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MODEL REDUCTION FOR VIBRO-ACOUSTIC ANALYSIS OF GEOMETRICALLY NONLINEAR STRUCTURES INTERACTING WITH ACOUSTIC CAVITIES

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Abstract. *This study investigates coupled reduced-order modeling to develop efficient methods for analyzing the vibro-acoustic response of geometrically nonlinear thin structures interacting with acoustic fluids, such as gas or liquid. Projection-based model reduction is adopted for the structural and fluid domain. To consider geometric nonlinearity, the reduced nonlinear restoring forces of the elastic structure are expressed as third order polynomials in modal coordinates. Consequently, the reduced system can be effectively solved using direct time-integration involving only the reduced coordinates. The study focuses on flat, thin structures where in-plane inertia effects can be neglected. For this specific case, the out-of-plane response can be represented by a set of bending modes, while the quasi-static in-plane response is considered implicitly. The proposed modeling strategy is validated by means of numerical examples investigating the vibro-acoustic response of thin panels interacting with acoustic cavities.*

Keywords: Model Order Reduction, Dynamic Substructuring, Geometrically Nonlinear, Finite Element Model, Fluid-Structure Interaction, Acoustics.

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1 INTRODUCTION

Vibro-acoustic analysis of thin-walled structures interacting with acoustic fluids is a recurring challenge in many engineering applications, particularly within the aerospace, automotive, and naval industries. The response of such structures is typically affected by bending-stretching coupling effects due to geometric nonlinearity. In addition, acoustic coupling introduces further complexity. The coupled response of fluid-structure systems is often analyzed using the finite element (FE) method. However, solving the nonlinear response of full-order FE models is often computationally expensive, particularly in the context of iterative design or multiple load case scenarios.

To address these challenges, projection-based model order reduction techniques have emerged as powerful tools for reducing computational demands while preserving essential dynamic characteristics. Similar to modal techniques commonly used in linear dynamics, nonlinear reduced order models can be obtained by projecting the governing equations (cf. Eq. 1) onto a modal reduction basis (e.g., see [1]). However, generating an accurate reduction basis is significantly more challenging for nonlinear systems. Several techniques have been proposed for this purpose, including approaches based on modal derivatives [2, 3, 4, 5, 6, 7] and data-driven methods [1, 8].

In this paper, we extend the strongly coupled model reduction approach, as proposed in [9], to vibro-acoustic systems that account for geometric nonlinearity in the structural domain. To incorporate these effects, the reduced nonlinear restoring forces of the structure are expressed as third-order polynomials in modal coordinates, enabling efficient time-domain simulations [3, 5, 6, 7, 10]. The present study focuses on vibro-acoustic problems involving flat structures subjected to transverse forcing, with negligible in-plane inertia. For this specific case, the nonlinear response can be accurately captured using a reduced basis of out-of-plane bending modes, with in-plane effects implicitly included through static condensation, as further described in Section 2.

2 MODEL REDUCTION OF GEOMETRICALLY NONLINEAR STRUCTURE

The equations of motion for a geometrically nonlinear structure, formulated using the FE method, can be expressed as [11]:

$$\begin{cases} \mathbf{M}_s\ddot{\mathbf{u}}(t) + \mathbf{C}_s\dot{\mathbf{u}}(t) + \mathbf{f}(\mathbf{u}(t)) = \mathbf{g}_s(t), \\ \mathbf{u}(t_0) = \mathbf{u}_0, \\ \dot{\mathbf{u}}(t_0) = \mathbf{v}_0. \end{cases} \tag{1}$$

Here, $\mathbf{M}_s$ and $\mathbf{C}_s$ are the $n_s \times n_s$ mass and damping matrices, $\mathbf{f}_s$ and $\mathbf{g}_s$ are the $n_s \times 1$ internal and external force vectors, respectively, and $\mathbf{u}$ is the $n_s \times 1$ displacement vector (dot notation indicates differentiation with respect to time). Additionally, $\mathbf{u}_0$ and $\mathbf{v}_0$ represent the initial displacements and velocities at the initial time t_0. Throughout this paper, subscripts and superscripts s and f denote the structural and fluid domains, respectively. For simplicity, time dependencies are omitted in the following equations.

2.1 Reduction basis and polynomial structure of nonlinear internal forces

By linearizing the nonlinear system around its static equlibrium point, assumed here to be $\mathbf{u}_{eq} = \mathbf{0}$, the first N_d^s eigenmodes can be obtained by solving the generalized eigenvalue problem:

$$(\mathbf{K}_s - \lambda_k\mathbf{M}_s)\,\boldsymbol{\phi}_k = \mathbf{0}, \quad k = 1, 2, \ldots, N_d^s \tag{2}$$

where

$$\mathbf{K}_s = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\bigg|_{\mathbf{u}=\mathbf{0}}$$

and $\boldsymbol{\phi}_k$ and λ_k are the kth eigenmode and eigenvalue, respectively, and $\omega_k = \sqrt{\lambda_k}$ is the corresponding eigenfrequency. Moreover, it is assumed that $N_d^s \ll n_s$.

Analogous to model reduction in linear dynamics, the full-order displacement vector can be approximated using the following transformation:

$$\mathbf{u} \approx \boldsymbol{\Phi}_d \mathbf{q} \tag{3}$$

where

$$\begin{aligned} \boldsymbol{\Phi}_d &= \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \dots & \boldsymbol{\phi}_{N_d^s} \end{bmatrix}, \\ \mathbf{q} &= \begin{bmatrix} q_1 & q_2 & \dots & q_{N_d^s} \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

are the truncated modal matrix and reduced modal coordinate vector, respectively. Furthermore, the eigenmodes are scaled to be mass-orthonormal, i.e., such that:

$$\begin{aligned} \boldsymbol{\Phi}_d^{\mathrm{T}} \mathbf{M}_s \boldsymbol{\Phi}_d &= \mathbf{I}_d, \\ \boldsymbol{\Phi}_d^{\mathrm{T}} \mathbf{K}_s \boldsymbol{\Phi}_d &= \boldsymbol{\Lambda}_d = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_{N_d^s}) \end{aligned}$$

where $\mathbf{I}_d$ is the $N_d^s \times N_d^s$ identity matrix.

Using Galerkin projection, a reduced order system can be formed by substituting Eq. 3 into Eq. 1 and premultiplying with $\boldsymbol{\Phi}_d^{\mathrm{T}}$, which gives:

$$\begin{cases} \ddot{\mathbf{q}} + \mathbf{D}_s \dot{\mathbf{q}} + \bar{\mathbf{f}}(\mathbf{q}) = \boldsymbol{\Phi}_d^{\mathrm{T}} \mathbf{g}_s \\ \mathbf{q}_0 = \boldsymbol{\Phi}_d^{\mathrm{T}} \mathbf{M}_s \mathbf{u}_0 \\ \dot{\mathbf{q}}_0 = \boldsymbol{\Phi}_d^{\mathrm{T}} \mathbf{M}_s \mathbf{v}_0 \end{cases} \tag{6}$$

where the orthgonality properties of the eigenmodes have been utilized, and where proportional damping is assumed, i.e., such that $\mathbf{D}_s = \boldsymbol{\Phi}_d^{\mathrm{T}} \mathbf{C}_s \boldsymbol{\Phi}_d = \mathrm{diag}(2\zeta_1\omega_1, \dots, 2\zeta_{N_d^s}\omega_{N_d^s})$, where ζ_j are modal damping ratios. Furthermore, the reduced nonlinear internal forces are given by $\bar{\mathbf{f}}(\mathbf{q}) = \boldsymbol{\Phi}_d^{\mathrm{T}} \mathbf{f}(\boldsymbol{\Phi}_d \mathbf{q})$.

Assuming large displacements and small (or moderate) strains, the reduced internal forces can be well approximated as cubic polynomials in modal coordinates, as demonstrated in [1, 3, 4, 5, 6, 7, 10, 12, 13, 14]. In fact, for St Venant Kirchhoff materials, it can be shown that the reduced nonlinear internal forces are fully resolved by cubic polynomials, e.g., see [3, 5]. Specifically, the reduced internal force vector can be split into a linear and nonlinear part, as:

$$\bar{\mathbf{f}}(\mathbf{q}) = \boldsymbol{\Lambda}_d \mathbf{q} + \bar{\mathbf{f}}_{\mathrm{nl}}(\mathbf{q}). \tag{7}$$

where the nonlinear part can be written in terms of quadratic and cubic terms; namely:

$$\left[\bar{\mathbf{f}}_{\mathrm{nl}}(\mathbf{q})\right]_k = \sum_{i=1}^{N_d^s} \sum_{j=i}^{N_d^s} \alpha_{ij}^k q_i q_j + \sum_{i=1}^{N_d^s} \sum_{j=i}^{N_d^s} \sum_{l=j}^{N_d^s} \beta_{ijl}^k q_i q_j q_l. \tag{8}$$

where $k = 1, \ldots, N_d^s$, and α_{ij}^k and β_{ijl}^k are nonlinear stiffness coeffcients. Thus, it follows that the equaiton of motions for the nonlinear reduced order model (Eq. 6) can be expressed as:

$$\ddot{q}_k + 2\zeta_k\omega_k\dot{q}_k + \omega_k^2 q_k + \sum_{i=1}^{N_d^s}\sum_{j=i}^{N_d^s}\alpha_{ij}^k q_i q_j + \sum_{i=1}^{N_d^s}\sum_{j=i}^{N_d^s}\sum_{l=j}^{N_d^s}\beta_{ijl}^k q_i q_j q_l = \boldsymbol{\phi}_k^{\mathrm{T}}\mathbf{g} \tag{9}$$

where, again, $k = 1, \ldots, N_d^s$. Note that, by expressing the reduced nonlinear internal forces in terms of cubic polynomials in modal coordinates, it is not necessary to recover the full-order displacement vector when solving the nonlinear system using direct time-integration, which significanlty reduces the computational cost [7].

Expressions for the nonlinear stiffness coefficients, α_{ij}^k and β_{ijl}^k, can be derived and implemented on the element level (e.g., see [3]). This approach is often referred to as an intrusive (direct) identification procedure. However, in this study, the stiffness coefficients were determined in a non-intrusive (indirect) manner using the STEP (STiffness Evaluation Procedure), first proposed in [12]. This approach will be further discussed in the next section.

2.2 Static condensation and identification of nonlinear stiffness coefficients

As compared to linear dynamics applications, a significantly larger set of eigenmodes is generally needed to adequately represent the geometrically nonlinear response [5]. In particular, for thin-walled structures, additional membrane modes are generally required, with eigenfrequencies far above the frequency content of the forcing. However, in some applications, considering only the quasi-static response of such high-frequency modes may be sufficient. This leads to the concept of static condensation of membrane modes (e.g., see [5, 7, 15]).

Let us consider flat structures subjected to low-frequency transverse forcing. For this specific case, the modal basis can be divided into N_d^b bending and N_d^m membrane modes, with superscripts b and m indicating bending and membrane coordinates, respectively (thus, the total number of modes is $N_d^s = N_d^b + N_d^m$). The structural membrane modes are not explicitly excited by either the external forcing or the acoustic pressure at interfaces. Furthermore, for flat structures, quadratic coefficients involving only bending coordinates are zero due to the symmetry of the restoring forces [10]. Therefore, as suggested in [5, 10, 13], a simplified form of the equation of motion may be adopted, where the system dynamics is expressed solely in terms of the bending coordinates, as:

$$\ddot{q}_r^b + 2\zeta_k\omega_r\dot{q}_r^b + \omega_r^2 q_r^b + \sum_{i=1}^{N_d^b}\sum_{j=i}^{N_d^b}\sum_{k=j}^{N_d^b}\hat{\beta}_{ijk}^r q_i^b q_j^b q_k^b = \boldsymbol{\phi}_k^{\mathrm{T}}\mathbf{g} \tag{10}$$

where $r = 1, \ldots, N_d^b$, $\hat{\beta}_{ijk}^r$ are the so-called *condensed* stiffness coeffcients, and q_r^b are modal coordinates representing the amplitude of bending eigenmodes.

This approach, described in detail in [5], implies that in-plane inertia effects are neglected. Accordingly, Eq. 10 is sometimes referred to as the *condensed* system equations. However, it should be emphasized that, although the dynamics is expressed solely in terms of the bending coordinates, the structure is not constrained in the in-plane direction. Instead, the statically condensed response of in-plane coordinates are considered implicitly. More specifically, the quasi-static in-plane motion is enslaved by the out-of-plane coordinates. This is due to the specific procedure used for determining the condensed stiffness coefficients $\hat{\beta}_{ijk}^r$, which will be further described as follows.

An FE model representing a flat structure, e.g., using beam or shell elements, can be formulated such that the out-of-plane bending modes only involve physical out-of-plane displacements (i.e., out-of-plane displacements and rotations around the in-plane axes). Hence, the physical displacements can be expressed as:

$$\underbrace{\begin{bmatrix} \mathbf{u}^b \\ \mathbf{u}^m \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} \mathbf{\Phi}_d^b & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}_d^m \end{bmatrix}}_{\mathbf{\Phi}_d} \underbrace{\begin{bmatrix} \mathbf{q}^b \\ \mathbf{q}^m \end{bmatrix}}_{\mathbf{q}} \tag{11}$$

where $\mathbf{\Phi}_d^b$ and $\mathbf{\Phi}_d^m$ include entries of the modal basis vectors corresponding to the out-of-plane $\mathbf{u}^b$ and in-plane $\mathbf{u}^m$ displacements, respectively.

The STEP, also referred to as the enforced displacement (ED) method [6], uses that the full-order internal force vector $\mathbf{f}$ can be obtained for prescribed displacement fields, using any FE program capable of solving nonlinear static problems. In particular, this makes it possible to determine the corresponding reduced order internal forces by projecting the full-order internal forces onto the modal basis, i.e., $\bar{\mathbf{f}}(\mathbf{q}) = \mathbf{\Phi}_d^{\mathrm{T}}\mathbf{f}(\mathbf{u})$.

Using this concept, and noting that the eigenmodes are mutually orthogonal, an algorithm can be established where the nonlinear stiffness coefficients can be identified [13]. Furthermore, the condensed stiffness coefficients can be determined by enforcing the bending modes solely on the physical out-of-plane DOFs, while the in-plane displacements are allowed to move freely.

For instance, coefficents of the form $\hat{\beta}_{iii}^r$, i.e., those with three equal indices, can be determined from the following static problems:

$$\begin{cases} \mathbf{u}^b = a\boldsymbol{\phi}_i^b \\ \mathbf{g}^m = \mathbf{0} \end{cases} \tag{12}$$

where $i = 1, \ldots, N_d^b$, and a is an amplitude coefficient, which should be sufficiently large to trigger geometric nonlinearties ($t/10$ was chosen in this study, with t being the structure thickness). The corresponding full-order internal force vectors $\mathbf{f}_{\mathrm{FE},i}$ are thus obtained by solving the nonlinear static problems using any suitable FE code. Furthermore, due to the orthogonality properties of the eigenmodes, the load case implies that the modal coordinates are $q_i^b = a$, $q_j^b = 0\ \forall j \neq i$. It then follows that the condensed stiffness coefficients can be determined from the following algebraic equations:

$$\boldsymbol{\phi}_r^{\mathrm{T}}\mathbf{f}_{\mathrm{FE},i} = \omega_r^2 a + \hat{\beta}_{iii}^r a^3. \tag{13}$$

for all $r, i = 1, \ldots, N_d^b$.

The full set of condensed stiffness coefficients can be determined in a similar manner based on static load cases composed of one, two, and three modal basis vectors (see [12] for details on the STEP method). The procedure implies that the in-plane displacments are enslaved by the bending modes. Concequently, the quasi-static in-plane response is considered implicitly in the dynamic analysis.

For the remainder of this paper, modal bases in the structural domain are restricted exclusively to bending eigenmodes; accordingly, the superscript b is omitted to simplify notation, i.e., hereafter $\mathbf{\Phi}_d = \mathbf{\Phi}_d^b$, $\mathbf{q} = \mathbf{q}^b$, and $N_d^s = N_d^b$.

3 MODEL REDUCTION OF NONLINEAR VIBRO-ACOUSTIC SYSTEMS

In this section, we introduce the strongly coupled model reduction method for vibro-acoustic systems, as proposed in [9]. First, linear structure-acoustic problems are considered. Then, the reduced-order system is adjusted to account for geometric nonlinear effects in the structural domain, using the approach outlined in Section 2. Structures undergoing large displacements and small strains, while interacting with an acoustic fluid such as gas or liquid, are considered. The fluid is considered compressible, inviscid, and irrotational, with free-surface gravity effects neglected. Note that, under these assumptions, the fluid pressure and density remain constant at equilibrium. In this regard, the methodology is suitable for analyzing acoustic fluids interacting with thin-walled structures, typically influenced by bending-stretching coupling effects due to geometric nonlinearity. Furthermore, to simplify the notation, damping is not considered in the derivation.

An FE formulation of undamped linear vibro-acoustic problems can be expressed in terms of the following asymmetric coupled equations, where displacement and pressure are used as primary variables for the structure and fluid domain, respectively [16, 17]:

$$\mathbf{A}\ddot{\mathbf{d}} + \mathbf{B}\mathbf{d} = \mathbf{g} \tag{14a}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{M}_s & \mathbf{0} \\ \rho c^2 \mathbf{C}^{\mathrm{T}} & \mathbf{M}_f \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{K}_s & -\mathbf{C} \\ \mathbf{0} & \mathbf{K}_f \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} \mathbf{g}_s \\ \mathbf{g}_f \end{bmatrix} \tag{14b}$$

where $\mathbf{M}$ and $\mathbf{K}$ constitute the mass and stiffness matrices, respectively, and $\mathbf{u}$, $\mathbf{p}$, and $\mathbf{g}$ represent the displacement vector for the structural domain, the pressure vector for the fluid domain, and the external force vector. Additionally, ρ and c are the fluid density and speed of sound, and $\mathbf{C}$ is a spatial coupling matrix (which should be distinguished from the damping matrix $\mathbf{C}_s$ in Eq. 1). A detailed derivation of Eq. 14, which is known as the standard (or basic) FSI-formulation can, e.g., be found in [17].

As proposed in [9], a strongly coupled model reduction of the system, Eq. 14, can be achieved by approximating the displacements and pressures as:

$$\begin{cases} \mathbf{u} \approx \bar{\mathbf{u}} = \mathbf{\Phi}_d \mathbf{q} + \mathbf{\Psi}\bar{\mathbf{p}} \\ \mathbf{p} \approx \bar{\mathbf{p}} = \tilde{\mathbf{\Xi}}_d \mathbf{r} \end{cases} \tag{15}$$

where $\mathbf{\Phi}_d$ contains a reduced set of bending eigenmodes of the structure, given by Eq. 2, and $\mathbf{q}$ is the associated modal coordinate vector. Furthermore, the columns of $\mathbf{\Psi} = \mathbf{K}_s^{-1}\mathbf{C}$ constitute constraint modes, that considers the relationship between pressures and displacements (see [9] for a detailed derivation). In accordance with the strongly coupled model reduction method [9], the pressure distribution of the fluid domain is approximated using a truncated set of eigenmodes:

$$\tilde{\mathbf{\Xi}}_d = \begin{bmatrix} \tilde{\boldsymbol{\xi}}_1 & \tilde{\boldsymbol{\xi}}_2 & \cdots & \tilde{\boldsymbol{\xi}}_{N_d^f} \end{bmatrix} \tag{16}$$

which are determined from the following generalized eigenvalue problem:

$$\left(\mathbf{K}_f - \gamma_i \tilde{\mathbf{M}}_f\right) \tilde{\boldsymbol{\xi}}_i = \mathbf{0}, \quad k = 1, 2, \ldots, N_d^f \tag{17}$$

where

$$\tilde{\mathbf{M}}_f = \mathbf{M}_f + \rho c^2 \mathbf{C}^{\mathrm{T}} \mathbf{\Psi} + \mathbf{\Psi}^{\mathrm{T}} \mathbf{M}_s \mathbf{\Psi}. \tag{18}$$

As demonstrated in detail in [9], replacing the fluid domain mass matrix $\mathbf{M}_f$ with $\tilde{\mathbf{M}}_f$ in the generalized eigenvalue problem significantly improves the convergence of the reduced-order model. The associated modal coordinate vector are given as $\mathbf{r} = \begin{bmatrix} r_1 & r_2 & \cdots & r_{N_d^f} \end{bmatrix}^{\mathrm{T}}$ (cf. Eq. 15).

In this work, a slight modification was introduced to improve numerical stability, which is particularly important when solving nonlinear systems using direct time-integration (see further Section 4); namely, the constraint modes $\mathbf{\Psi}$ were made mass- and stiffness-orthogonal as:

$$\mathbf{\Psi}_{\perp} = \left(\mathbf{I} - \mathbf{\Phi}_d \mathbf{\Phi}_d^{\mathrm{T}} \mathbf{M}_s\right) \mathbf{\Psi} \tag{19}$$

Thus, the subspace already spanned by the structural modes $\mathbf{\Phi}_d$ is removed from the constraint modes. Additionally, orthogonality with respect to the structural eigenmodes is required to determine the nonlinear stiffness coefficients in a nonintrusive manner using the STEP, as discussed in Section 2.2.

The strongly coupled model order reduction is then obtained by introducing the following reduced transformation:

$$\left\{ \begin{array}{c} \mathbf{u} \\ \mathbf{p} \end{array} \right\} \approx \mathbf{T} \left\{ \begin{array}{c} \mathbf{q} \\ \mathbf{r} \end{array} \right\} \tag{20}$$

where the transformation matrix is given by

$$\mathbf{T} = \left[\begin{array}{cc} \mathbf{\Phi}_d & \mathbf{\Psi}_{\perp} \tilde{\mathbf{\Xi}}_d \\ 0 & \tilde{\mathbf{\Xi}}_d \end{array} \right]. \tag{21}$$

Using Galerkin projection, it then follows that a nonlinear reduced system can be expressed as:

$$\bar{\mathbf{A}} \left\{ \begin{array}{c} \ddot{\mathbf{q}} \\ \ddot{\mathbf{r}} \end{array} \right\} + \bar{\mathbf{B}} \left\{ \begin{array}{c} \mathbf{q} \\ \mathbf{r} \end{array} \right\} + \left\{ \begin{array}{c} \bar{\mathbf{f}}_{\mathrm{nl}}(\mathbf{q}) \\ \mathbf{0} \end{array} \right\} = \mathbf{T}^{\mathrm{T}} \left\{ \begin{array}{c} \mathbf{g}_s \\ \mathbf{g}_f \end{array} \right\} \tag{22}$$

where

$$\bar{\mathbf{A}} = \mathbf{T}^{\mathrm{T}} \mathbf{A} \mathbf{T} = \left[\begin{array}{cc} \mathbf{I}_d & \mathbf{0} \\ \rho c^2 \tilde{\mathbf{\Xi}}_d^{\mathrm{T}} \mathbf{C}^{\mathrm{T}} \mathbf{\Phi}_d & \tilde{\mathbf{\Xi}}_d^{\mathrm{T}} \tilde{\mathbf{M}}_{f,\perp} \tilde{\mathbf{\Xi}}_d \end{array} \right], \quad \bar{\mathbf{B}} = \mathbf{T}^{\mathrm{T}} \mathbf{B} \mathbf{T} = \left[\begin{array}{cc} \mathbf{\Lambda}_d & -\mathbf{\Phi}_d^{\mathrm{T}} \mathbf{C} \tilde{\mathbf{\Xi}}_d \\ \mathbf{0} & \tilde{\mathbf{\Gamma}}_d \end{array} \right] \tag{23}$$

and where

$$\tilde{\mathbf{M}}_{f,\perp} = \mathbf{M}_f + \rho c^2 \mathbf{C}^{\mathrm{T}} \mathbf{\Psi}_{\perp} + \mathbf{\Psi}_{\perp}^{\mathrm{T}} \mathbf{M}_s \mathbf{\Psi}_{\perp} \approx \mathbf{M}_f + \rho c^2 \mathbf{C}^{\mathrm{T}} \mathbf{\Psi}_{\perp}, \quad \tilde{\mathbf{\Gamma}}_d = \tilde{\mathbf{\Xi}}_d^{\mathrm{T}} \mathbf{K}_s \tilde{\mathbf{\Xi}}_d. \tag{24}$$

Furthermore, the nonlinear part of the reduced internal forces of the structural domain is given by:

$$\left[\bar{\mathbf{f}}_{\mathrm{nl}}(\mathbf{q})\right]_r = \sum_{i=1}^{N_d^s} \sum_{j=i}^{N_d^s} \sum_{k=j}^{N_d^s} \hat{\beta}_{ijk}^r q_i q_j q_k \tag{25}$$

for all $r = 1, \ldots, N_d^s$ (cf. Eq. 10). Note that, as indicated by Eq. 22, the nonlinear part of the internal forces is only considered for the bending eigenmodes $\mathbf{\Phi}_d$. Thus, in this work, it is assumed that geometric nonlinearity can be neglected for the orthogonalized constraint modes (i.e., $\mathbf{\Psi}_{\perp} \tilde{\mathbf{\Xi}}_d$). Consequently, it is essential to include a sufficiently large set of bending eigenmodes $\mathbf{\Phi}_d$ to capture the structure nonlinear behaviors, and, moreover, to construct the nonlinear reduced model using the orthogonalized modes $\mathbf{\Psi}_{\perp}$ instead of $\mathbf{\Psi}$, which is used in the original formulation of the strongly coupled model reduction [9].

The nonlinear reduced system (Eq. 22) can be solved using direct time-integration, e.g., using the Newmark-β method. Furthermore, damping can be included by introducing Rayleigh damping for the structural domain, such that $\mathbf{C}_s = \alpha \mathbf{M}_s + \beta \mathbf{K}_s^t$, where $\mathbf{K}_s^t$ is the structure tangent stiffness and α and β are the Rayleigh damping coefficients. Finally, it should also be

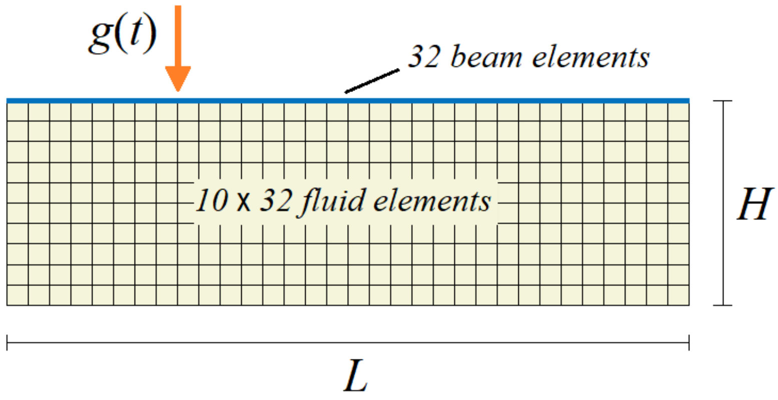


Figure 1: FE model of two-dimensional fluid cavity and flexible beam.

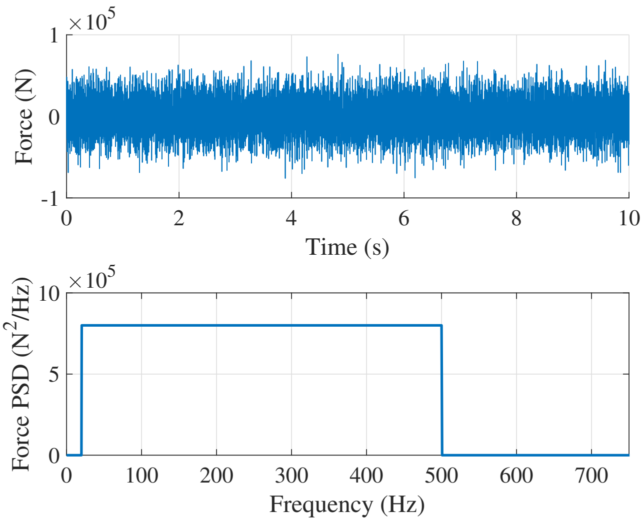


Figure 2: Random excitation corresponding to band-limited noise in the frequency range 20 to 500 Hz, with standard deviation of approximately 20 kN.

noted that the nonlinear part of the reduced internal force vector $\bar{\mathbf{f}}_{\text{nl}}(\mathbf{q})$ can be approximated by considering only a subset of the N_d^s structural modal coordinates, and thereby reduce the number of nonlinear stiffness coefficients. In many applications, geometric nonlinear effects, such as bending-stretching coupling, are typically most pronounced for bending modes with low frequency and large displacement amplitude.

4 NUMERICAL EXAMPLE

In this section, the proposed modeling strategy is evaluated by means of a numerical example of a two-dimensional fluid box with one flexible side, similar to the one introduced by Sandberg et al. [17]. The box has width $L = 1.5$ m and height $H = 0.45$ m, see Figure 1. The density and speed of sound for the fluid are $\rho = 1000$ kg/m^3 and $c = 1500$ m/s, and the Young's modulus and density of the flexible structure are 211 GPa and 2000 kg/m^3, respectively.

The structure was modeled using two-dimensional beam elements, having rectangular crosssection with height 0.01 m and width 1 m (i.e., in the out-of-plane direction of the 2D plane). The beam rotational and translational DOFs were fixed at the endpoints. Geometric nonlinearity, resulting in bending-stretching coupling effects for the beam structure, was considered. However, the displacements were assumed sufficiently small such that the fluid domain can be represented by linear acoustic elements.

The nonlinear structural-acoustic problem was investigated using two FE models; one model was implemented in MATLAB and another was generated using the commercial software Abaqus. In both models, the structural and fluid domain were modeled by 32 beam elements and 320 acoustic elements, see Figure 1. Beam elements, denoted B21 in Abaqus, were used, which are formulated using large deformation theory. By exporting the beam elements' stiffness and mass matrices from Abaqus, these elements were used in the model implemented in MATLAB as well as in Abaqus. However, the acoustic domain was modeled using different acoustic element types in the respective model; namely, in the MATLAB FE model, 4-node elements were used, whereas 8-node elements, denoted AC2D8, were used in the Abaqus model. Furthermore, the acoustic elements implemented in MATLAB were formulated using pressure DOFs, which is advantageous for generating strongly coupled reduced order models, as discussed in [9].

The dynamic response of the geometrically nonlinear structure-fluid system was analyzed by means of direct time-integration, using Newmark implicit time-integration, assuming constant average acceleration within time-increments. Equilibrium in each time increment was enforced by means of Newton-Raphson iterations. A random excitation $\mathrm{g}(t)$, represented by a point load corresponding to band-limited noise in the frequency range of 20 Hz to 500 Hz, was applied at a distance of 0.375 m ($L/4$) from the left end of the beam, as depicted in Figure 1 and further illustrated in Figure 2. Two excitation magnitudes were considered, corresponding to standard deviations of approximately 10 kN and 20 kN, respectively. Further, Rayleigh damping was applied to the structural domain, with Rayleigh coefficients $\alpha = 9.67$ and $\beta = 2.45 \cdot 10^{-5}$ resulting in less than 4% damping between 20 and 500 Hz for the structure.

It should be noted that the mean pressure of a closed cavity modelled using acoustic elements is not constrained. Consequently, the mean pressure may drift in an unphysical manner when solving the transient dynamic response using direct time-integration. In contrast, when analyzing linear systems using, e.g., frequency domain harmonic analysis, this is typically not a problem, as the quasistatic mean pressure does not affect the response at higher frequencies. However, for nonlinear problems, the quasi-static pressure deforms and stiffens the structure, i.e., the principle of superposition does not apply.

To address this issue, the mean pressure can be constrained to a fixed value by imposing constraint conditions in the FE model (e.g., see [11]). However, the constraint equation then involves all the pressure DOFs which can be inconvenient and may lead to computationally expensive analyses. Furthermore, it was found to be problematic to implement such a constraint equation in Abaqus, and, at the same time, ensure that the fluid-structure coupling was enforced correctly. Therefore, in this study, the uniform pressure mode of the cavity was instead constrained in a simplified manner by imposing a zero pressure boundary condition at the cavity centroid. As manifested by the frequency response functions for the linearized system in Figure 3, the resonances within the frequency range of interest (20 to 500 Hz) is well-approximated when zero pressure is enforced at the cavity centroid. The pressure constraint significantly affects the first mode of the fluid-structure system. However, the 2nd to 7th modes, which are shown in Figure 4 were almost identical for the constrained and unconstrained cavity—both with respect to the eigenfrequencies and the mode shapes.

Using the methodology described in Sections 2 and 3, a strongly coupled nonlinear reduced

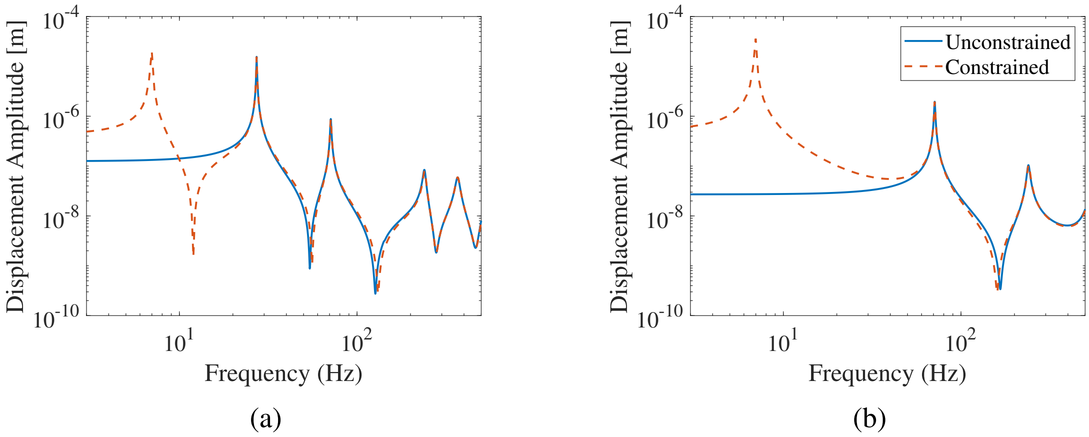


Figure 3: Frequency response functions for unit loading; displacement amplitude at (a) load position, and (b) beam midpoint. The response is shown for linearized model with unconstrained pressures (unconstrained) and with prescribed zero pressure at cavity centroid (constrained).

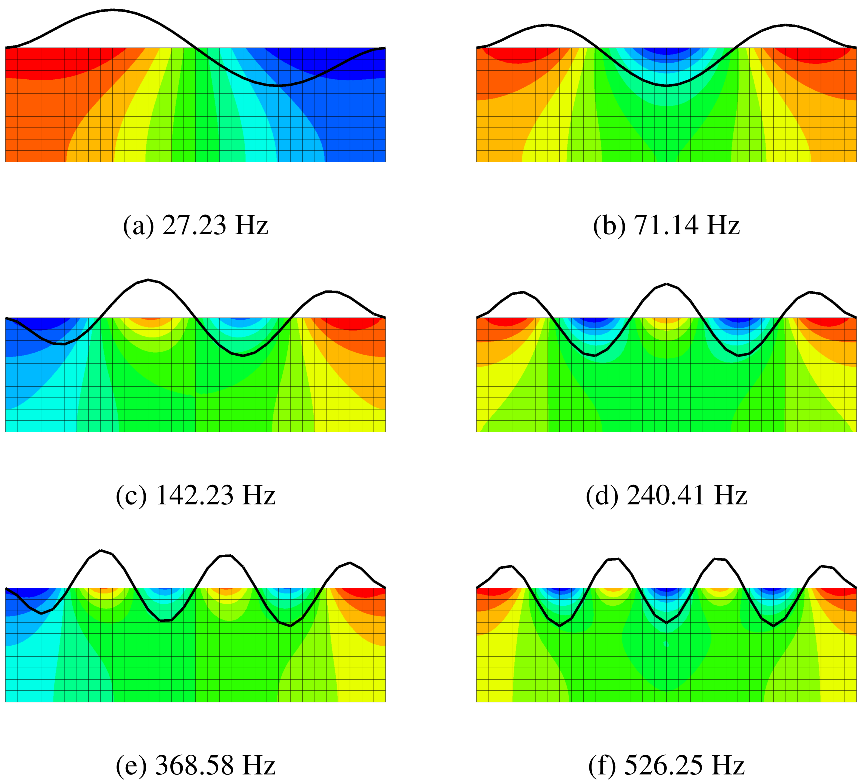


Figure 4: Mode shapes for 2nd to 7th coupled mode as provided by linearized Abaqus model with zero pressure enforced at fluid cavity centroid.

order model were generated based on the FE model implemented in MATLAB. The reduction basis was constructed using 10 modes for the structure and fluid domain, respectively, which was judged to be sufficient based on the convergence for the linearized system, as illustrated in Figure 5. Thus, the reduced model include a total of 20 modal coordinates. Furthermore, recall that the acoustic elements in the MATLAB model uses pressure as the primary variable, which ensures an efficient implementation of the strongly coupled model reduction procedure [9].

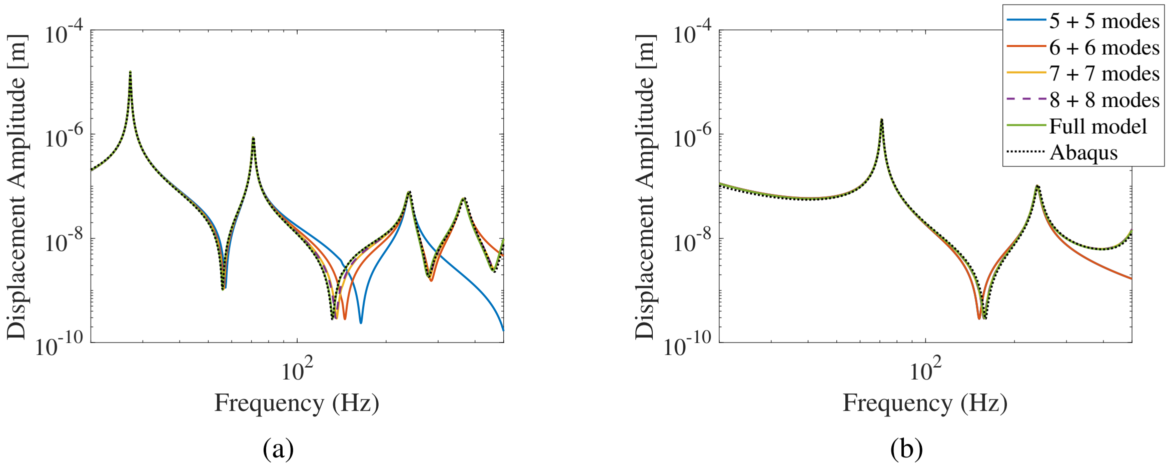


Figure 5: Frequency response function for unit loading; displacement amplitude at (a) beam midpoint, and (b) load point. The response is shown for strongly coupled linear reduced order models generated using $d_s + d_f$ mode shapes, where d_s and d_f represent the number of structural and fluid modes, respectively. The full-order linear response is shown for the Abaqus model as well as the model implemented in MATLAB.

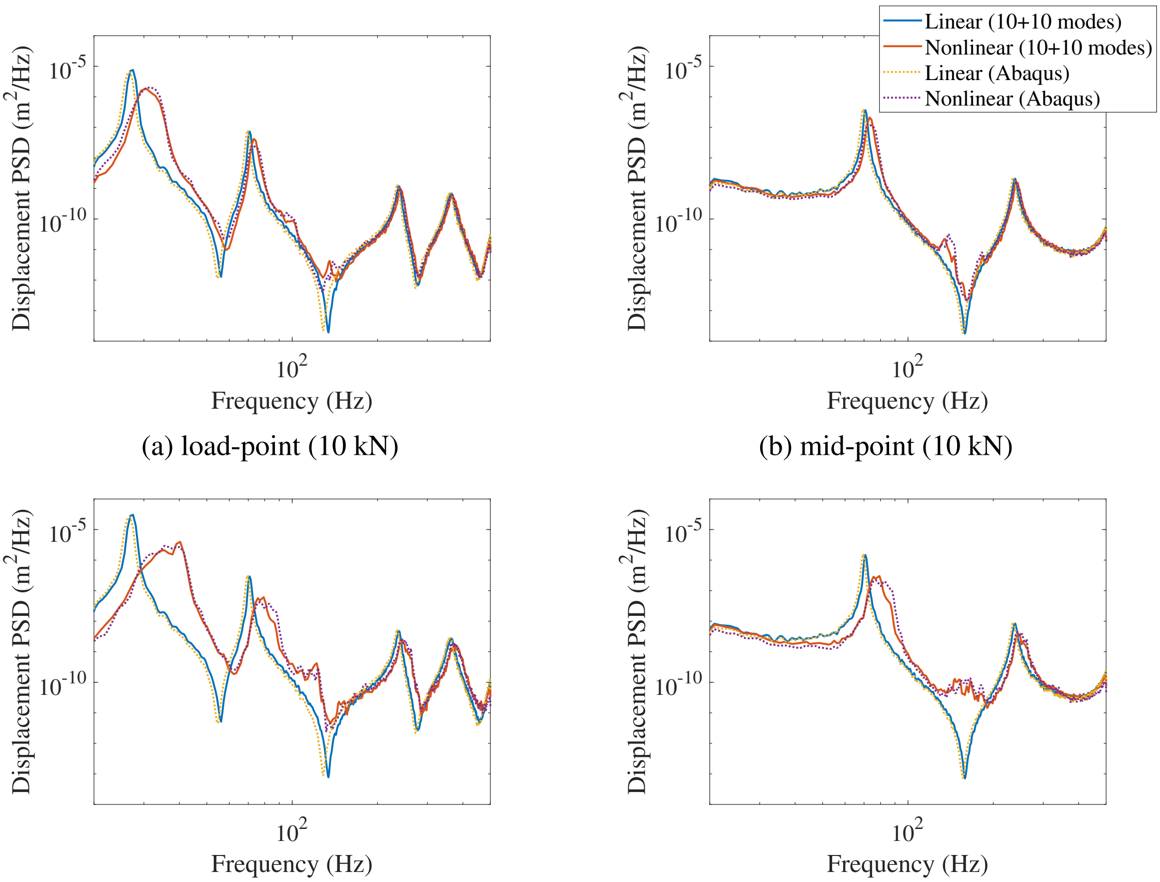
The structure has small thickness compared to the span width, which leads to bending-stretching coupling effects due to geometric nonlinearity. To consider this in the reduced model, the condensed nonlinear stiffness for the first 6 bending eigenmodes of the structure were determined using the STEP procedure. Note that, by using the condensed stiffness coefficients, the quasi-static in-plane motion is implicitly accounted for, as described in Section 2.2.

The response was evaluated using estimated power spectral densities (PSDs) derived from displacement time histories. To obtain sufficient quality of the PSDs, the analysis total time and time increment were set to 10 s and 0.1 ms, respectively, resulting in a total of 100 000 time steps. The PSDs were then determined by dividing the displacement time history into overlapping segments, applying a Hanning window to each segment, and averaging the resulting spectra to reduce variance. Further, the first 0.1 s of the time records, which were considered to be a transient (non-stationary) phase, were removed before the displacement PSDs were estimated.

As shown in Figure 6, the results provided by the nonlinear reduced model is in fairly good agreement with the corresponding results provided by the nonlinear full-order Abaqus model. The discrepancies are believed to be mainly due to the neglected in-plane inertia in the reduced order model. However, some deviations are found both in the linear and nonlinear analysis, which is likely due to the fact that different acoustic elements are used in the respective model, i.e., four-node elements formulated using pressure as the primary variable was used in MATLAB whereas eight-node elements AC2D8 were used in Abaqus. Finally, it should also be noted that, due to the low damping used, the peak values at resonances are influenced by the frequency increment size in the estimated PSDs.

5 CONCLUSIONS

In this paper, a projection-based reduced-order modeling strategy was developed for geometrically nonlinear structures interacting with acoustic fluids. The methodology builds on the strongly coupled model reduction approach [9], extended to account for geometric nonlinearity by expressing the reduced internal forces as third-order polynomials in modal coordinates.



(a) load-point (10 kN)

(b) mid-point (10 kN)

(c) load-point (20 kN)

(d) mid-point (20 kN)

Figure 6: Estimated PSDs for vertical displacement at beam mid-point and load-point, respectively. The response is shown for the strongly coupled, nonlinear reduced order model generated using the first 10 eigenmodes of each domain. Results are shown for random excitation with standard deviation 10 kN and 20 kN, respectively (cf. Figure 2).

The study focuses on flat structures subjected to transverse forcing, for which the nonlinear response can be accurately represented using a reduced basis of bending eigenmodes. In-plane inertia effects are neglected, and the quasi-static in-plane response is implicitly included through static condensation.

Numerical results show good agreement between the reduced-order model and full-order FE simulations. The proposed approach significantly reduces the number of system variables, enabling computationally efficient time-domain analysis of coupled vibro-acoustic systems. Moreover, to further improve computational efficiency, nonlinear effects may be considered for only a subset of the bending modes included in the reduction basis.

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