamilton's principle states that the motion of the undamped beam element during the time interval $t_0 < t < t_1$ is such that the time integral of the energy L is stationary. In other words, it states

$$\int_{t_0}^{t_1} \delta L\,dt = 0, \text{ subject to } \delta w = 0 \text{ at } t = t_0, t_1 \tag{14}$$

This condition leads to the governing equation (2), considering $c = 0$, and, additionally, to the following boundary conditions at $x = 0, L$

$$\text{either } EIw'' = 0 \Rightarrow w'' = 0 \text{ (Natural B.C.)} \tag{15a}$$

$$\text{or } \delta w' = 0 \Rightarrow w' \text{ is specified (Geometric B.C.)} \tag{15b}$$

and

$$\text{either } EIw''' = 0 \Rightarrow w''' = 0 \text{ (Natural B.C.)} \tag{16a}$$

$$\text{or } \delta w = 0 \Rightarrow w \text{ is specified (Geometric B.C.)} \tag{16b}$$

When non-conservative systems are considered, there is no longer an energy functional that is conserved in time. However, a separate functional can be introduced to account for non-conservative systems within the framework of Hamilton's principle, well known as Rayleigh's dissipation functional, which can be defined, in this case, as

$$D = \frac{1}{2}\int_0^L c\dot{w}^2 \, dx \tag{17}$$

Making use of Rayleigh's dissipation functional, an extended version of the Hamilton's principle states that the motion of the damped beam element during the time interval $t_0 < t < t_1$ is such that the following condition holds

$$\int_{t_0}^{t_1} \left(\delta L + \frac{\partial D}{\partial \dot{w}} \right) \delta w \, dt = 0, \text{ subject to } \delta w = 0 \text{ at } t = t_0, t_1 \tag{18}$$

which leads to the complete governing equation (2) and to the set of boundary conditions (15,16). This variational statement is, therefore, herein taken as the primal variational principle.

3.2 Two- and One-Field Dual Variational Principles

An alternative, or complementary, form of the primal variational principle presented above can be formulated. Consider the following two-field complementary energy functional of the undamped beam

$$L_c^*(p, M) = T_c(p) - U_c(M) \tag{19}$$

with T_c and U_c the complementary kinetic and complementary strain energies, respectively, given by

$$T_c(p) = \frac{1}{2}\int_0^L \frac{p^2}{m} \, dx \tag{20}$$

and

$$U_c(M) = \frac{1}{2}\int_0^L \frac{M^2}{EI} \, dx \tag{21}$$

A solution for the bending moment field that satisfies equation (9) can be decomposed as

$$M = M_h + M_p \tag{22}$$

with M_h a solution to the homogeneous form of the equation and M_p a particular solution to the equation, *i.e.*

$$M_h'' + \dot{p} + \frac{c}{m} p = 0 \tag{23a}$$

$$M_p'' = f \tag{23b}$$

Considering the following variable transformation

$$r'' = -p \tag{24}$$

yields

$$\dot{r}'' = -\dot{p} \tag{25}$$

which, taking into account the decomposition assumed for the bending moment, gives

$$M_h = \dot{r} + \frac{c}{m} r \tag{26}$$

This shows that the introduced variable r represents a bending-moment impulse.

As for M_p, double integration of the moving load f with respect to x gives

$$M_p(x,t) = g(t)H(x - vt)x + c_1(t)x + c_2(t) \tag{27}$$

Making use of equation (8), the deflection of the beam associated with the particular solution comes out as

$$w_p(x,t) = \int\int \frac{M_p}{EI}\, dx\, dx + c_3(t)x + c_4(t) \tag{28}$$

which gives

$$w_p(x,t) = \frac{1}{EI}\left(\frac{1}{6} g(t)(x - vt)^2(x + 2vt)H(x - vt) + \frac{c_1(t)}{6} x^3 + \frac{c_2(t)}{2} x^2\right) + c_3(t)x + c_4(t) \tag{29}$$

The associated cross-section rotation results, therefore, as

$$w_p'(x,t) = \frac{1}{EI}\left(\frac{1}{2} g(t)(x^2 - v^2t^2)H(x - vt) + \frac{c_1(t)}{2} x^2 + c_2(t)x\right) + c_3(t) \tag{30}$$

The integration functions $c_1(t)$, $c_2(t)$, $c_3(t)$ and $c_4(t)$ are obtained from the boundary conditions of the specific problem under analysis. It is worth mentioning that this particular solution, w_p, is the solution to the corresponding static beam problem, *i.e.*, the problem for which both the velocity , $\dot{w}$, and the acceleration, $\ddot{w}$, are null.

The variable transformation (24) allows to rewrite the two-field functional $L_c^*(p, M)$ for the undamped beam case as the following one-field, or pure, energy functional

$$L_c(r, \dot{r}) = \frac{1}{2}\int_0^L \frac{r''^2}{m}\, dx - \frac{1}{2}\int_0^L \frac{(\dot{r} + M_p)^2}{EI}\, dx \tag{31}$$

This energy functional resembles the one derived in [9] for the free vibrations of Euler-Bernoulli beams.

The stationarity condition of L_c reads

$$\int_{t_0}^{t_1} \delta L_c\, dt = 0, \text{ subject to } \delta r = 0 \text{ at } t = t_0, t_1 \tag{32}$$

and leads to equation (8) and, additionally, to the following boundary conditions at $x = 0, L$

$$\text{either } \frac{r''}{m} = 0 \Rightarrow r'' = 0 \text{ (Natural B.C.)} \tag{33a}$$

$$\text{or } \delta r' = 0 \Rightarrow r' \text{ is specified (Geometric B.C.)} \tag{33b}$$

and

$$\text{either } \frac{r'''}{m} = 0 \Rightarrow r''' = 0 \text{ (Natural B.C.)} \tag{34a}$$

$$\text{or } \delta r = 0 \Rightarrow r \text{ is specified (Geometric B.C.)} \tag{34b}$$

For the damped beam element, the following condition is required

$$\int_{t_0}^{t_1} \left(\delta L_c + \int_0^L \frac{M}{EI} \frac{c}{m} \delta r \, dx \right) dt = 0, \text{ subject to } \delta r = 0 \text{ at } t = t_0, t_1 \tag{35}$$

It can be shown that, under the subsidiary condition of equilibrium (9), this variational statement leads to equation (8) and to the boundary conditions (33,34). It is, therefore, taken herein as the dual variational principle.

4 DUAL FINITE ELEMENT FORMULATION AND IMPLEMENTATION ASPECTS

A new numerical approach based on a novel finite element formulation for space discretization together with an unconditionally Newmark's scheme for time integration is herein proposed. The novel finite element formulation is based on the dual variational principle presented above (35). The approach can be applied to the dynamic analysis of either simple or continuous linear-elastic planar beams subjected to a single or a series of either moving or fixed time-dependent loads. The approach was implemented in Wolfram Mathematica® for both frequency- and time-domain analyses.

4.1 Space Discretization: Dual Finite Element Formulation

A new two-node beam element is herein proposed. The element is derived within the framework of a novel finite element formulation for space discretization, which is based on the dual variational principle presented above (35).

Since C^1 continuity is required (*i.e.*, approximations that are continuous and have continuous first-order derivatives), the bending-moment impulse and its x-derivative (or shear-force) are taken as the nodal degrees-of-freedom. Hermite cubic polynomials are, therefore, adopted for r. It is worth noting that, the standard displacement-based (or primal) finite element formulation also requires a C^1 interpolation for the beam deflection w. The simplest primal finite element formulation is also based on a two-node beam element with two degrees-of-freedom per node, which are the transverse displacement and its x-derivative (or rotation).

The approximation of the bending-moment impulse at the element level, r_e, is written as

$$r_e^h(x, t) = \boldsymbol{N}_r(x) \boldsymbol{r}_e(t) \tag{36}$$

with $\boldsymbol{N}_r$ the matrix that contains the approximation functions, taken as mentioned above as Hermite cubic polynomials, and $\boldsymbol{r}_e$ the vector that collects the element degrees-of-freedom for r_e.

The derivatives of r_e can be written, therefore, as

$$r_e^{h''} = \boldsymbol{N}_r''\boldsymbol{r}_e, \quad \dot{r}_e^h = \boldsymbol{N}_r\dot{\boldsymbol{r}}_e \tag{37}$$

The discrete form of the Euler-Lagrange equations associated with the dual variational principle (35) come out as the following governing system of equations defined at the element level

$$\boldsymbol{\mathcal{M}}_e\ddot{\boldsymbol{r}}_e + \boldsymbol{\mathcal{C}}_e\dot{\boldsymbol{r}}_e + \boldsymbol{\mathcal{F}}_e\boldsymbol{r}_e = \boldsymbol{d}_e \tag{38}$$

with

$$\boldsymbol{\mathcal{M}}_e = \int_0^{L_e} \frac{\boldsymbol{N}_r^T\boldsymbol{N}_r}{EI}\, dx \tag{39a}$$

$$\boldsymbol{\mathcal{C}}_e = \int_0^{L_e} \frac{c\boldsymbol{N}_r^T\boldsymbol{N}_r}{mEI}\, dx \tag{39b}$$

$$\boldsymbol{\mathcal{F}}_e = \int_0^{L_e} \frac{\boldsymbol{N}_r''^T\boldsymbol{N}_r''}{m}\, dx \tag{39c}$$

$$\boldsymbol{d}_e = -\int_0^{L_e} \frac{\boldsymbol{N}_r^T\dot{M}_p}{EI}\, dx \tag{39d}$$

The element matrices $\boldsymbol{\mathcal{M}}_e$, $\boldsymbol{\mathcal{C}}_e$ and $\boldsymbol{\mathcal{F}}_e$ are (4×4) matrices, and the element vector $\boldsymbol{d}_e$ is a (4×1) vector.

Assembling the element equations (38), gives the following governing system of global equations

$$\boldsymbol{\mathcal{M}}\ddot{\boldsymbol{r}} + \boldsymbol{\mathcal{C}}\dot{\boldsymbol{r}} + \boldsymbol{\mathcal{F}}\boldsymbol{r} = \boldsymbol{d} \tag{40}$$

The eigenfrequencies of the proposed dual model, ω_i, can be obtained from the following condition

$$det(\boldsymbol{\mathcal{M}}\omega - \boldsymbol{\mathcal{F}}) = 0 \tag{41}$$

4.2 Time Integration: Newmark's Method

The governing system of global equations (40) is transformed into a set of algebraic equations by means of Newmark's method. In particular, the approximated bending-moment impulse and its first time derivative are taken at the end of the time step t_{n+1} as

$$\boldsymbol{r}_{n+1} = \boldsymbol{r}_n + \dot{\boldsymbol{r}}_n\Delta t + \big((\frac{1}{2} - \beta)\ddot{\boldsymbol{r}}_n + \beta\ddot{\boldsymbol{r}}_{n+1}\big)\Delta t^2 \tag{42a}$$

$$\dot{\boldsymbol{r}}_{n+1} = \dot{\boldsymbol{r}}_n + \big((1-\gamma)\ddot{\boldsymbol{r}}_n + \gamma\ddot{\boldsymbol{r}}_{n+1}\big)\Delta t \tag{42b}$$

where Δt is the time step given by

$$\Delta t = t_{n+1} - t_n \tag{43}$$

and β and γ are the control parameters of the algorithm. It is required that $2\beta \geq \gamma \geq 1/2$ to ensure an unconditionally stable numerical scheme. For $\beta = 1/4$ and $\gamma = 1/2$, which is the combination adopted within the framework of the proposed formulation, the algorithm corresponds to the average acceleration method.

The governing system of global equations at the time step t_{n+1} reads

$$\boldsymbol{\mathcal{M}}\ddot{\boldsymbol{r}}_{n+1} + \boldsymbol{\mathcal{C}}\dot{\boldsymbol{r}}_{n+1} + \boldsymbol{\mathcal{F}}\boldsymbol{r}_{n+1} = \boldsymbol{d}_{n+1} \tag{44}$$

Hence, on insertion of (42) into (44) yields

$$\ddot{\boldsymbol{r}}_{n+1} = \boldsymbol{\mathcal{M}}^{*-1}\boldsymbol{d}^*_{n+1} \tag{45}$$

with

$$\boldsymbol{\mathcal{M}}^* = \boldsymbol{\mathcal{M}} + \boldsymbol{\mathcal{C}}\gamma\Delta t + \boldsymbol{\mathcal{F}}\beta\Delta t^2 \tag{46a}$$

$$\boldsymbol{d}^*_{n+1} = \boldsymbol{d}_{n+1} - \boldsymbol{\mathcal{C}}\Big(\dot{\boldsymbol{r}}_n + \big((1-\gamma)\ddot{\boldsymbol{r}}_n\big)\Delta t\Big) - \boldsymbol{\mathcal{F}}\Big(\boldsymbol{r}_n + \dot{\boldsymbol{r}}_n\Delta t + \big((\frac{1}{2}-\beta)\ddot{\boldsymbol{r}}_n\big)\Delta t^2\Big) \tag{46b}$$

The homogeneous part of the element bending-moment solutions, M^h_h, can be retrieved from the obtained bending-moment impulses $\boldsymbol{r}$. The total bending-moment solutions of the beam elements arise from $M^h = M_p + M^h_h$. The element shear-forces, Q^h, are computed from equation (10). The resulting element bending-moments, M^h, and shear-forces, Q^h, satisfy all equilibrium conditions. The obtained solutions are, therefore, said to be equilibrated solutions.

The associated element deflections, w^h_h, can be computed from equation (8), which requires double integration of M^h_h with respect to x. The resulting integration constants, two per element, are obtained by setting the kinematical boundary conditions $w^h_h(0,t) = w^h_h(L,t) = 0$ and the interlement compatibility conditions $[\![w^h_h]\!]_{int} = 0$. The total deflections of the beam elements are obtained from $w^h = w_p + w^h_h$.

It is worth mentioning that, although it appears that the proposed dual (equilibrium-based) finite element formulation has the ability to produce the exact solutions, since the obtained bending-moments are continuous as well as satisfy all equilibrium boundary conditions, and the obtained deflections are continuous as well as satisfy all kinematical boundary conditions, this is not the case, since equation (7) is not satisfied in strong form at the element level. In fact, the displacement fields that would arise from integration in time of equation (7) differ from the ones computed from the integration is space of equation (8).

5 NUMERICAL TESTS

To validate and assess the accuracy and effectiveness of the proposed finite element formulation, a clamped-hinged (CH) beam as illustrated in Figure 1 was considered. The beam was analysed using both the primal and dual finite element formulations. Uniform meshes of 2, 4, 8 and 16 two-node finite elements with Hermite cubic polynomials were adopted within the framework of both formulations. All the required integrations were performed exactly.

The beam parameters were set as: length $L = 2\,m$, bending stiffness $EI = 1000\,Nm^2$, mass per unit length $m = 100\,kg/m$, and viscous damping coefficient $c = 100\,Ns/m^2$. The loading parameters were taken as: speed $v = 5\,m/s$ and amplitude $g = 100\,N$.

Within the framework of Newmark's method, the following parameters were considered: $\beta = 1/4$ and $\gamma = 1/2$. The assumed time step was $\Delta t = 1/2000\,s$.

A frequency-domain analysis, with $c = 0$, was first considered. The numerical results obtained using both primal and dual formulations for the first five eigenfrequencies of the beam are displayed in Table 1. As it can be observed, the dual formulation leads to more accurate solutions than the primal one, despite the fact that both formulations make use of the same number of degrees-of-freedom. Also, both formulations lead to approximate eigenfrequencies that converge from above with respect to their corresponding exact solutions.

A time-domain analysis was afterwards considered, in which approximate bending-moment, shear-force and deflection solutions were computed using both formulations in the time range $t \in [0,2]s$.

n_e		ω_1	ω_2	ω_3	ω_4	ω_5
2	P	12.301909	46.174026	123.043485		
	D	12.265060	44.131810	107.169201	197.034079	
4	P	12.196863	39.747105	84.272284	158.252902	258.009925
	D	12.196275	39.711429	83.835978	155.809214	247.605731
8	P	12.189651	39.517097	82.560694	141.639528	217.449063
	D	12.189641	39.516534	82.553323	141.591539	217.240958
16	P	12.189193	39.501732	82.424433	140.981199	215.223318
	D	12.189192	39.501723	82.424317	140.980444	215.220008

Table 1: CH beam - Primal (P) and dual (D) eigenfrequencies (rad/s).

The mid-span bending-moment solutions were first computed within the framework of the time-domain analysis considering the time range $t \in [0, 2]s$ using both formulations on the 2- and 8-finite element meshes. As it can be observed from Figure 2, the dual formulation produces more accurate solutions than the primal one. The solution that would be obtained on a static analysis is also represented in Figure 2 for comparison purposes and, as it can be seen, the maximum dynamic bending-moment is higher than the static one.

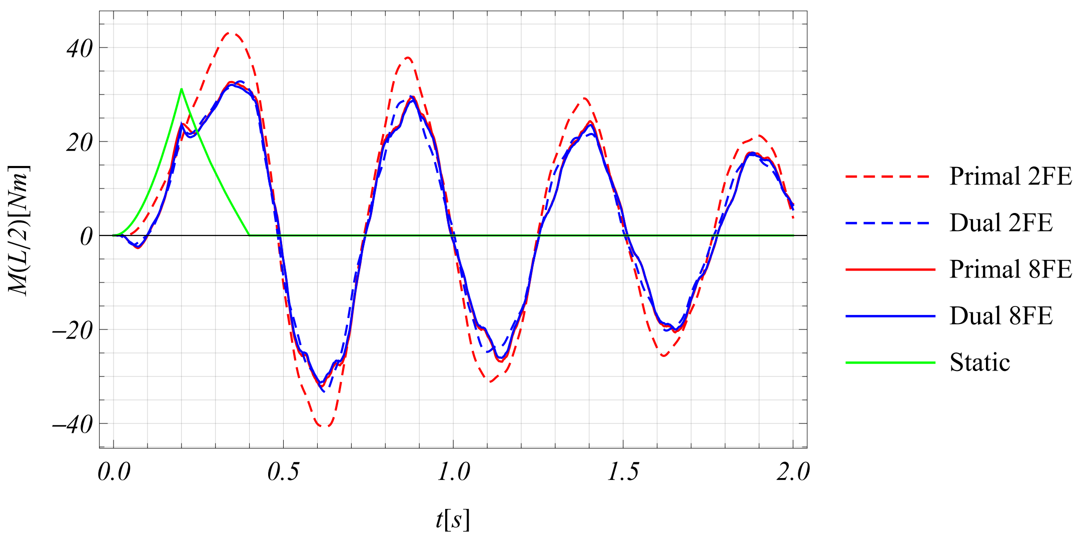

Figure 2: CH beam - Mid-span bending-moment solutions in the time range $t \in [0, 2]s$.

The bending-moments at $x = L$ were also computed using both formulations on the 2- and 8-finite element meshes. The obtained results are depicted in Figure 3. As it can be observed, while the dual formulation leads to bending-moment fields that exactly satisfy the equilibrium boundary condition $M(L) = 0$ on both meshes, the primal one does not. The same conclusion would arise for finer meshes as well.

Also the mid-span deflections were computed using both formulations on the 2-finite element mesh. The obtained solutions are represented in Figure 4, which shows that both formulations provide similar accuracies for the deflections. This was expected, since both formulations provide approximate solutions for the deflections that are continuous as well as satisfy all kinematical boundary conditions.

Finally, the bending-moment and shear-force distributions obtained on the 4-finite element

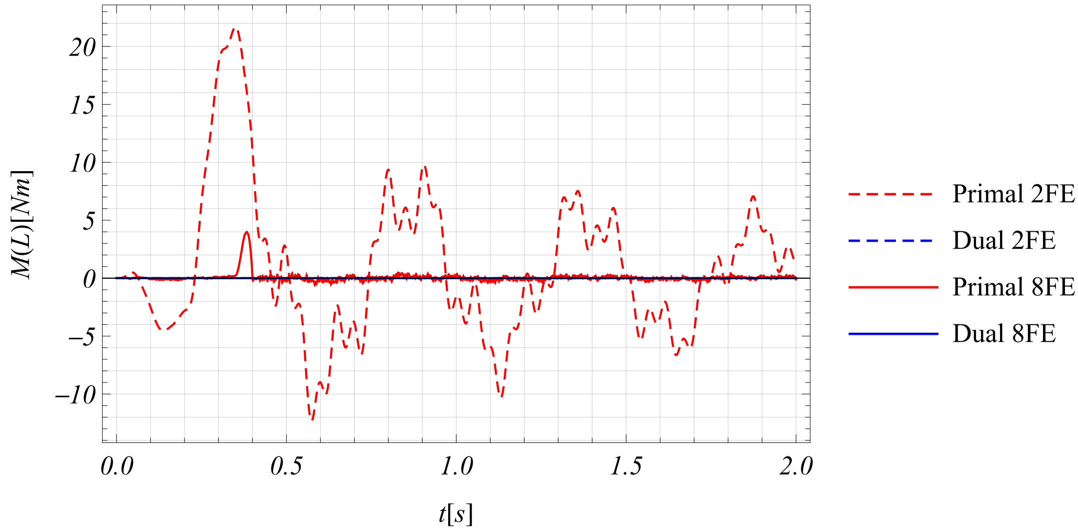

Figure 3: CH beam - Bending-moment solutions at $x = L$ in the time range $t \in [0, 2]s$.

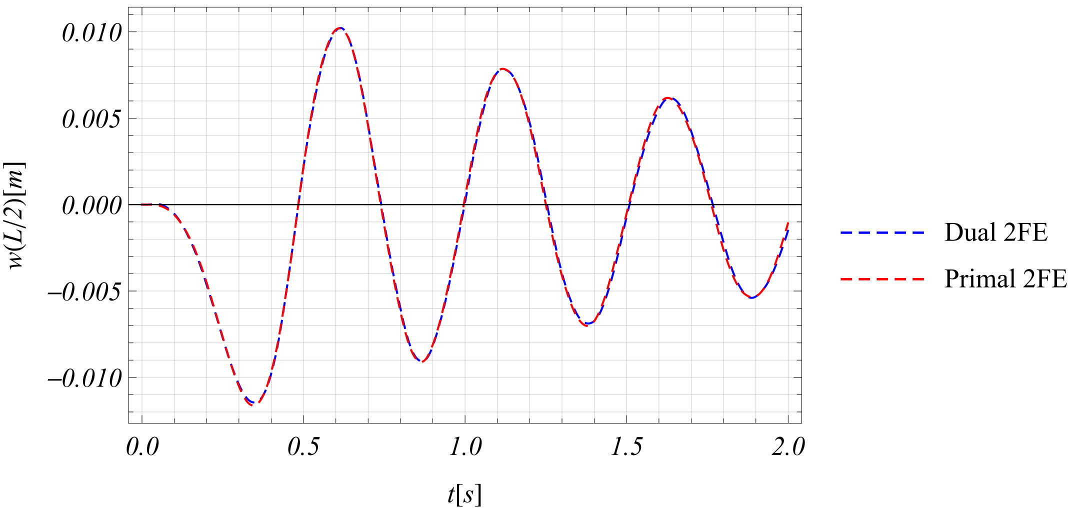

Figure 4: CH beam - Mid-span deflection solutions in the time range $t \in [0, 2]s$.

mesh for $t = 0.08s$ and $t = 0.18s$ are shown in Figures 5 and 6, respectively. Clearly, in contrast to the results provided by the primal formulation, the dual formulation produces continuous bending-moment and shear-force distributions that automatically satisfy the equilibrium boundary conditions. As shown in Figure 6, the discontinuities of the shear-force distributions produced by the dual formulation only occur at $x = vt$, the position of the applied load for the considered instant of time, and match exactly the magnitude of the applied load. On the contrary, the primal formulation is not able to represent the exact shear-force discontinuity resulting from the applied concentrated load.

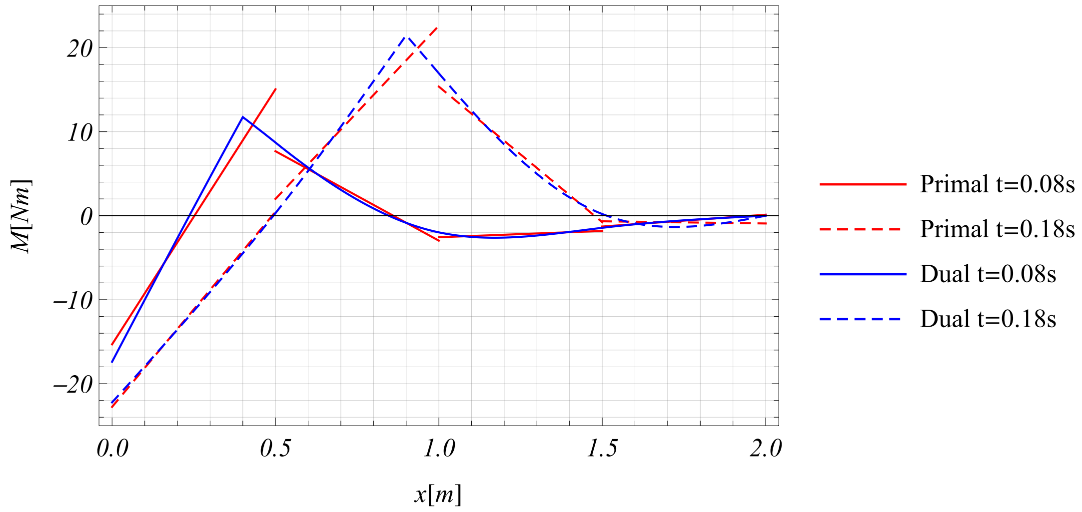

Figure 5: CH beam - Bending-moment distributions on the 4-finite element mesh for $t = 0.08s$ and $t = 0.18s$.

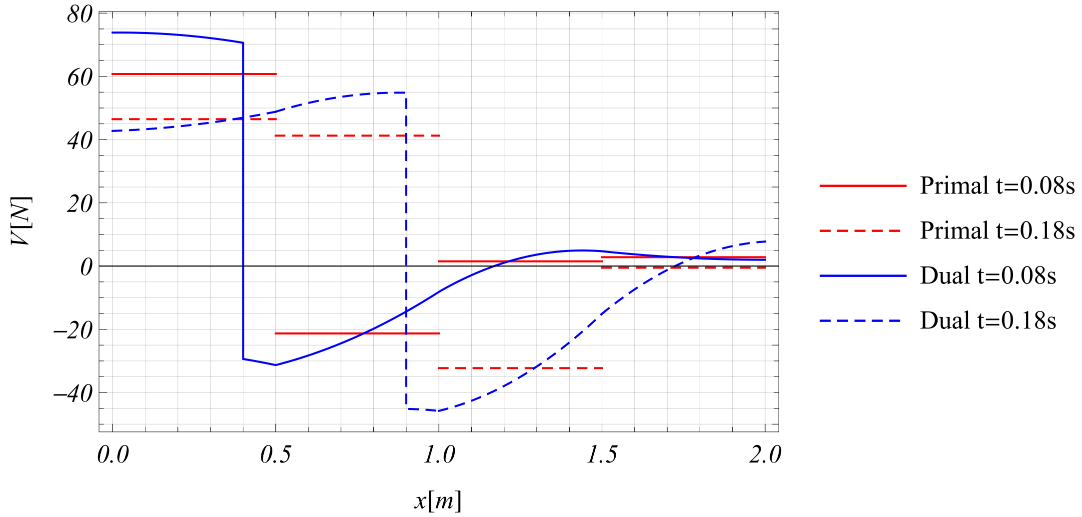

Figure 6: CH beam - Shear-force distributions on the 4-finite element mesh for $t = 0.08s$ and $t = 0.18s$.

6 CONCLUSIONS

The dynamic analysis of Euler-Bernoulli linear-elastic beams with internal viscous damping under moving loads was numerically addressed by resorting to a new finite element formulation. The proposed formulation was derived from a complementary variational principle, which can be viewed as the dual principle of the well known Hamilton's principle. Based on this formulation, a novel two-node beam element was derived in which the bending-moment impulse and its x-derivative were taken as the fundamental nodal degrees-of-freedom. The accuracy and effectiveness of the formulation were numerically assessed in both frequency and time domains by comparison of the obtained results with those provided by the standard two-node displacement-based formulation with the same number of nodal degrees-of-freedom. Integration in time was carried out by adopting an unconditionally stable version of Newmark's method. In the frequency-domain, it was numerically demonstrated that both formulations produce approximate eigenfrequencies that converge from above with respect to their corresponding exact solutions, as well as more accurate solutions were obtained by resorting to the proposed formulation. In the time-domain, the new formulation proved capable of not only rendering continuous bending-moment and shear-force distributions, a desired feature in the structural design field, but also demonstrated a superior accuracy than that provided by the standard two-node displacement-based formulation.

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