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STATICALLY ADMISSIBLE SHELL INTERNAL FORCES FOR THE STABILITY ANALYSIS OF THE DOME OF PISA CATHEDRAL LOADED BY VERTICAL AND HORIZONTAL LOADS

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Abstract. *The present contribution illustrates some recent results concerning the stability analysis of the oval-based pointed masonry dome of the renowned Pisa Cathedral subjected to both vertical and horizontal loads. The study is aimed at determining lower bounds of the horizontal collapse load by making use of the static theorem of limit analysis. Heyman's hypotheses are adopted, i.e., masonry is assumed to have unlimited compressive strength and zero tensile strength while collapse due to sliding is assumed to be prevented. The structural analysis is carried out by modelling the dome as a thin shell; an ad-hoc solution procedure is enforced which is able to define optimised statically admissible stress fields. The results in terms of stress distribution and safety factor are discussed.*

Keywords: Masonry Domes, Limit Analysis, Shell Theory, Convex Optimisation.

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1 INTRODUCTION

The dome of Pisa Cathedral is a 12th-century masonry construction of great value from both the historical and architectural standpoints, currently part of the UNESCO World Heritage 'Piazza del Duomo'. It also represents an intriguing case study for the construction history and structural analysis. A multidisciplinary research project started in 2016 is still being carried out by the authors with the aim of determining the structural response of the dome through a thorough assessment of its geometry and constituent materials, and by taking into account historical and architectural aspects as well. The activities carried out as part of the project involved conducting several steps, in particular: (*i*) an archival investigation of the historical development of the construction [1, 2, 3, 4], (*ii*) a detailed high-density survey of the geometry using both laser scanning and photogrammetric techniques [5], (*iii*) in-situ and laboratory tests on construction materials aimed at determining their mechanical and chemical-physical properties [6], (*iv*) the definition of a suitably designed geometric model that could wholly and analytically describe the shape of the dome [7] and (*v*) a series of structural analyses using different mechanical models and analysis methods [6, 7, 8, 9, 10]. The problem of modelling the mechanical response of complex systems such as historic masonry structures is a topic widely addressed by the scientific community. The inherent difficulties of this issue have led to the development of a myriad of different analysis and modelling methods, as witnessed by the vast production of scientific literature on the subject [11, 12]. While this breadth of modelling techniques testifies to the importance of the problem, it also highlights that a universally recognised strategy for studying these types of constructions is still lacking [13]. A line of research that often proves effective in providing helpful information about the safety level of masonry structures and which is generally characterised by a lower degree of complexity is the one that uses limit analysis theorems. This approach, which roots in the past, was formalised in modern terms by Jacques Heyman in his 1966 seminal work [14]. From this work, a whole series of analysis methods, now well established in the literature, have been derived, such as the Thrust Line Method [15, 16], the Thrust Network Analysis [17, 18] and the Thrust Surface Analysis [19, 20]. All these methods have been the subject of many studies and have been reformulated and extended to overcome some of their limitations or broaden their scope. Another more recent approach that presents advantages over previous techniques makes use of shell theory to model vaulted masonry structures [21, 22, 23, 24, 25]. The present contribution, which is set in the latter approach, is aimed at illustrating some results concerning the estimate of the safety level of the construction, which have been recently obtained by means of a new analysis methodology. Assuming that the masonry satisfies the well-known Heyman hypotheses, *i.e.*, it has no tensile strength, infinite compressive strength and sliding failure is prevented, limit analysis tools and methods are readily exploited to study the stability of masonry structures and to determine their collapse load. The employed methodology models the dome as a thin shell and searches for statically admissible stress distributions, allowing for both membrane forces and bending moments, through an expressly developed optimisation procedure. The determination of statically admissible stress fields ensures the stability of the structure by virtue of the static theorem of limit analysis. Some first results are presented, concerning the stability of the dome of Pisa Cathedral when subjected to increasing horizontal loads in addition to its own weight. The system of horizontal forces we consider, proportional to mass distribution, can represent the load due to seismic action, at least in a first approximation. The estimates of the dome's safety level are compared with the results of other theoretical and numerical analyses documented in the literature.

2 HORIZONTAL COLLAPSE LOAD ANALYSIS VIA SHELL-FORCES METHOD

The safety level of the dome against horizontal loads is assessed by estimating the maximum horizontal load multiplier compatible with the self-weight. To this aim, an analysis method developed by the authors and previously employed to study the stability of masonry domes against vertical loads is used [10]. In this section we briefly recall the main features of the method and the nomenclature adopted.

2.1 The equilibrium problem for a no-tension shell

The dome is modelled as a shear-deformable thin shell (*e.g.* [26]), whose geometry is fully represented by its middle surface and thickness. For the sake of simplicity, the thickness is supposed to be constant and much smaller than the characteristic dimension of the middle surface. The dome middle surface Σ is embedded in the three-dimensional Euclidean space $\mathbb{E}^3$, and it is assumed that it admits a parametric representation in terms of a curvilinear coordinate pair (θ^1, θ^2)

$$\Sigma : \mathbf{x} = \mathbf{x}(\theta^1, \theta^2), \qquad (\theta^1, \theta^2) \in \Theta \subset \mathbb{R}^2 \tag{1}$$

being $\mathbf{x}$ the position vector and Θ the coordinates' domain. The parameterisation induces a local natural (covariant) basis on the surface, given by

$$\mathbf{a}_1 = \frac{\partial \mathbf{x}(\theta^1, \theta^2)}{\partial \theta^1}, \qquad \mathbf{a}_2 = \frac{\partial \mathbf{x}(\theta^1, \theta^2)}{\partial \theta^2}, \tag{2}$$

and completed by the unit normal vector, given by the Gauss map

$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}, \tag{3}$$

to form a basis of $\mathbb{E}^3$. In applications involving masonry domes, it is usually convenient to choose the parameterisation in such a way that the coordinate lines coincide with meridians and parallels, *i.e.*, $\theta^1 = const.$ corresponds to a meridian, $\theta^2 = const.$ to a parallel. The internal stress state in the shell is described by the force tensor $\mathbf{N}$ and couple force tensor $\mathbf{M}$, which admit the following representations

$$\mathbf{N} = n^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + n^{3\alpha}\mathbf{a}_3 \otimes \mathbf{a}_\alpha, \quad \mathbf{M} = m^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta \,, \tag{4}$$

with Greek indices assuming values 1, 2 and where Einstein summation convention is used. The contravariant components $n^{\alpha\beta}$, $n^{3\alpha}$, $m^{\alpha\beta}$, are commonly called membrane force, transverse shear and moment components, respectively. The equilibrium equations for a shell can be written in terms of the contravariant components of the force and couple force tensors as (see [27])

$$\begin{cases} n^{\beta\alpha}|_\alpha - b^\beta_{\ \alpha} n^{3\alpha} + p^\beta = 0 \,, \\ b_{\beta\alpha} n^{\beta\alpha} + n^{3\alpha}|_\alpha + p^3 = 0 \,, \\ m^{\beta\alpha}|_\alpha - n^{3\beta} + c^\beta = 0 \,, \\ \epsilon_{\alpha\beta}(n^{\beta\alpha} - b^\alpha_{\ \lambda} m^{\beta\lambda}) = 0 \,, \end{cases} \tag{5}$$

where, p^β, p^3 and c^β are the external load and couple components, $b^\beta_{\ \alpha}$ are the curvature mixed components, $b_{\beta\alpha}$ are the second fundamental form covariant components, $\epsilon_{\alpha\beta}$ is the two-dimensional alternating symbol and $|_\alpha$ denotes the covariant derivative. The equilibrium problem is completed by the boundary conditions

$$\mathbf{n}_\partial = n^{\alpha\beta}\nu_\beta \mathbf{a}_\alpha + n^{3\alpha}\nu_\alpha \mathbf{a}_3, \quad \mathbf{m}_\partial = m^{\alpha\beta}\nu_\beta \mathbf{a}_\alpha, \tag{6}$$

where $\mathbf{n}_\partial$ and $\mathbf{m}_\partial$ are the known external force and couple force vectors acting on the edges of the shell that are free of constraints, here denoted by $\partial_N \Sigma$, and ν_α are the covariant components of the unit vector $\boldsymbol{\nu}$ normal to the boundary.

In the model, equilibrium is enforced in the initial configuration. The tensor $\mathbf{M}$ and the surface part of $\mathbf{N}$, namely $n^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta$, are assumed to be symmetric. Masonry is assumed to fulfil Heyman's hypotheses: normal stresses on elements orthogonal to the middle surface cannot be tensile. Compressive stresses can be unbounded, as well as shearing stresses (since sliding failure cannot occur). It has been shown in [28] that the internal force field is compatible with Heyman's hypotheses provided it belongs to the set

$$A = \left\{ (n^{\alpha\beta}, n^{3\alpha}, m^{\alpha\beta}) \mid (hn^{\alpha\beta} + m^{\alpha\beta})\, \mathbf{a}_\alpha \otimes \mathbf{a}_\beta \in \mathrm{Sym}^-,\ (hn^{\alpha\beta} - m^{\alpha\beta})\, \mathbf{a}_\alpha \otimes \mathbf{a}_\beta \in \mathrm{Sym}^- \right\}, \tag{7}$$

which also ensures $n^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta \in \mathrm{Sym}^-$, being Sym^- the space of negative semidefinite second order tensors, and where $2h$ denotes the thickness of the shell. An internal force distribution satisfying equations (5) in Σ and (6) on $\partial_N \Sigma$ and belonging to the set A is called statically admissible.

The static theorem of limit analysis ensures that if at least one statically admissible force field exists then the structure is not in a collapse condition. It is clear that the force field corresponding to the highest safety factor, *i.e.*, the highest load multiplier has to be sought to avoid underestimating the load-bearing capacity of the structure. The solution procedure for searching optimised statically admissible force fields is described in the following section.

2.2 Solution procedure

The first requirement for a statically admissible force field is to comply with the equilibrium equations and boundary conditions (5) and (6). The equilibrium problem is statically undetermined since the unknowns, represented by the eight components $n^{\alpha\beta}$, $n^{3\alpha}$ and $m^{\alpha\beta}$, outnumber the equations (six). The strategy for determining statically admissible optimised force field starts by choosing as redundant forces the two components n^{11} and m^{11} (the hoop normal force and bending moment), and by writing them as function series:

$$n^{11}(\theta^1, \theta^2) = \sum_{i=1}^{N} C_i \phi_i(\theta^1, \theta^2), \quad m^{11} = \sum_{j=1}^{M} D_j \psi_j(\theta^1, \theta^2), \tag{8}$$

where $\phi_i(\theta^1, \theta^2)$ and $\psi_j(\theta^1, \theta^2)$ are known trigonometric basis functions, while C_i, D_j are the unknown coefficients of the linear combinations. By this way, the redundant force components are considered as additional loading terms and the unknowns of the problem become n^{12}, n^{22}, n^{31}, n^{32}, m^{12} and m^{22}. Taking advantage of the linearity of the equilibrium equations, the actual system can be decomposed into statically determined sub-systems, each solved by the collocation method (*e.g.* [29]). The numerical method used enables providing a simple evaluation of the local error in the solution, as well as easily checking the admissibility conditions for the material, as will be shown in the following. The maximum statically admissible lateral load multiplier λ is pursued by solving the following convex optimisation problem

$$\begin{aligned} \min_{C_i, D_j, \lambda} \quad & -\lambda, \\ s.t. \quad & (hn^{\alpha\beta} + m^{\alpha\beta})\, \mathbf{a}_\alpha \otimes \mathbf{a}_\beta \in \mathrm{Sym}^- \\ & (hn^{\alpha\beta} - m^{\alpha\beta})\, \mathbf{a}_\alpha \otimes \mathbf{a}_\beta \in \mathrm{Sym}^- \\ & \lambda \geq 0, \end{aligned} \tag{9}$$

where the cost function is linear and the constraints, ensuring compliance with the material requirements expressed by (7), are convex.

3 CASE STUDY: THE DOME OF PISA CATHEDRAL

The dome of Pisa Cathedral is a structure of extraordinary artistic, historical, and architectural importance. Still, it is also a fascinating case study in terms of construction techniques and structural response. In this paper, we limit ourselves to describing the geometry of the dome, but the interested reader may find it helpful to refer to [2] and the bibliographical references therein to deepen the above-mentioned aspects.

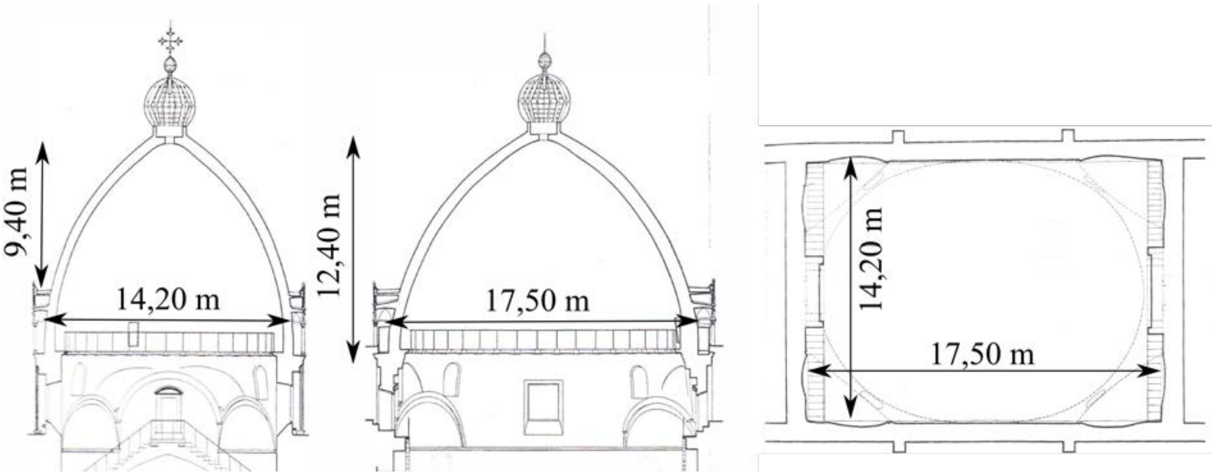

Figure 1: Cross sections on the major and minor axes and floor plan of the dome; taken from [2] (dimensions added to original).

The shape of the Cathedral (see Figure 1) is peculiar, presenting an oval base and an ogival profile. There is a hole at the top, surmounted by a 'ball' covered with lead and sustained by a frame made of wood and iron. The dome's geometry has been carefully reconstructed by means of a laser scanner survey conducted within the research project [6]. As a result, the middle surface of the dome is expressed in parametric form as

$$\mathbf{x}(\alpha, z) = q(\alpha, z) \cos\alpha \, \mathbf{i}_1 + q(\alpha, z) \sin\alpha \, \mathbf{i}_2 + z \, \mathbf{i}_3 \tag{10}$$

with

$$q(\alpha, z) = \frac{\left(d + \sqrt{e^2 - z^2}\right)\left(f + \sqrt{g^2 - z^2}\right)}{\sqrt{\left(d + \sqrt{e^2 - z^2}\right)^2 \sin^2(\alpha) + \left(f + \sqrt{g^2 - z^2}\right)^2 \cos^2(\alpha)}}, \tag{11}$$

having set $\theta^1 = \alpha$, $\theta^2 = z$, with $\alpha \in [0, 2\pi]$ and $z \in [3\,m, 12.05\,m]$, $d = -5.57\,m$, $e = 13.99\,m$, $f = -8.11\,m$, $g = 15.43\,m$.

The scheme adopted for the analyses is shown in Figure 2. The dome is subjected to its self-weight and to a horizontal action given by the mass distribution multiplied by a scale factor λ; since eccentricity of the body loads with respect to the middle surface is small, distributed couples c^1 and c^2 are not entered into the model. The boundary load at the top corresponds to the weight of the 'ball'.

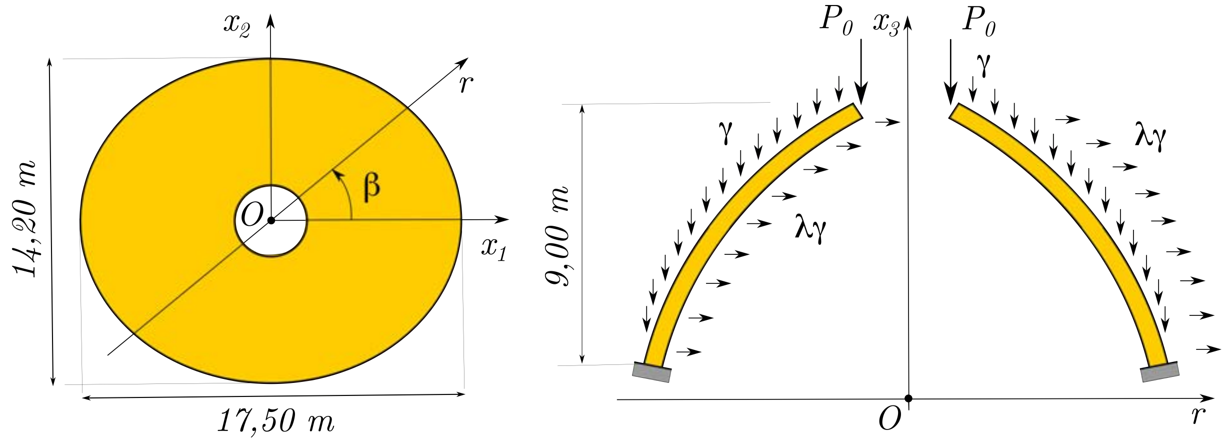

Figure 2: Mechanical scheme of the problem.

Since the dome is not axisymmetric, the maximum horizontal load multiplier depends on the direction of the horizontal action, here denoted by the angle β. The solution obtained in the case where $\beta = 0°$ (horizontal forces directed along the x_1 axis) is illustrated below.

3.1 Internal Force Components

The analyses have been carried out for different numbers of basis functions. Convergence of the method has been checked; moreover, it has been verified that the maximum lateral load multiplier increases with the number of basis functions as expected. The solution shown in the following has been obtained by considering 45 constants C_i, while m^{11} has been set to zero because it has been verified that its presence does not appreciably affect the solution whilst the computational time increases. The maximum lateral load multiplier found is $\lambda_{max} = 0.360$. The optimised force field is shown in Figures 3, 4 and 5. Two vertical sections of the force fields diagrams are also shown in Figure 6 and Figure 7.

The solution shows a high gradient of the compressive hoop forces near the base in the region $x_1 < 0$, where they reach values one order of magnitude greater than the compressive forces in the meridians. This seems to suggest that a concentration of compressive forces along a half-ring in the $x_1 < 0$ region near the abutments is optimal from a static point of view. It is also worth noting that, contrary to the case where only vertical loads are present, the transverse shear forces are of the same order of magnitude as the meridian compressive forces.

The analysis was also conducted for $\beta = 90°$, *i.e.* when the horizontal loads act along the x_2 axis, which is the weakest direction for the dome. In this case the collapse multiplier reduces to $\lambda_{max} = 0.245$, while the trend of the stresses resembles qualitatively that obtained for $\beta = 0°$ rotated by 90°.

The obtained results seem to agree with analogous theoretical and numerical results documented in the literature. In [30] a horizontal multiplier 0.42 was found for a hemispherical dome with thickness-to-radius ratio equal to 0.2. In [31] the horizontal multiplier fell within the range 0.22 to 0.35 for a hemispherical dome with thickness-to-radius ratio equal to 0.06.

3.2 Admissibility Conditions and Error estimation

The admissibility of the force field can be evaluated by computing the principal values of the membrane forces and the eccentricity, evaluated with its sign. The principal forces (Figure 8)

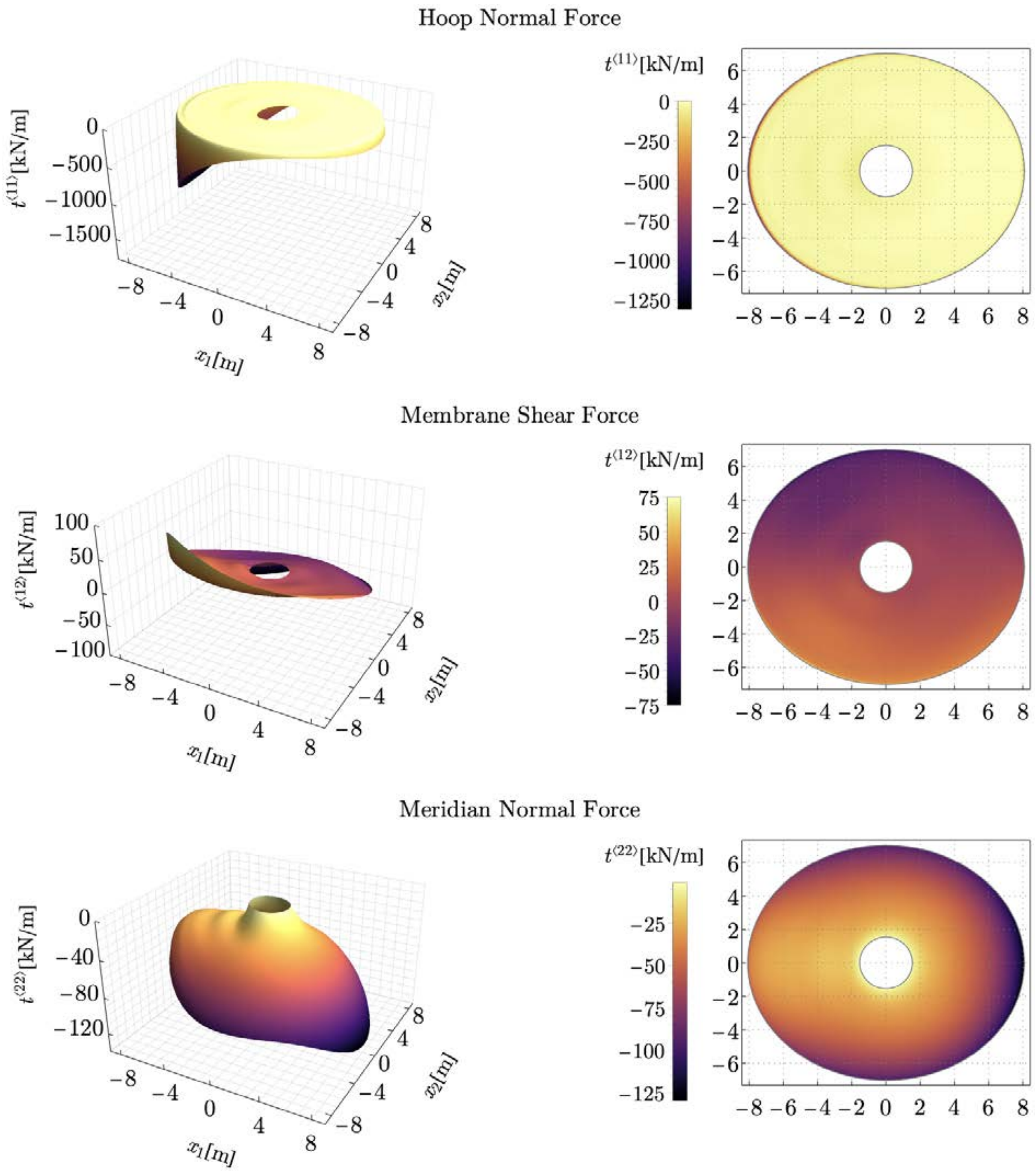

Figure 3: Membrane force components.

are the eigenvalues of the membrane force tensor; the eccentricity are computed as

$$e_{max}^{min} = \frac{\mathrm{Tr}(\mathbf{N}^{-1}\mathbf{M}) \pm \sqrt{(\mathrm{Tr}(\mathbf{N}^{-1}\mathbf{M}))^2 - \mathrm{Det}(\mathbf{N}^{-1}\mathbf{M})}}{2} \,. \tag{12}$$

The diagrams of the eccentricity along two vertical sections are shown in Figure 9.

As already mentioned, the collocation method allows obtaining an approximate solution made of a suitable combination of analytical expressions, which can be used to assess the error committed in solving the equilibrium equations. By writing the equations in dimensionless

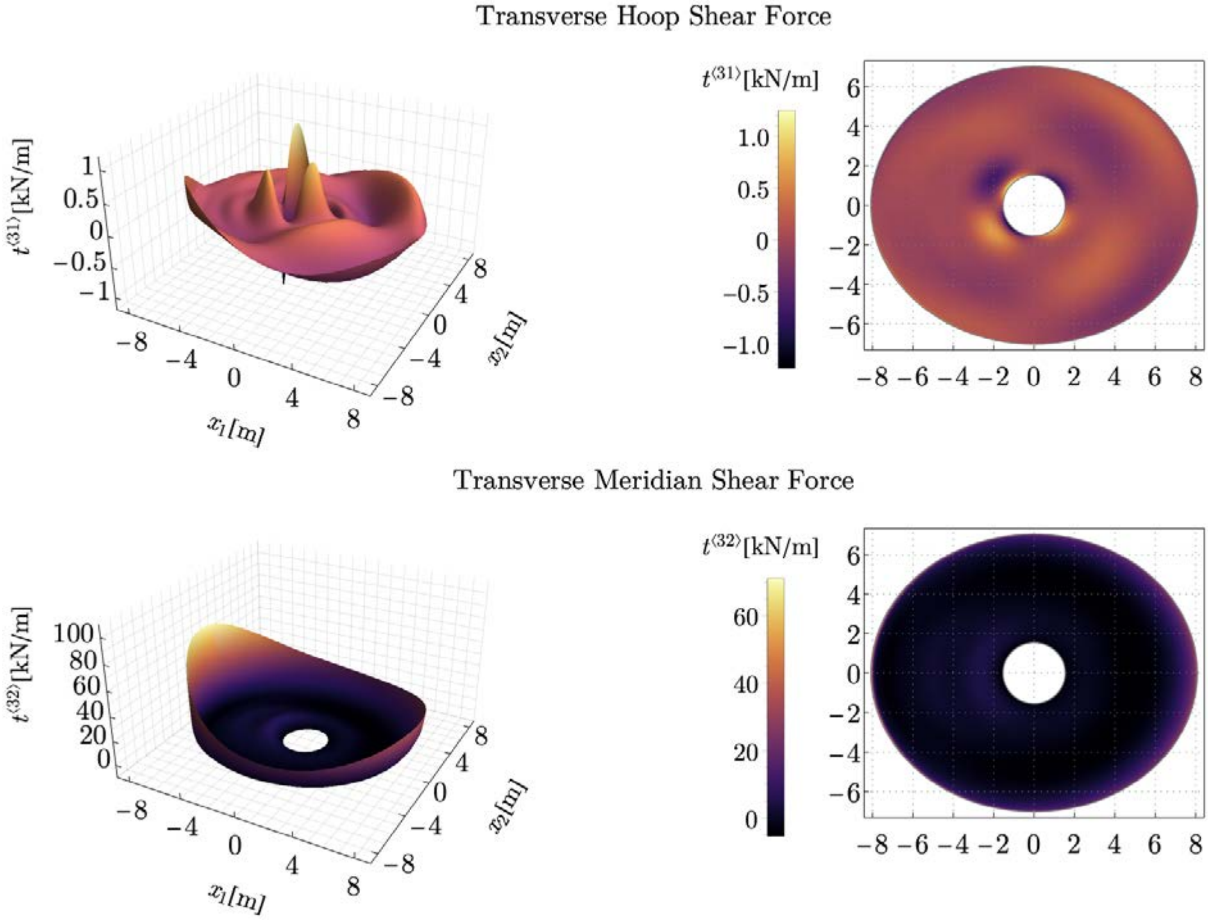

Figure 4: Transverse shear components.

form, it has been verified that the maximum error is less than 1%, except in small regions around the top hole and base where it is as high as 7%. This error is considered fully acceptable for our purposes, considering the uncertainties in the evaluation of the actual load distribution.

Some remarks are in order. In the region $x_1 < 0$, *i.e.*, upstream with respect to the horizontal action, the meridian forces are smaller compared to the case where only vertical loads are present (see [10]). In the same region, a relevant increase in meridians shear forces is observed. Furthermore, the compressive membrane forces are very high in the parallels near the base. Although further studies are essential, these first findings seem to suggest that Heyman's hypotheses could result to be not fully conservative, contrary to the case in which only vertical loads act on the structure, and that more suitable failure criteria could be needed to properly describe the behaviour of masonry in the presence of horizontal actions.

4 CONCLUSIONS

The work illustrates some first results concerning the estimation of the collapse load of the dome of Pisa Cathedral. In the analysis, the dome is subjected to its self-weight and to a system of horizontal forces proportional to the mass by a scale factor. The maximum load multiplier compatible with the stability of the structure is estimated by modelling the dome as a thin shell satisfying Heyman's hypotheses. In other words, the shell is considered unable of transmitting tensile forces whilst compressive forces, as well as transverse shear forces, can be

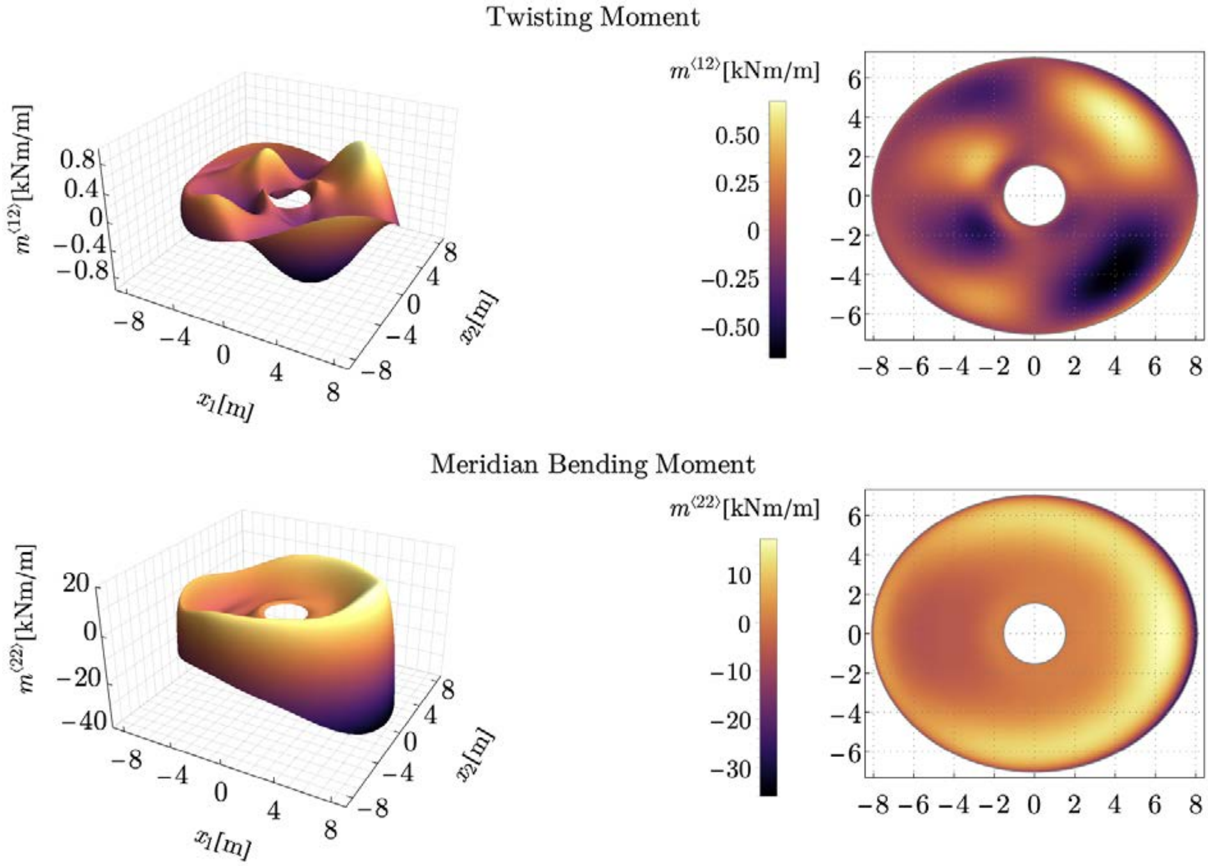

Figure 5: Moment components.

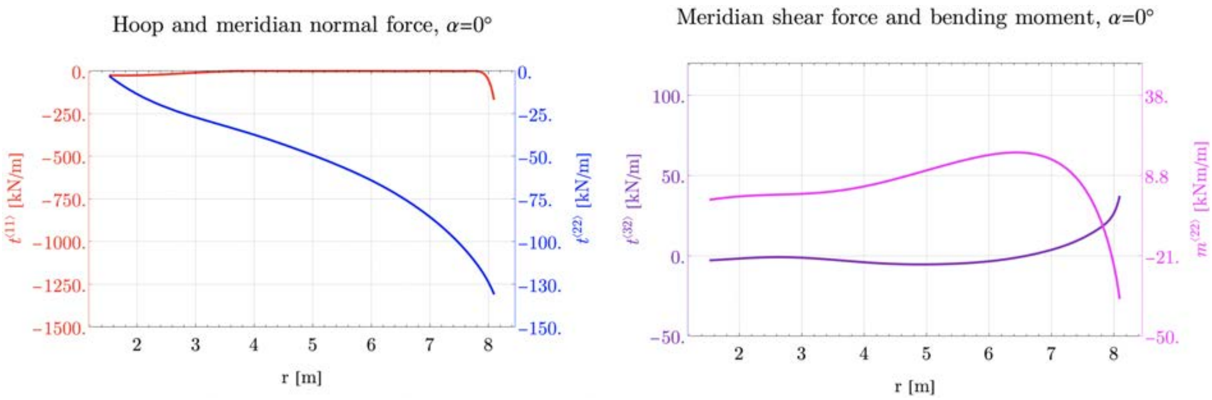

Figure 6: Membrane normal forces (left), meridian shear and bending moment (right) on the vertical section $\alpha = 0°$.

of any magnitude. An analysis method already developed by the authors and used to study the stability of domes subjected to vertical loads only has been applied for the first time to the case where horizontal loads, alongside to gravitational loads, are present. The method makes use of an ad-hoc developed solution procedure capable of determining optimised statically admissible stress fields.

The obtained values of the maximum load multiplier turned out to be strongly dependent on

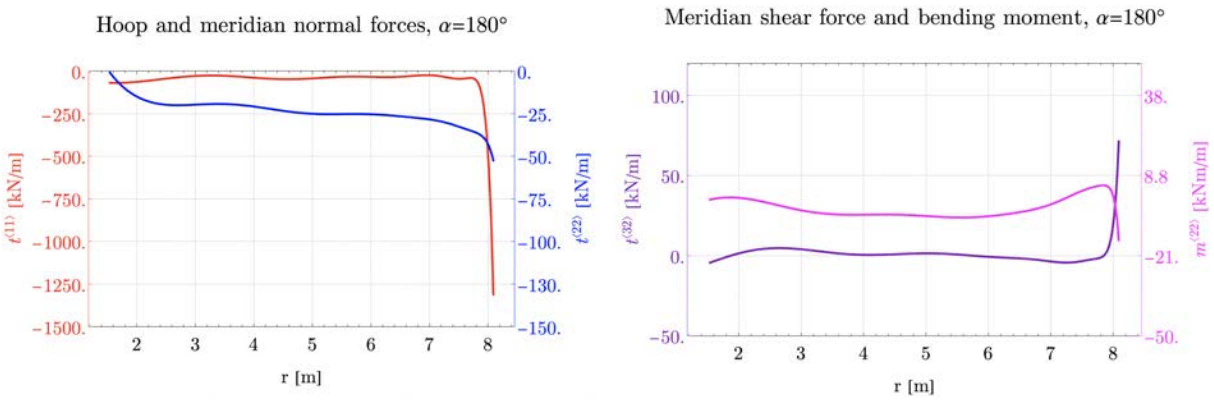

Figure 7: Membrane normal forces (left), meridian shear and bending moment (right) on the vertical section $\alpha = 180°$.

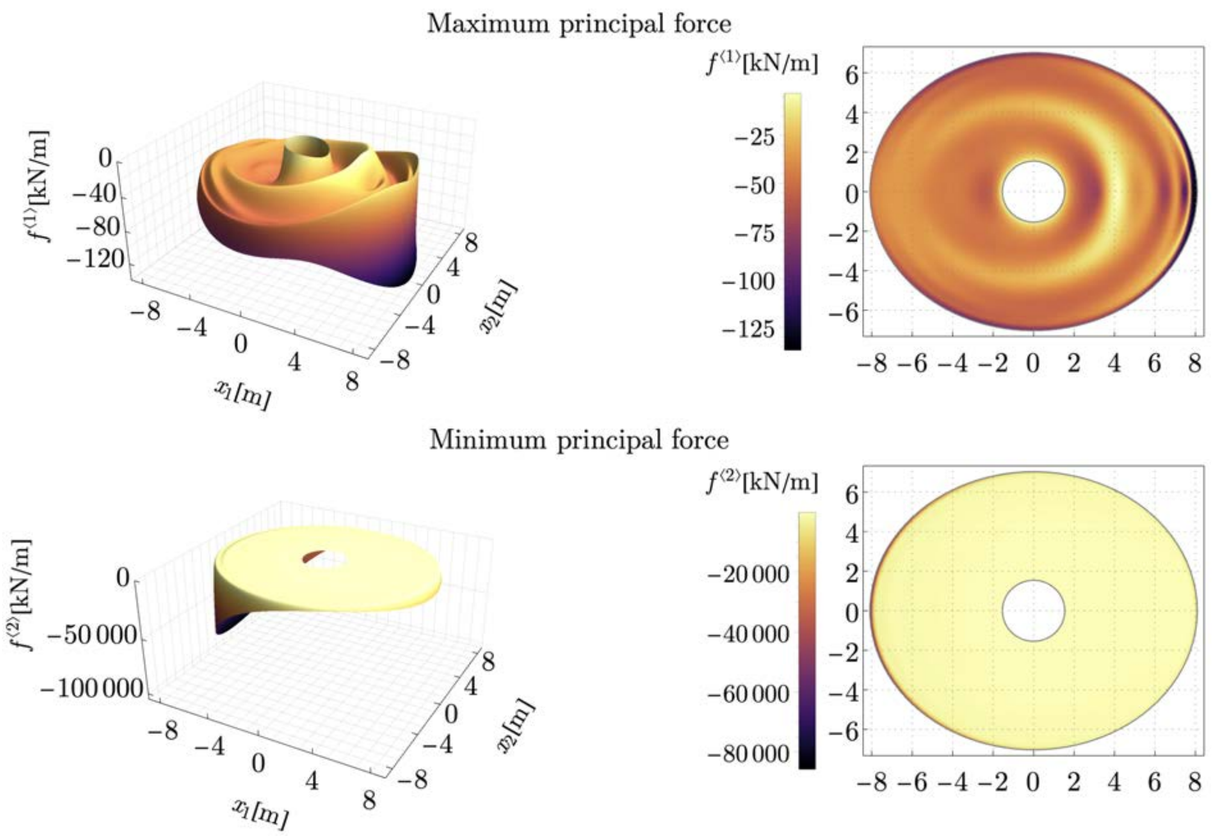

Figure 8: Principal membrane forces.

the direction of loading, being equal to 0.360 in the case where the loads act along the major axis of the dome and to 0.245 in the case where they act along the minor axis. These values seem to agree with analogous theoretical and numerical results obtained under Heyman's hypotheses and documented in the literature.

The solution illustrated in the present contribution highlights large hoop compressive forces concentrated in a narrow strip near the abutments. Moreover, transverse shear forces of the same order of magnitude as compressive forces are observed in the meridians upstream with

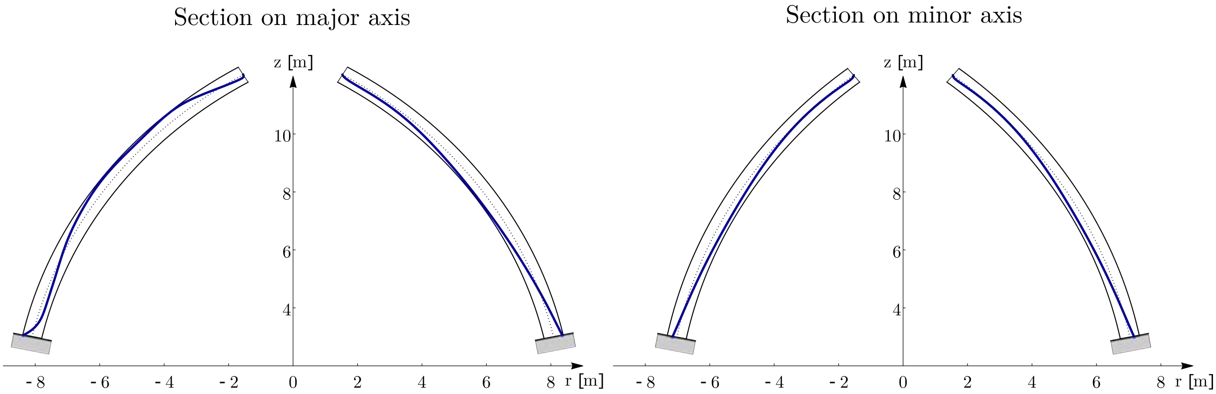

Figure 9: Maximum modulus eccentricity surface on two vertical sections at $\alpha = 0°$ (right) and $\alpha = 90°$ (left).

respect to the horizontal action. Although further studies are essential, these first findings seem to suggest that in the presence of horizontal actions Heyman's hypotheses could result to be not fully conservative and that more suitable failure criteria could be needed to properly describe the behaviour of masonry.

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