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# A PROBABILISTIC BASED ADAPTIVE REFINEMENT FOR RELIABILITY ASSESMENT USING KRIGING METAMODEL

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**Abstract.** *The paper presents a novel, adaptively, refined Kriging model that is faster and more stable compared to previous variants of the method. In the context of structural reliability analysis, the aim of the proposed the Kriging metamodel is able to effectively control the accuracy of the model by continuously improving the model and hence its capacity to classify a sample in the failure, or in the safe domain. The model uses a probabilistic refinement procedure instead of a sampling algorithm in order to produce new support points that are used to re-train the surrogate model. The selection is based on the so-called "most probable misclassification" function which allows to create a probability density function from which the algorithm produces samples close to failure surface. This function assesses the prediction quality using a proba-bility-based metric, instead of calculating the prediction error. The refinement procedure should be terminated when the Kriging model is reliable enough. A probabilistic-based stopping criterion is also used for measuring the accuracy of the Kriging model. The efficiency of the proposed Kriging method is demonstrated with the aid of case study examples.*

**Keywords:** Kriging, Reliability analysis, Adaptive refinement, Surrogate model.

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## 1 INTRODUCTION

In the field of structural reliability assessment, determining the reliability integral is essential. This integral relies on the performance function, which, in practical engineering scenarios, is usually implicit and relies on time-consuming numerical simulations. To mitigate this challenge, surrogate model methods are employed, replacing the actual performance function with a model that provides quick response estimates. These methods are constantly becoming more popular. Kriging surrogate models [24, 25, 26] are widely used in reliability, uncertainty quantification and optimization [22]. The method was first adopted for structural reliability analysis by Kaymaz *et al.* [1].

Echard *et al.* [20] introduce a method called AK-MCS, which consists of an Active learning reliability method combining Kriging and Monte Carlo Simulation. In this paper a learning function, called $U$, is proposed for selecting new support points. Fauriat *et al.* [21] presented a strategy to adapt the AK-MCS method for system reliability. The method AK-SYS, "Active learning and Kriging-based SYStem reliability method", is introduced. Considering a fully probabilistic approach, the probabilistic classification function proposed by Pincheny *et al.* [9], was introduced in Kriging metamodels by Dubourg *et al.* [5], either in combination with Monte Carlo Simulation (MCS), [5, 6], or with Subset Simulation (SS) [7, 8]. In addition, in Dubourg *et al.* [5] the Kriging metamodel is used to define a sub-optimal probability density function (PDF) for the Importance Sampling method. Moreover, Zhang *et al.* [24] used the probabilistic classification function in order to introduce a refinement method for Kriging, which determines the *most probable region* in a probabilistic viewpoint, choosing subsequent samples in this region.

A new Kriging metamodel for structural reliability assessment is herein presented. The paper proposes a simple and efficient refinement method, which creates a more robust and reliable Kriging surrogate model compared to previous implementations of the method. The updating of the Kriging model is based on a new learning function, the *most probable misclassification function*. With the aid of this function new support points are selected, in regions where the model is not accurate enough. Moreover, a probability-based stopping criterion is introduced based on this function instead of calculating the prediction error. This criterion leads to a better training of the Kriging model and avoiding add superfluous points during the refinement. This procedure is accurate and fast and can be easily implemented.

## 2 RELIABILITY ANALYSIS

Structural reliability analysis aims on calculating, or predicting, the probability of a system's performance function $G(\mathbf{x})$, being violated during the course of its lifetime [13]. The uncertainties associated with an entire system, such as material properties, loading, and boundary conditions, are quantified in terms of a $n$-dimensional random variables $\mathbf{x} = [x_1, \ldots, x_n]^T$. If $f_{\mathbf{x}}(\mathbf{x})$ is the joint probability density function (PDF) of vector $\mathbf{x}$. If $p_f$ is the failure probability of the system. This probabilistic approach yields a generalized reliability formulation suitable for many structural reliability analysis problems:

$$p_f = \int_{G(\mathbf{x})<0} f_{\mathbf{x}}(\mathbf{x})\, dx = \int_{\mathbb{R}^n} I_{G<0}(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x})\, dx \tag{1}$$

where $I_{G<0}(\mathbf{x})$ is the failure indicator boolean function, it is equal to one if $G(\mathbf{x}) > 0$ and zero otherwise.

For solution of the fundamental integral of Eq.1, many different methods have been developed over the past few decades. In general, the existing reliability methods can be classified into three categories. The first category are analytical methods based on the Taylor-series expansion of the performance function, e.g. the First Order Reliability Method (FORM) and the Second Order Reliability Method (SORM) [12]. The second category are the Monte Carlo [11, 10, 14] simulation methods, among which are the Importance Sampling [17, 16] and the Subset Simulation method [3]. The Monte Carlo simulation (MCS) technique is a basic reference approach and is widely used in engineering practice. Subset Simulation (SS) [2] is an efficient and elegant variant of the MC method for simulating rare events and for estimating the corresponding small tail probabilities.

The current study focuses on surrogate model methods, which belong to the third category. Surrogate model methods replace the limit-state function by a surrogate model. The use of a surrogate minimizes the computational cost since the many function evaluations required are performed by the surrogate model thus become inexpensive. A novel Kriging surrogate model with adaptive refinement is inroduced, as discussed in the rest of the paper.

## 3 KRIGING METAMODEL

### 3.1 Basic Kriging definitions

Given an initial design of experiments (DoE) $\mathbf{S}_0 = [\mathbf{x}_1, \ldots, \mathbf{x}_{N_0}]^T$ with $\mathbf{x}_i \in \mathbb{R}^n$ $(i = 1, 2, \ldots, N_0)$, and $\mathbf{G} = [G(\mathbf{x}_1), G(\mathbf{x}_2), \ldots, G(\mathbf{x}_{N_0})]^T$, with $G(\mathbf{x}_i) \in \mathbb{R}$ the values of the limit-state function $G(\mathbf{x})$ and $N_0$ the size of the training sample. Kriging is based on the idea that the performance function $G$ can be approximated by $\widehat{G}(\mathbf{x})$ as follows:

$$\widehat{G}(\mathbf{x}) = \boldsymbol{\beta}^T \mathbf{f}(\mathbf{x}) + Z(\mathbf{x}), \tag{2}$$

where $\boldsymbol{\beta}^T \mathbf{f}(\mathbf{x})$ is the trend of the Gaussian Process (GP), and $\boldsymbol{\beta}^T = [\beta_1, \ldots, \beta_p]^T$ is the regression coefficient of the basis function $\mathbf{f}(\mathbf{x}) = [f_1, \ldots, f_p]^T$, where $p$ is the number of basis functions. The term $\sigma_G^2$ consists of the variance of the Gaussian process and $Z(\mathbf{x})$ denotes a zero-mean stationary Gaussian Process with unit variance that relates to a covariance matrix as:

$$Cov(\mathbf{x}_i, \mathbf{x}_j) = \sigma_G^2 R(\mathbf{x}_i - \mathbf{x}_j, \boldsymbol{\vartheta}). \tag{3}$$

where $R(\mathbf{x}_i - \mathbf{x}_j, \boldsymbol{\vartheta})$ corresponds to the correlation function between the points $\mathbf{x}_i$ and $\mathbf{x}_j$. Moreover, $\boldsymbol{\vartheta}$ is the vector with the parameters of the correlation function. A widely used autocorrelation functions is the *anisotropic squared exponential model*.

$$R(\mathbf{x_i} - \mathbf{x_j}, \boldsymbol{\vartheta}) = \exp\left[ - \sum_{k=1}^{n} \left( \frac{x_i^k - x_j^k}{\vartheta_k} \right)^2 \right] \tag{4}$$

According to the principle of Kriging, $G(\mathbf{x})$ can be described by the PDF of a Gaussian distribution and hence:

$$\widehat{G}(\mathbf{x}) \sim \mathcal{N}(\mu_{\widehat{G}(\mathbf{x})}, \sigma_{\widehat{G}(\mathbf{x})}) \tag{5}$$

The expected value $\mu_{\widehat{G}(\mathbf{x})}$ and the variance $\sigma_{\widehat{G}(\mathbf{x})}$ of $\widehat{G}(\mathbf{x})$ at point $\mathbf{x}$ can be obtained as

$$\mu_{\widehat{G}(\mathbf{x})} = \mathbf{f}(\mathbf{x})^T \widehat{\boldsymbol{\beta}} + \mathbf{r}(\mathbf{x})^T \mathbf{R}^{-1} (\mathbf{G} - \mathbf{F}\widehat{\boldsymbol{\beta}}) \tag{6}$$

$$\sigma^2_{\widehat{G}(\mathbf{x})} = \sigma_G^2 (1 - \mathbf{r}(\mathbf{x})^T \mathbf{R}^{-1} \mathbf{r}(\mathbf{x}) + \mathbf{u}(\mathbf{x})^T (\mathbf{F}^T \mathbf{R}^{-1} \mathbf{F})^{-1} \mathbf{u}(\mathbf{x})) \tag{7}$$

where $\mathbf{r}(\mathbf{x})$ is the vector of cross-correlations between the prediction point $\mathbf{x}$ and each one of the observations whose elements read as $\mathbf{r}_i(\mathbf{x}) = \mathbf{R}(\mathbf{x}-\mathbf{x_i}, \boldsymbol{\vartheta}), i = 1, \ldots, N_0$; $\mathbf{R}$ is their correlation matrix defined by $R_{ij} = R(\mathbf{x}_i, \mathbf{x}_j)\ i, j = 1, \ldots, N$;$\mathbf{F}$ is the observation matrix of the Kriging metamodel trend defined by $F_{ij} = f_j(\mathbf{x_i}), i = 1, \ldots, N_0, j = 1, \ldots, p$. The generalized least squares solution $\widehat{\boldsymbol{\beta}}$ and the vector $\mathbf{u}(\mathbf{x})$ respectively read as follows:

$$\widehat{\boldsymbol{\beta}} = (\mathbf{F}^T\mathbf{R}^{-1}\mathbf{F})^{-1}\mathbf{F}^T\mathbf{R}^{-1}\mathbf{G}, \tag{8}$$

$$\mathbf{u}(\mathbf{x}) = \mathbf{F}^T\mathbf{R}^{-1}\mathbf{r}(\mathbf{x}) - \mathbf{f}(\mathbf{x}) \tag{9}$$

At this stage one can easily prove that $\mu_{\widehat{G}(\mathbf{x})} = g(\mathbf{x}_i)$ and $\sigma_{\widehat{G}(\mathbf{x})} = 0$ for $i = 1, \ldots N_0$ thus meaning that the Kriging surrogate interpolates the observations. Given a choice for the regression and correlation models, the optimal set of parameters $\beta, \vartheta$ and $\sigma_G^2$ can then be estimated using DACE [23]. For implementation details, you can refer to the work by Lophaven *et. al* [23], which provides further information on the specific algorithms and techniques used in solving the optimization problem for Kriging.

### 3.2 The probabilistic classification function

When a limit-state function $G(\mathbf{x})$ is simulated as a Gaussian process, such as the Kriging predictor $\widehat{G}(\mathbf{x})$, the classification can be performed in a probabilistic way with the aid of the *probabilistic classification function* $\pi(\mathbf{x})$ [5, 9]. The probabilistic classification function $\pi(\mathbf{x})$ is defined as:

$$\pi(\mathbf{x}) = \mathcal{P}(\widehat{G} < 0) = \Phi\left(\frac{0 - \mu_{\widehat{G}(x)}}{\sigma_{\widehat{G}(x)}}\right) \tag{10}$$

It provides the probability that the prediction $\widehat{G}(\mathbf{x})$ is negative. Note that $\mathcal{P}$ denotes the probability measure associated with the epistemic uncertainty of the surrogate model. Eq. 10 allows to calculate the failure probability of Eq. 1 by integrating $f_{\mathbf{X}}(\mathbf{x})$ in the domain $\mathbb{R}^n$, as follows:

$$p_{f\varepsilon} = \int_{\mathbb{R}^n} \pi(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, dx \tag{11}$$

For the case of Monte Carlo simulation, using the sample $\mathbf{X}$, $p_{f\varepsilon}$ can be estimated as:

$$\widehat{p}_{f\varepsilon} = \frac{1}{N}\sum_{i=1}^{N} \pi\left(\mathbf{x}^{(i)}\right) \tag{12}$$

where $[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}]^T$ is set of samples of vector $\mathbf{X}$.

## 4 MODEL REFINEMENT

### 4.1 Proposed adaptive refinement algorithm

In the context of structural reliability assessment, the aim of refinement algorithms is to effectively control the accuracy of the Kriging model by continuously improving the model and hence its capacity to classify a sample in the failure, or in the safe domain. The efficiency of the proposed Kriging metamodel strongly depends on the model refinement technique that is based on the most probable misclassification function. The aim of the proposed strategy is to increase the accuracy of the model, which is related with capacity of the Kriging model to made accurate predictions in the vicinity of the limit-state surface $G(\mathbf{x}) = 0$.

The algorithm initiates with a generation of *Monte Carlo* population $\mathbf{X}$ in the design space. This sample is not evaluated on the performance function, but it represents a set of points to be evaluated on the most probable misclassification function. Moreover, from this sample the failure probability will be calculated. The metamodel is build from an initial design of experiments (DoE) denoted by $\mathbf{S}_0$, while it is usually chosen with an LHS scheme. The initial DoE depend on the number of r.vs, and it is preferred to be relatively small. Next, the Kriging model is trained using DACE toolbox [23]. A refinement procedure that will update the support points is then adopted. The selection of new supports points is based on the use of the *most probable misclassification* function $\mathcal{C}(\mathbf{x})$. The most probable misclassification function expresses the probability that the Kriging prediction $\widehat{G}(\mathbf{x})$ (Eq. 2) has been erroneously classified and is defined as the complementary of the classification function $\pi(\mathbf{x})$ (Eq. 10). Therefore, for every sample $\mathbf{x}^{(i)}$ we calculate the mean and the standard deviation of the Kriging prediction $\mu_{\widehat{G}(\mathbf{x}^{(i)})}$, $\sigma_{\widehat{G}(\mathbf{x}^{(i)})}$ (Eqs. 6 and 7) and the most probable misclassification function is obtained as:

$$\mathcal{C}(\mathbf{x}^{(i)}) = 2 * min\{\pi(\mathbf{x}^{(i)}); 1 - \pi(\mathbf{x}^{(i)})\} \tag{13}$$

Therefore, with the aid of $\mathcal{C}(\mathbf{x})$ the regions with the most probable misclassified samples have been identified.

The most probable misclassification function is then used to transform the PDF of the random variables as follows:

$$\widehat{f_{\mathbf{X}}}(\mathbf{x}^{(i)}) \propto \mathcal{C}(\mathbf{x}^{(i)}) f_{\mathbf{X}}(\mathbf{x}^{(i)}) \tag{14}$$

The new PDF $\widehat{f_{\mathbf{X}}}(\mathbf{x}^{(i)})$ receives large values, close to $G(\mathbf{x}^{(i)}) = 0$ and proportionately decays away from $G$, analogously to $f_{\mathbf{X}}(\mathbf{x}^{(i)})$. Therefore, Eq. 14 shifts the PDF to the areas that mostly affect the failure probability. The vector of the support points $\mathbf{S}_i$ is enriched with the new points $\mathbf{s}_i$ that can be generated either using a Markov chain Monte Carlo (MCMC) simulation technique [5], or with a more efficient iterative algorithm, here proposed, that does not require sampling from $\widehat{f_{\mathbf{X}}}(\mathbf{x}^{(i)})$.

The refinement algorithm selects the candidate support points $\mathbf{s}_i$ in order to refine the model from the entries of $\mathbf{x}^{(i)}$. For every sample $\mathbf{x}^{(i)}$, the Kriging model calculates the misclassification function $C(\mathbf{x}^{(i)})$ (Eq. 13). Every sample is then "modified" according to Eq. 14. The whole sample is subsequently clustered using the $K$-means algorithm [18, 19] assigning weights $w(\mathbf{x}^{(i)})$ to every sample point, discarding points with weight less than 0.1. Small values of $\mathcal{C}(\mathbf{x})$ declare that the Kriging model is enough accurate in this area, thus discarding these points is ensured that the new points are not selected from areas, where the model is accurate. For the crude Monte Carlo simulation method, the weights are equal to the most probable misclassification function, i.e. $w(\mathbf{x}^{(i)}) = \mathcal{C}(\mathbf{x}^{(i)})$, while the definition of the weights may differ if other reliability methods are adopted.

Using the $K$-means algorithm, the available data are divided to $n$ clusters, $\mathbf{\Omega}^{(cl)}$, $cl = 1, 2, \ldots, n$, where $n$ is the number of random variables. For every cluster the point $\omega_o^{cl}$ which is the closest to the cluster center $O^{(cl)}$ is chosen. The selection of the points is thus based on the formula:

$$\omega_o^{cl} = \arg\min \|\mathbf{\Omega}^{(cl)} - O^{(cl)}\|_2. \tag{15}$$

The weights $\omega_o^{cl}$ should also respect the constraint:

$$w(\omega^{(cl)}) \geq 0.5 \max[w(\mathbf{\Omega}^{(cl)})] \tag{16}$$

where $\max[w(\mathbf{\Omega}^{(cl)})]$ is the maximum weight within the cluster. The next point is chosen if the condition 16 is not satisfied. Finally, the vector of the new support points $\mathbf{s}_i$ will be:

$$\mathbf{s}_i = \{\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(n)}\}^T \tag{17}$$

and the vector of support points is updated as:

$$\mathbf{S}_i = \{\mathbf{S}_0, \ldots, \mathbf{s}_{i-1}, \mathbf{s}_i\}^T = \{\mathbf{S}_{i-1}, \mathbf{s}_i\}^T \tag{18}$$

Once $\mathbf{S}_i$ is obtained, the Kriging model is re-trained using the new vector of support points $\mathbf{S}_i$.

### 4.2 Probabilistic-based stopping criterion

The refinement procedure should be terminated when the Kriging model is reliable enough. The following probability is used for measuring the accuracy of the Kriging model, as it calculated using the most probable misclassification function.

$$p_{\mathcal{C}} = \int_{\mathbb{X}} \mathcal{C}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, dx \tag{19}$$

The proposed probabilistic-based stopping criterion is defined as the ratio between the $p_{\mathcal{C}}$ and $p_{f\varepsilon}$, i.e. $\varepsilon_p = p_{\mathcal{C}}/p_{f\varepsilon}$. Probability $p_{\mathcal{C}}$ can be estimated from the same sample, that the failure probability (Eq. 11) is estimated. For the case of Monte Carlo simulation, using the sample $N$, $p_{\mathcal{C}}$ can be estimated as:

$$\widehat{p}_{\mathcal{C}} = \frac{1}{N} \sum_{i=1}^{N} \mathcal{C}\big(\mathbf{x}^{(i)}\big) \tag{20}$$

By definition $\widehat{p}_{\mathcal{C}}/\widehat{p}_{f\varepsilon} \leq 1$, as also seen from Eqs. 1, 10, 13 and 19.

A low $\varepsilon_p$ value, implies that the number of erroneously classified samples is small. The tolerance for stopping the refinement procedure was empirically found equal to 5%, which offers a good balance between accuracy and efficiency, since it is reduced often after a few refinement iterations.

### 4.3 Steps of Kriging refinement

The steps of the proposed adaptive refinement algorithm can be summarized as follows:

1. Assume the matrix $\mathbf{X}$ (section 3) of $N$ Monte Carlo simulations,
2. For every simulation/sample $\mathbf{x}^{(i)} \in \mathbf{X}$, calculate the most probable misclassification function (Eq. 13),
3. For every entry $\mathbf{x}^{(i)}$ of $\mathbf{X}$, obtain weights $w(\mathbf{x}^{(i)})$ equal to the missclassification function (keep points with $w \geq 0.1$)
4. Find $n$ clusters using the $K$-means algorithm and the calculated weights,
5. Find the closest point $\omega^{(cl)}$ of every cluster center (Eq. 15) and the vector of new support points $\mathbf{s}_i$ (Eq. 17),
6. Evaluate the limit-state function $G$ for the entries of $\mathbf{s}_i$.
7. Augment the training sample $\mathbf{S}_i$ and update the Kriging model,
8. Check the stopping criterion, if $\varepsilon_p$ is less than $5\%$ the refinement procedure is terminated, otherwise a new refinement iterations is required.

## 5 NUMERICAL EXAMPLES

In order to assess the performance of the proposed Kriging model, a test example is considered, based on mathematical model.. The implementation of the Kriging model is based on the Matlab toolbox DACE [23]. The autocorrelation function used is the *anisotropic squared exponential model* and the preferred *zero order polynomial function* is the regression model.

### 5.1 Two-dimensional, piecewise performance function

The first example is a four-boundary series system with the following performance function:

$$G(\mathbf{x}) = min \begin{Bmatrix} k + \frac{(x_1 - x_2)^2}{10} - \frac{x_1 + x_2}{\sqrt{2}} \\ k + \frac{(x_1 - x_2)^2}{10} + \frac{x_1 + x_2}{\sqrt{2}} \\ x_1 - x_2 + \frac{m}{\sqrt{2}} \\ -x_1 + x_2 + \frac{m}{\sqrt{2}} \end{Bmatrix}, \mathbf{x} \in \mathbb{R}^2 \tag{21}$$

where $k$, $m$ are integers that control the shape of the problem, while the random vector $\mathbf{x} = [x_1, x_2]^T$ consists of two independent and identically distributed (iid) standard normal variables. Integers for the problem considered $k, m$ are set to $k = 3$ and $m = 7$ [20]. In order to solve this problem will apply the two proposed models, demonstrating all the necessary implementations details.

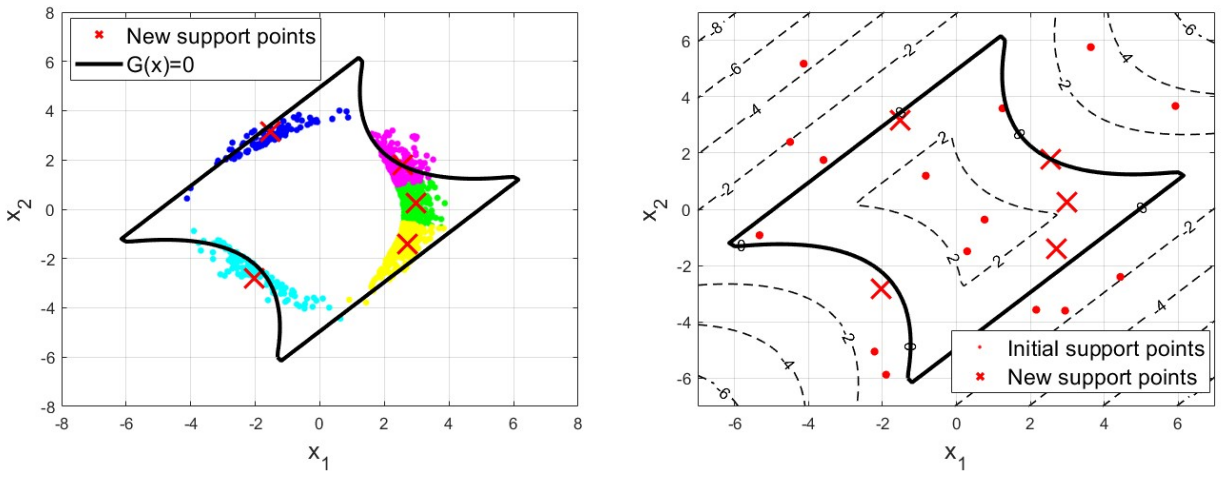
First, we apply the adaptively refined Kriging model. The number of initial support points was chosen using the empirical relationship $\mathbf{S}_0 = \max(2n; 15) = 15$. The fifteen initial support points are selected using LHS sampling for training the Kriging model and are shown in Fig. 1). Therefore five new supports point are generated with the aid of the proposed refinement strategy. Iterations were performed until the probabilistic-based refinement criterion ($\varepsilon_p < 5\%$) is fulfilled.

Figures 1, 2 and 3, demonstrate step by step the probabilistic-based refinement strategy. More specifically, contours of $G(\mathbf{x})$ are shown in combination with the initial supports points $\mathbf{S}_0$, and the additional points $\mathbf{s}_i$. During the first iteration (Fig. 1), the whole sample is divided in five clusters, and hence five new supports points are sought. Each cluster's set of points is shown with different color, while the new points are shown with a red cross, and these are the closest points of each cluster center (Eq. 15). In Fig. 1b, the existing training dataset is updated with the new points and a new Kriging model is constructed. At this first iteration the number of function calls used for reconstructing the Kriging model is twenty ($N_{call}$), while the probability of failure as function of the function evaluations is shown at in Fig. 4. The stopping criterion is not fulfilled at this iteration, thus the refinement should be continued.

In the second refinement iteration, the same steps are followed. Comparing with Fig. 2a some points of the sample have been discarded from the clustering procedure, since their weights are less than $10\%$ (4.3). Hence, as shown in Fig. 3b the new sample is smaller but closer to the failure surface. The sample is again divided to five clusters, new supports points (denoted with a red cross) are selected and the existing training dataset is updated. A new refinement step is required since $\varepsilon_p > 5\%$. The required number of function evaluations for the third iteration is twenty-five (Fig. 4).

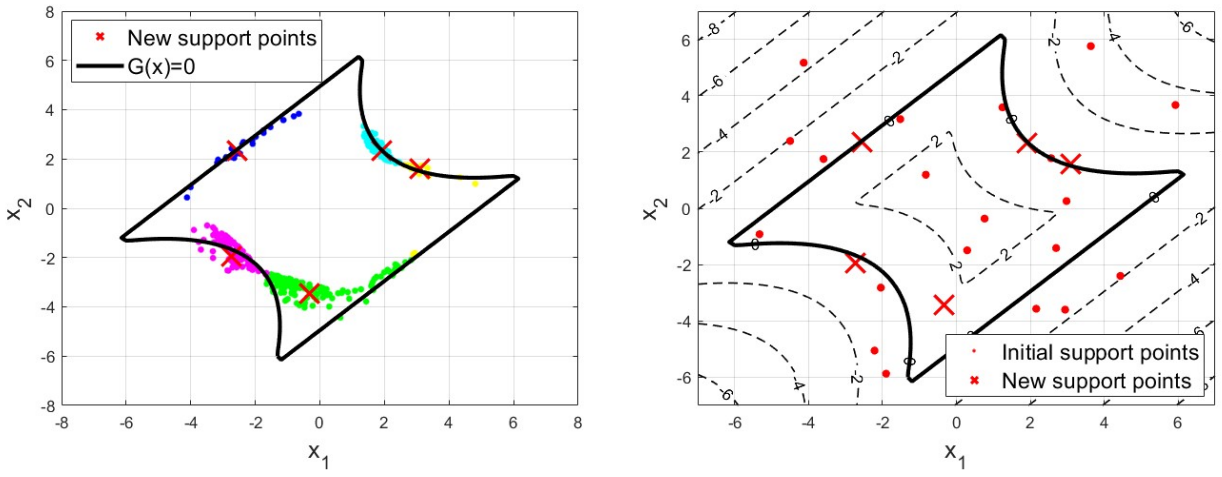
The results of iteration three are shown in Fig. 3. We should denote that in Fig. 3a the number of points is smaller, as the model have become more accurate. Thus, the number of points with weight less than 10% has been increased. In this iteration, the number of support points is thirty. The refinement strategy is terminated, since the stopping criterion is less than

5%(Fig. 4). Notice the the sample is considerably smaller compared to Fig. 1a and Fig. 2a




(a) (b)

Figure 1: Selection of new support points at **iteration one**. (a) Data clustering using $K$-means algorithm and selection of new support points (red cross). (b) Representation of new and initial supports points ($\mathbf{s}_i$ and $\mathbf{S}_0$) to the contours of the limit-state function.



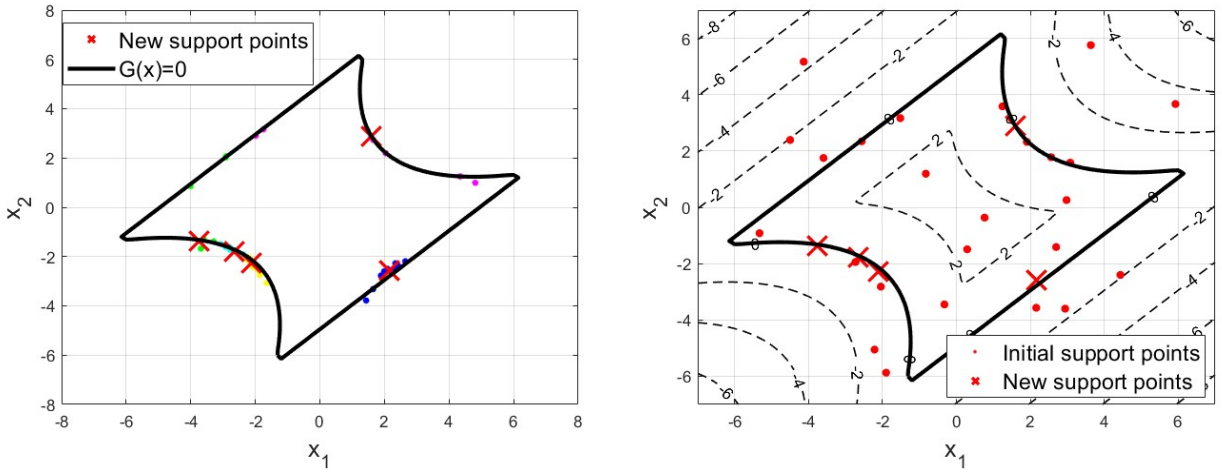

(a) (b)

Figure 2: Selection of new support points at **iteration two**. (a) Data clustering using $K$-means algorithm and selection of new support points (red cross). (b) Representation of new and initial supports points ($\mathbf{s}_i$ and $\mathbf{S}_0$) to the contours of the limit-state function.

Figure 4 shows the number of function evaluations that were required for building the model, while in the vertical the values of the estimated failure probability $p_{f\varepsilon}$ are shown at each iteration, with targeted $\delta_\varepsilon = 1.5\%$.

As shown in Fig. 4 before the refinement, the estimated $p_{f\varepsilon}$ is not accurate, but after the first refinement iteration, the estimation quickly converges to its true value. At iteration five, the

accuracy of the model $\varepsilon_p$ is less than $5\%$. The failure probability converges to the actual failure probability. The crude MCS method on the real $G(\mathbf{x})$ which required $N = 10^6$ evaluations, the Kriging model requires only $N_{call} = 30$ function evaluations.




(a) (b)

Figure 3: Selection of new support points at **iteration five**. (a) Data clustering using $K$-means algorithm and selection of new support points (red cross). (b) Representation of new and initial supports points ($\mathbf{s}_i$ and $\mathbf{S}_0$) to the contours of the limit-state function.

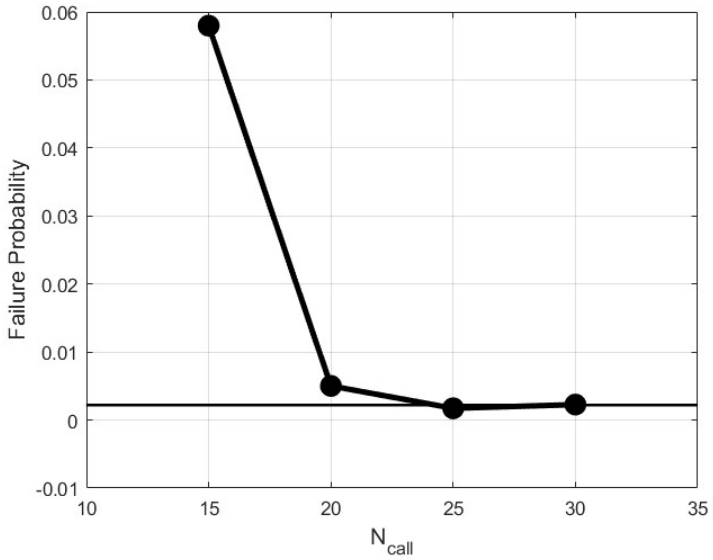



Figure 4: Convergence history of Kriging-based MCS with targeted $\delta_\varepsilon = 1.5\%$.

Results show that the use of the probabilistic based criterion, which proposed in this work, works in tandem with the proposed refinement method and it leads to high accuracy surrogate model, as the failure probability approximates the actual probability (Table 1). The proposed Kriging model is compared with the AK-MCS that was adopted in reference [20]. The results

are summarized in Table 1. The information is given as the number of calls to the performance function $N_{call}$, the failure probability, and the c.o.v.. The proposed strategy reduces the computational cost, and it is accurate enough. The Kriging method of Echard *et al.* [20] is slightly more accurate, but it required more function evaluations. The proposed probabilistic-based refinement technique leads to $15 + 3 \times 5 = 30$ function evaluation, while Echard *et al.* [20] to 96, thus the computational cost is significantly larger. The respective failure probabilities occur $2.253 \cdot 10^{-3}$ and $2.230 \cdot 10^{-3}$ for the two methods, thus it is almost the same to the reference probability that was obtained after $10^6$ function evaluations with the crude Monte Carlo method. of MCS ($2.233 \cdot 10^{-3}$).

Table 1: Results for the application of the proposed methodology and AK-MCS method to the piecewise polynomial function for $k = 3, m = 7$.

| Method | MCS | B.Echard *et al.* [20] | MCS-Kr. |
|---|---|---|---|
| $k = 3, m = 7$ | | | |
| $N_{call}$ | $10^6$ | 96 | 30 |
| $p_f(\times 10^{-3})$ | 2.233 | 2.230 | 2.253 |
| $\delta_\varepsilon$ | $< 2\%$ | $< 2\%$ | $< 2\%$ |

## 6 CONCLUSIONS

In this paper a novel Kriging metamodel for structural reliability assessment is introduced. The study proposes a simple and effective adaptive refinement method for improving the Kriging surrogate model. For this reason a new learning function is proposed, which is called the most probable misclassification function. This function can identify the regions where the model is not enough accurate, i.e. the failure surface, while new supports points in this area are selected. Additionally, a probability-based stopping criterion, based on this function, is introduced as an alternative to calculating prediction errors for the evaluation of the model. This criterion improves the training of the Kriging model and prevents the inclusion of unnecessary points during the refinement process. Overall, this process is quick and precise, and it is simple to include into computer programs.

## REFERENCES

[1] I. Kaymaz, Application of Kriging method to structural reliability problems. *Structural Safety*, **27**, 133–151, 2005.

[2] S.K. Au, J.L. Beck, Estimation of small failure probabilities in high dimensions by subset simulation. *Probabilistic Engineering Mechanics*, **16**, 263-277, 2001.

[3] S.K. Au, J.Ching,J.L.Beck, Application of subset simulation methods to reliability benchmark problems. *Structural Safety*, **29**, 183-193, 2007.

[4] S.K.Au, J.L.Beck, A new adaptive importance sampling scheme for reliability calculations. *Structural Safety*, **21**, 135-158, 1999.

[5] V.Dubourg, B.Sudret, F.Deheeger, Metamodel-based importance sampling for structural reliability analysis. *Probabilistic Engineering Mechanics*, **33**, 47-57, 2013.

[6] V.Dubourg, B.Sudret, Metamodel-based importance sampling for reliability sensitivity analysis. *Structural Safety*, **49**, 27-36, 2014.

[7] V. Dubourg, B. Sudret, J.M. Bourinet, Reliability-based design optimization using kriging surrogates and subset simulation. *Structural Multisciplinary Optimization*, **44**, 673-690, 2011.

[8] B. Sudret, V. Dubourg, J.M. Bourinet, Enhancing meta-model-based importance sampling by subset simulation. *In: 16th IFIP working conference on reliability and optimization of structural systems*, 2012.

[9] J. Bect, D. Ginsbourger, L. Li, V. Picheny, E. Vazquez, Sequential design of computer experiments for the estimation of a probability of failure, **Statistics and Computing**, **22**, 773-793, 2012.

[10] C. Bucher, *Computational Analysis of Randomness in Structural Mechanics*, Springer-Verlag, 2001.

[11] J.S. Liu, *Monte Carlo Strategies in Scientific Computing*, Book Series, 3. CRC Press, 2009.

[12] A.S. Nowak, K.R. Collins, *Reliability of Structures*, CRC Press London, 2012.

[13] R.E. Melchers, *Structural Reliability Analysis and Prediction*, John Wiley and Sons England, 1999

[14] R. Rubinstein, D. Kroese *Simulation and the Monte Carlo method*, Wiley Series in Probability and Statistics. Wiley, 2008.

[15] Matlab, *The language of technical computing*. 2022.

[16] S.K. Au, J.L. Beck, A new adaptive importance sampling scheme, *Structural Safety* **21**, 135-158, 1999.

[17] S.Engelund, R.Rackwitz, A benchmark study on importance sampling techniques in structural reliability. *Structural Safety*, **12**, 255-276, 1993.

[18] J. MacQueen, Some methods for classification and analysis of multivariate observations. *In:Le Cam LM, Neyman J, editors. Proceedings of the Berkeley symposium on mathematical statistics and probability*, **1**, 281-297, 1979.

[19] S. Lloyd, Least squares quantization in PCM, *IEEE Transactions on Information Theory*, **28**, 129-137, 1982.

[20] B. Echard, N.Gayton, M.Lemaire, AK-MCS: An active learning reliability method combining Kriging and Monte Carlo Simulation. *Structural Safety*, **33**, 145-154, 2011.

[21] W. Fauriat, N.Gayton, AK-SYS: An adaptation of the AK-MCS method for system reliability, *Reliability Engineering & System Safety*, **123**, 137-144, 2014.

[22] P. Wang, Z. Lu, Z. Tang, An application of the Kriging method in global sensitivity analysis with parameter uncertainty. *Applied Mathematical Modelling*, **37**, 9, 6543-6555, 2013.

[23] S. Lophaven, H. Nielsen, J. Søndergaard J, DACE a matlab Kriging toolbox. *Technical University of Denmark*, 2002

[24] L. Zhang, Z. Lu, P. Wang, Efficient structural reliability analysis method based on advanced Kriging model. *Applied Mathematical Modelling*, **39** 781-793, 2015.

[25] J. Vahedi, M.R. Ghasemi, M. Miri, Structural reliability assessment using an enhanced adaptive Kriging method. *Structural Engineering and Mechanics*, **66** 677-691, 2018.

[26] J. Wang, Z. Sun, R. Cao, An efficient and robust Kriging-based method for system reliability analysis. *Reliability Engineering & System Safety*, **216**, 107953, 2021.