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A REDUCED ORDER FORMULATION FOR THE ELASTODYNAMIC BOUNDARY ELEMENT METHOD

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Abstract. *The boundary element (BE) method is a powerful tool for solving dynamic soil-structure interaction (SSI) problems in unbounded domains, enabling accurate prediction of environmental vibration and seismic wave propagation. At its core, the BE method relies on fundamental solutions or Green's functions, which reduce the need for extensive boundary discretization. However, the resulting fully populated system matrices pose a significant computational challenge, requiring substantial memory and processing power.*

This study introduces a reduced order BE formulation based on Proper Generalized Decomposition (PGD) to represent Green's functions in a separable, low-rank format. The decomposition treats source/receiver positions, wavenumber, and frequency as independent coordinates, with each mode computed iteratively using a Greedy Tucker Algorithm (GTA) to enhance convergence. By decoupling source and receiver terms, the reduced order formulation provides an alternative to multipole Green's functions. The low-rank approximations of Green's functions allow to evaluate the BE system in a computationally efficient way. The field variables are computed in an iterative setting without fully assembling the populated BE system matrices.

We present the proposed formulation through an example of a surface foundation on an elastic medium. The results are compared with those obtained from the classical BE method. The conclusions highlight a comparative analysis of solution accuracy and computational efficiency, focusing on memory requirements and CPU time.

Keywords: Boundary element method, Green's function, Proper Generalized Decomposition, Greedy Tucker Algorithm, reduced order formulation.

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1 Introduction

The boundary element (BE) method is a well-established numerical approach for modeling static and dynamic boundary value problems. Examples include soil-structure interaction and wave propagation where the BE method inherently incorporates wave radiation to infinity. Moreover, BE methods require discretization of the domain boundary only, thereby leading to a reduction in the spatial problem dimension.

Despite these advantages, classical BE methods are hindered by computational challenges. The resulting system matrices are fully-populated, unsymmetrical and dense, leading to high memory and CPU requirements. These limitations obstruct the application of the BE method to large-scale problems. Several fast BE methods [1] have been developed in the past decades to improve the computational efficiency, including the fast multipole method (FMM) [2, 3] and methods based on hierarchical matrices [4, 5].

The fast-multipole method [2, 3] is based on the reformulation of the fundamental solutions or Green's functions using a multipole expansion in terms of source and receivers by recasting of the position vector. It has proven to be very efficient if analytical expressions of the Green's functions are available, e.g. for a full space. However, when the latter are used for problems including a layered halfspace, a considerable amount of boundary elements are required for the discretization of the layer interfaces and the free surface, limiting the size of the problem that can be treated. Moreover, the separation of Green's functions only converges when the source and receiver are sufficiently separated.

BE methods based on hierarchical matrices [4, 5] are essentially algebraic tools to approximate the BE matrices. Assembly of the $\mathscr{H}$–matrix involves forming a hierarchical cluster tree from the BE mesh, with each cluster recursively partitioned into two children via Principal Component Analysis (PCA) until they contain no more than a prescribed number of elements or nodes. Cluster pairs are identified as either admissible (far field interactions) or inadmissible (near field interactions). Matrix blocks for admissible cluster pairs are approximated by low-rank approximations, while inadmissible cluster pairs are computed exactly due to the singular behaviour of the Green's functions. A prominent bottleneck persists in the computation of Green's functions for multiple source-receiver configurations.

This study explores the potential of model order reduction techniques to enhance the computational efficiency of the classical BE method. As an alternative to FMM and $\mathscr{H}$-BE , we employ Proper Generalized Decomposition (PGD) [6, 7] to approximate Green's functions in a decomposed manner. PGD is an a priori model reduction technique that constructs a reduced representation by assuming a separable form of the multi-dimensional field, with components computed iteratively in a greedy manner utilizing a Greedy Tucker Algorithm (GTA) [8, 9]. The problem parameters are included as coordinates in the proposed algorithm providing great flexibility in the solutions offered. Specifically, separation of source and receiver terms allows efficient treatment of near field interactions, serving as an alternative to multipole Green's functions. The resulting low-rank Green's functions enable a reformulation of the boundary integral equation (BIE) that enhances both the computational efficiency and flexibility of the BE method.

The outline of the paper is as follows. The detailed formulation for a reduced order BE formulation is explained in section 2. The computation of the Green's function in the required decomposed manner is also briefly discussed in the same section. The numerical implementation of the proposed methodology is demonstrated in section 3 where we analyze the case of a surface foundation on elastic medium. The computational performance of the proposed algorithm is assessed in terms of accuracy, memory and CPU requirements followed by a dis-

cussion of the reduced order BE formulations and their potential for application to larger-scale problems. Finally, section 4 summarizes the main conclusions.

2 Methodology

2.1 Problem outline

This section presents the formulation of the reduced order BE method. Analyzing a relatively simple problem, the formulation is applied to a surface foundation of width B on an elastic medium subject to horizontal tractions $\hat{t}_y^{\mathbf{n}}(x,\omega)$ as demonstrated in figure 1. We restrict to 2D out-of-plane (SH) wave propagation, characterized by a scalar wave field $\hat{u}_y(x,z,\omega)$. A hat above a variable denotes its representation in the frequency domain domain.

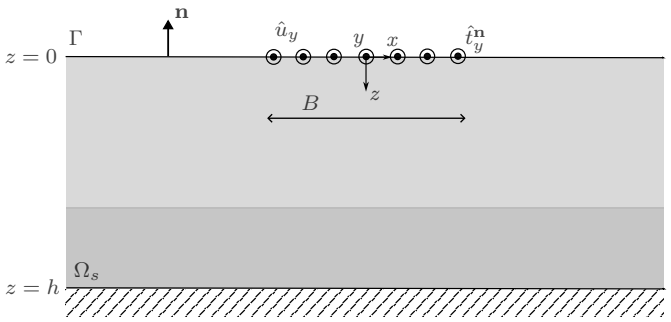

Figure 1: Surface foundation on an elastic medium.

The dynamic reciprocity theorem lays the basis for the classical BE method, which by means of an integral equation links unknown field variables, i.e. the displacement $\hat{u}_y(x_\mathrm{s},\omega)$ at the interface $z = 0$ of the soil and foundation to a known elastodynamic state, i.e. the Green's function $\hat{u}_{yy}^{\mathrm{G}}(x_\mathrm{s},x,\omega)$ in the domain:

$$\hat{u}_y(x_\mathrm{s},\omega) = \int_{-B/2}^{B/2} \hat{u}_{yy}^{\mathrm{G}}(x_\mathrm{s},x,\omega)\ \hat{t}_y^{\mathbf{n}}(x,\omega)\,\mathrm{d}x \tag{1}$$

To be able to employ the BIE, it needs to be regularized and discretized. Spatial discretization of the boundary using appropriate boundary elements allows the traction to be written in terms of interpolation functions $M_k(x)$ as:

$$\hat{t}_y^{\mathbf{n}}(x,\omega) = \sum_k M_k(x)\,\hat{t}_k^{\mathbf{n}}(\omega) \tag{2}$$

where $\hat{t}_k^{\mathbf{n}}(\omega)$ denotes the nodal tractions. Introducing the discretization into equation (1) yields the following discretized BIE:

$$\hat{u}_y(x_\mathrm{s},\omega) = \int_{-B/2}^{B/2} \hat{u}_{yy}^{\mathrm{G}}(x_\mathrm{s},x,\omega)\ \sum_k M_k(x)\hat{t}_k^{\mathbf{n}}(\omega)\,\mathrm{d}x \tag{3}$$

In order to obtain an algebraic system of equations for the solution of the unknown nodal displacements, a point collocation is applied. The interpolation nodes $x_\mathrm{s} = x_m$ are selected

as collocation points, resulting in as many equations as unknowns. Evaluation of x_s at the collocation points yields the following:

$$\hat{\mathbf{U}}\underline{\hat{\mathbf{t}}} = \underline{\hat{\mathbf{u}}} \tag{4}$$

where the vectors $\underline{\hat{\mathbf{t}}}$ and $\underline{\hat{\mathbf{u}}}$ collect the nodal displacements $\hat{u}_k$ and tractions $\hat{t}_k^{\mathbf{n}}$ of the BE mesh. The resulting BE system matrix $\hat{\mathbf{U}}$ is dense, fully-populated and unsymmetric, demanding high memory and CPU requirements and therefore limiting its applicability to moderate sized problems. We propose a reduced order BE formulation based on the decomposed Green's functions. First, low rank approximations are obtained for Green's functions and then the classical BIE is reformulated using these Green's functions.

2.2 Low-rank approximation of Green's functions

The Green's functions or the fundamental solutions of the soil domain are approximated using a PGD algorithm [6, 7] which incorporates the source/receiver positions and frequency as coordinates. These separated components greedily build up their respective dimension space by addition of a new rank-one approximation in each iteration through adaptive rank selection according to the Greedy Tucker Algorithm [8]. To render these approximations useful in BE methods, source and receiver are also explicitly separated to manipulate the classical BIE to our favour, which is explained in detail in subsection 2.3. The Green's displacement tensor is generally approximated by the following Tucker representation:

$$\hat{u}^{\mathrm{G}}_{yy}(x_\mathrm{s}, z_\mathrm{s}, x, z, \omega) \approx \sum_{r_{x_\mathrm{s}}=1}^{R_{x_\mathrm{s}}} \sum_{r_{z_\mathrm{s}}=1}^{R_{z_\mathrm{s}}} \sum_{r_x=1}^{R_x} \sum_{r_z=1}^{R_z} \sum_{r_\omega=1}^{R_\omega} g_{r_{x_\mathrm{s}} r_{z_\mathrm{s}} r_x r_z r_\omega} u^{\mathrm{G}(x_\mathrm{s})}_{r_{x_\mathrm{s}}}(x_\mathrm{s})\, u^{\mathrm{G}(z_\mathrm{s})}_{r_{z_\mathrm{s}}}(z_\mathrm{s})\, u^{\mathrm{G}(x)}_{r_x}(x)\, u^{\mathrm{G}(z)}_{r_z}(z)\, \hat{u}^{\mathrm{G}(\omega)}_{r_\omega}(\omega) \tag{5}$$

where the problem coordinates include sources $(x_\mathrm{s}, z_\mathrm{s})$, receivers (x, z) and the frequency ω. R_* is the rank of approximation, $g_{r_{x_\mathrm{s}} r_{z_\mathrm{s}} r_x r_z r_\omega}$ is the core tensor and $u^{\mathrm{G}(x_\mathrm{s})}_{r_{x_\mathrm{s}}}(x_\mathrm{s})$, $u^{\mathrm{G}(z_\mathrm{s})}_{r_{z_\mathrm{s}}}(z_\mathrm{s})$, $u^{\mathrm{G}(x)}_{r_x}(x)$, $u^{\mathrm{G}(z)}_{r_z}(z)$, and $\hat{u}^{\mathrm{G}(\omega)}_{r_\omega}(\omega)$ are the factor matrices defined on their respective one-dimensional domains, $x_\mathrm{s} \in \mathcal{I}_{x_\mathrm{s}}$, $z_\mathrm{s} \in \mathcal{I}_{z_\mathrm{s}}$, $x \in \mathcal{I}_x$, $z \in \mathcal{I}_z$, and $\omega \in \mathcal{I}_\omega$.

For the analysis of a surface foundation on a soil stratum at a single frequency, the low-rank approximations for the Green's functions take the following form as $z_\mathrm{s} = 0$:

$$\hat{u}^{\mathrm{G}}_{yy}(x_\mathrm{s}, x, z) \approx \sum_{r_{x_\mathrm{s}}=1}^{R_{x_\mathrm{s}}} \sum_{r_x=1}^{R_x} \sum_{r_z=1}^{R_z} g_{r_{x_\mathrm{s}} r_x r_z} u^{\mathrm{G}(x_\mathrm{s})}_{r_{x_\mathrm{s}}}(x_\mathrm{s})\, u^{\mathrm{G}(x)}_{r_x}(x)\, u^{\mathrm{G}(z)}_{r_z}(z) \tag{6}$$

It is noted that both the rank and factor matrices along each dimension are unknown a priori and are solved in a greedy manner. Storing the Green's displacements in a Tucker format is computationally efficient in comparison to the conventional Green's functions. While the Tucker format is rich in information, the size of the core tensor increases with added dimensions, making it computationally expensive to store. However, for moderately sized problems, its performance proves to be optimal [8].

2.3 Reformulation of the boundary integral equation

Our proposed reduced-order BE formulation is built upon the introduction of low-rank approximations of Green's displacements into the classical BE framework. This enables a substantial computational simplification by recasting the BIE in terms of products of one-dimensional

vectors, rather than expensive full matrix-vector operations. In a classical BE formulation, displacement evaluation involves computation of the full Green's kernel for each source–receiver pair and frequency. By leveraging low-rank tensor decomposition, we exploit the separable structure of Green's displacements to reduce complexity and computational cost. The low-rank approximations form equation (6) can be further reduced for evaluation of displacements at the interface $z = 0$ of the foundation and the soil.

$$\hat{u}^{\mathrm{G}}_{yy}(x_{\mathrm{s}}, x, z = 0) \approx \sum_{r_{x_{\mathrm{s}}}=1}^{R_{x_{\mathrm{s}}}} \sum_{r_x=1}^{R_x} g'_{r_{x_{\mathrm{s}}} r_x} u^{\mathrm{G}(x_{\mathrm{s}})}_{r_{x_{\mathrm{s}}}}(x_{\mathrm{s}}) u^{\mathrm{G}(x)}_{r_x}(x) \tag{7}$$

where

$$g'_{r_{x_{\mathrm{s}}} r_x} = \sum_{r_z=1}^{R_z} g_{r_{x_{\mathrm{s}}} r_x r_z}\, u^{\mathrm{G}(z)}_{r_z}(z = 0) \tag{8}$$

is the modified/reduced core tensor obtained by collapsing the three-way core tensor along the mode representing the receivers in the z-direction, resulting in a compact representation that retains the physical fidelty of the Green's functions across the soil-foundation interface. Substituting this low-rank approximation (7) into the classical BIE (3) yields the reformulated expression:

$$\hat{u}_y(x_{\mathrm{s}}) = \int_{-B/2}^{B/2} \Big(\sum_{r_{x_{\mathrm{s}}}=1}^{R_{x_{\mathrm{s}}}} \sum_{r_x=1}^{R_x} g'_{r_{x_{\mathrm{s}}} r_x} u^{\mathrm{G}(x_{\mathrm{s}})}_{r_{x_{\mathrm{s}}}}(x_{\mathrm{s}}) u^{\mathrm{G}(x)}_{r_x}(x) \Big) \sum_k M_k(x) \hat{t}^{\mathbf{n}}_k(\omega)\, \mathrm{d}x \tag{9}$$

The low-rank approximation of Green's displacements enables a separation of source and receiver dependent terms within the BIE:

$$\hat{u}_y(x_{\mathrm{s}}) = \sum_{r_{x_{\mathrm{s}}}=1}^{R_{x_{\mathrm{s}}}} \sum_{r_x=1}^{R_x} g'_{r_{x_{\mathrm{s}}} r_x} u^{\mathrm{G}(x_{\mathrm{s}})}_{r_{x_{\mathrm{s}}}}(x_{\mathrm{s}}) \int_{-B/2}^{B/2} \sum_k u^{\mathrm{G}(x)}_{r_x}(x) M_k(x) \hat{t}^{\mathbf{n}}_k(\omega)\, \mathrm{d}x \tag{10}$$

The structure of this BIE reduces computational complexity and also facilitates reuse of precomputed integrals over the receiver domain. The boundary integral is evaluated using Gauss quadrature, where the spatial source basis $u^{\mathrm{G}(x)}_{r_x}(x)$ is interpolated at Gauss points and weighted against the imposed tractions $\hat{t}^{\mathbf{n}}_k(\omega)$, resulting in the projection of the factor matrix onto the BE mesh as:

$$\hat{u}_y(x_{\mathrm{s}}) = \sum_{r_{x_{\mathrm{s}}}=1}^{R_{x_{\mathrm{s}}}} \sum_{r_x=1}^{R_x} g'_{r_{x_{\mathrm{s}}} r_x} u^{\mathrm{G}(x_{\mathrm{s}})}_{r_{x_{\mathrm{s}}}}(x_{\mathrm{s}})\, \mathrm{U}^{(x)}_{r_x} \tag{11}$$

where $\mathrm{U}^{(x)}_{r_x}$ encapsulates the outcome of the integral in equation (10). Finally, the displacements at the foundation–soil interface are obtained as:

$$\hat{u}_y(x_{\mathrm{s}}) = \sum_{r_{x_{\mathrm{s}}}=1}^{R_{x_{\mathrm{s}}}} g''_{r_{x_{\mathrm{s}}}} u^{\mathrm{G}(x_{\mathrm{s}})}_{r_{x_{\mathrm{s}}}}(x_{\mathrm{s}}) \tag{12}$$

where

$$g''_{r_{x_{\mathrm{s}}}} = \sum_{r_x=1}^{R_x} g'_{r_{x_{\mathrm{s}}} r_x}\, \mathrm{U}^{(x)}_{r_x} \tag{13}$$

is the reduced core tensor obtained by collapsing the previously modified core tensor $g'_{r_{x_\mathrm{s}} r_x}$ along the receivers in the x-direction. Equation (12) provides an elegant way to directly compute the displacements at the soil-foundation interface without actually assembling the BE system matrix. Moreover, the proposed algorithm provides us the flexibility to incorporate the source/receiver positions, frequency and soil/foundation properties to solve diverse problems. The Green's functions can be computed once, stored in a reduced form and called into the BE analysis without needing their full-scale computation over an elaborate source-receiver grid.

3 Numerical example

In this section, we present numerical results obtained using the proposed reduced order BE formulation. The problem involves a surface foundation of width $B = 10\,\mathrm{m}$ resting on an elastic medium. The soil layer has a thickness $h = 50\,\mathrm{m}$, shear wave velocity $C_s = 100\,\mathrm{m/s}$ and density $\rho = 1800\,\mathrm{kg/m^3}$. A damping ratio $\xi = 0.02$ is considered for shear waves. The analysis is performed at a frequency of $10\,\mathrm{Hz}$. The low-rank approximations for the Green's displacements are initially computed in the wavenumber-frequency domain using a GTA algorithm for tensor linear systems, expressed by the following Tucker representation:

$$\tilde{u}^{\mathrm{G}}_{yy}(x_\mathrm{s}, p_x, z) \approx \sum_{r_{x_\mathrm{s}}=1}^{R_{x_\mathrm{s}}} \sum_{r_{p_x}=1}^{R_{p_x}} \sum_{r_z=1}^{R_z} g_{r_{x_\mathrm{s}} r_{p_x} r_z} u^{\mathrm{G}(x_\mathrm{s})}_{r_{x_\mathrm{s}}}(x_\mathrm{s})\, \tilde{u}^{\mathrm{G}(p_x)}_{r_{p_x}}(p_x)\, u^{\mathrm{G}(z)}_{r_z}(z) \tag{14}$$

A tilde above a variable denotes its representation in the wavenumber-frequency domain domain. The approximation procedure involves a structured sampling over source positions x_s, slowness p_x, and receiver locations in the x- and z-directions. To improve efficiency, the wavenumber domain is sampled in terms of slowness, defined as $p_x = k_x/\omega$, which is independent of the excitation frequency ω. Scaling the slowness p_x with the shear wave velocity C_s yields the dimensionless wavenumber $\bar{k}_x = k_x C_s/\omega = p_x C_s$. To efficiently approximate the Green's function in the wavenumber–frequency domain, a reduced subspace is constructed greedily by iteratively expanding along each tensor dimension. The approximation process is controlled by a stopping criterion based on a prescribed tolerance of 1×10^{-6} and a maximum number of 45 modes, which collectively influence the convergence behavior and accuracy of the solution. Figure 2 compares the Green's displacement obtained from a reference full-order model (FOM), using the direct stiffness method, and a reduced-order model (ROM) based on thin-layer matrices [10, 11] computed with varying numbers of modes. As shown, the accuracy of the ROM improves consistently with an increasing number of modes.

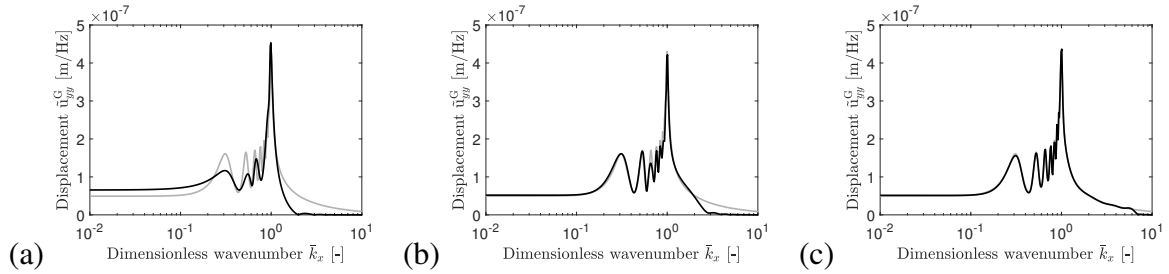

Figure 2: Displacement $\tilde{u}^{\mathrm{G}}_{yy}(x_\mathrm{s} = 0, z_\mathrm{s} = 0, p_x, z = 0)$ in the wavenumber-frequency domain using FOM (gray solid line) and ROM (black solid line) with (a) 10 modes, (b) 25 modes and (c) 45 modes.

Green's displacements in the spatial–frequency domain (6) are obtained by applying an inverse logarithmic Fourier transform [11] to equation (14). The low-rank structure enables this transformation to be applied directly to the relevant factor matrix, i.e. $\tilde{u}_{r_{p_x}}^{\mathrm{G}(p_x)}(p_x)$, rather than to the full Green's tensor. In contrast to the conventional approach, which requires inverse transforms over a full grid of size $n(x_{\mathrm{s}}) \times n(p_x) \times n(z)$, where $n(x_{\mathrm{s}})$ is the number of source positions, $n(p_x)$ the number of slowness samples and $n(z)$ the number of vertical discretization points, the proposed method performs the transformation only over R_{p_x}, rank-one components needed to approximate the factor matrix $\tilde{u}_{r_{p_x}}^{\mathrm{G}(p_x)}(p_x)$ within the prescribed accuracy. Figure 3 illustrates the transformed Green's displacements for both the full-order model (FOM) and the reduced-order model (ROM).

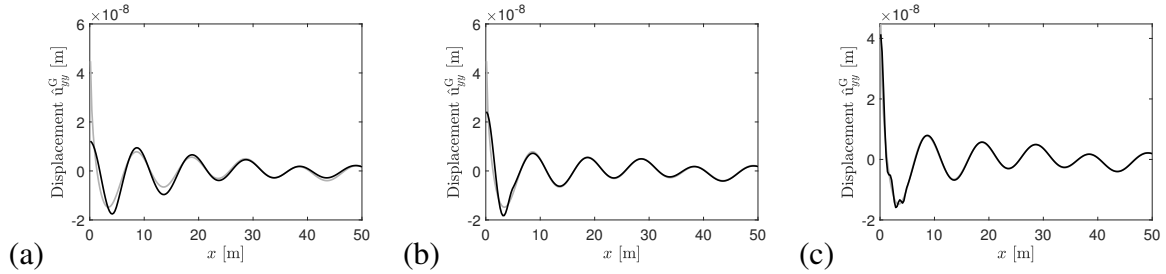

Figure 3: Displacement $\hat{u}_{yy}^{\mathrm{G}}(x_{\mathrm{s}} = 0, z_{\mathrm{s}} = 0, x, z = 0)$ in the spatial-frequency domain using FOM (gray solid line) and ROM (black solid line) with (a) 10 modes, (b) 25 modes and (c) 45 modes.

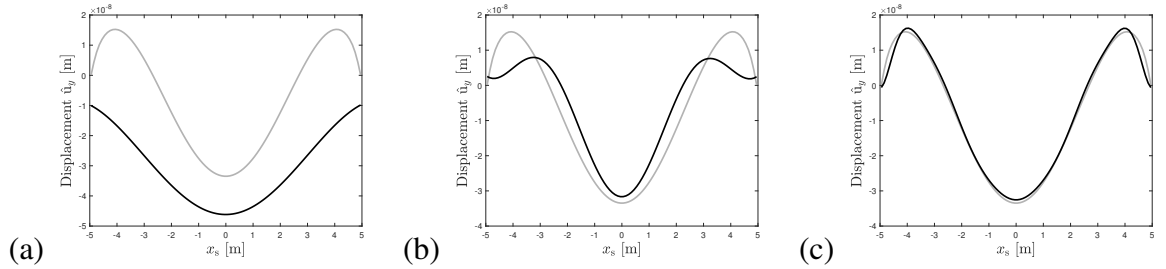

Figure 4: Displacement $\hat{u}_y(x_{\mathrm{s}})$ at the soil-foundation interface using FOM (gray solid line) and ROM (black solid line) with (a) 10 modes, (b) 25 modes and (c) 45 modes.

Following the formulation outlined in subsection 2.3, the low-rank representation of Green's functions is incorporated into the BIE to compute displacements at the soil–foundation interface. This approach avoids the need to assemble the full BE system matrix of size $N_{\mathrm{DOF}} \times N_{\mathrm{DOF}}$. Figure 4 compares the interface displacements obtained using the reduced order formulation and the classical BE full order solution [12]. The displacement shows that accuracy and convergence improve with an increasing number of modes. However, memory limitations constrain the number of modes and, consequently, the maximum resolution. With 45 maximum modes, the displacements are well-approximated, achieving a favourable balance between accuracy and memory requirement. The memory savings are substantial: the low-rank Green's function requires only 1.25 MB of memory, compared to 9.6 MB for the full-order Green's displacements. The CPU time for the classical BE solution is 0.004 seconds, while the reduced-order BE formulation takes 0.391 seconds. For a small-scale problem, the difference in runtime is not expected to be significant. A key limitation of the greedy Tucker algorithm is that the size of core tensor

grows with dimensionality, which can become a bottleneck. For larger-scale problems, more efficient decomposition techniques such as the hierarchical Tucker format may offer improved performance.

4 Conclusion

Model order reduction of the BE method offers a promising pathway for extending its applicability to large-scale, computationally demanding problems. The results presented in this paper demonstrate that the proposed reduced order formulation seeks to achieve an effective balance between accuracy and computational efficiency. Future extension of this work to incorporate more efficient tensor formats is expected to improve convergence and accuracy in high-dimensional settings.

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