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FREE VIBRATIONS OF FRAME STRUCTURES ADOPTING FUNCTIONALLY GRADED MATERIALS FOR DAMAGE MODELLING

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Abstract

The effects of the presence of damage in the free vibrations of structures have been widely investigated in the scientific literature adopting several strategies to model the damage. Most of the studies consider a concentrated reduction of the stiffness of the structural member due to damage still maintaining the hypothesis of homogeneous material. Anyway, the mechanical characteristics of the damaged portion of the beam cannot be considered homogeneous for example when an external corrosion of the material occurs. In the present study the change of mechanical characteristics in the damaged portion of a structural element has been originally modelled by means of functionally graded materials which allow to introduce a smooth change in the material properties.

In particular, the presence of damage is accounted for by means of the introduction of a beam element modelled as functionally graded, in which the material properties vary symmetrically across the height of the section with respect to the centroid. To simulate material degradation due to damage, it is assumed that the inner material is stiffer than the outer material. Since the damaged elements have a shorter length compared to the overall structural members, the effect of shear deformation is accounted for using the Timoshenko beam model.

The proposed damage modelling approach is applied to framed structures in presence of concentrated damage following the exact approach of the dynamic stiffness method in conjunction with the Wittrick and Williams algorithm. Numerical simulations have been carried out, varying the properties of the damaged element, and comparisons have been made with other commonly used damage models, such as the hinge with rotational spring and the reduction of the cross-sectional dimensions.

Keywords: Damage, functionally graded materials, vibrations, dynamic stiffness.

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1 INTRODUCTION

Structures subjected to repeated stress and exposure to atmospheric conditions may sustain damage of varying severity, which decreases their load-bearing capacity and can eventually lead to structural failure. A crucial aspect of all the analyses related to damaged structures is the development of an accurate and reliable numerical model for the entire structure, with a particular focus on the areas affected by damage. Numerous studies have been conducted to characterize and quantify local defects, as reported in the literature. All the models involve a reduction of the stiffness of the structural element by using one-dimensional continuum theories [1] or fracture mechanics methods [2, 3]. The damage can be either localized or diffused on a certain portion of the length of the structural element and its intensity can be related to the reduction of the reactive area of the cross section, for example due to the presence of a notch [4]. In the present study the damage is originally modelled by means of the introduction of a beam element made of functionally graded material (FGM). These materials exhibit unique characteristics due to their graded composition, where material properties (such as stiffness, strength, and thermal conductivity) vary gradually across the beam's length or thickness. Therefore, in FGMs, the material properties change smoothly from one surface to another, often based on a predefined gradient profile. This gradual variation is typically designed to optimize performance under specific loading or thermal conditions and helps to reduce stress concentration points, improving the beam's fatigue resistance compared to homogeneous materials, especially under cyclic loading conditions.

In the last years FGM have found several applications in aerospace, automotive, or structural components exposed to high thermal or mechanical stresses. Due to the possibility, by means of the use of FGM, of defining a continuous variation of mechanical properties which could be caused for example by the degradation or corrosion of the material due to damage, the present study proposes an original application of FGM to damage modelling.

In particular, it is assumed that in a certain portion of the structure the material properties vary through the height of the section with a symmetric law with respect to the centroid [5]. To simulate the degradation of the material due to damage, the inner material is assumed to be stiffer than the outer one. Several ratios between the Young moduli and the densities of the outer and inner materials can be considered in order to simulate the intensity of damage; damage extension can be taken into account varying the length of the FG element.

The proposed damage model has been adopted for the evaluation of natural frequencies and modes of vibration of damaged frames. Since the damaged elements, modelled by means of FGM, have shorter length with respect to the entire structural members, the effect of shear deformation has been taken into account by means of the Timoshenko beam model which allows to a stiffness reduction along the beam section also in terms of shear.

The eigen-properties of the damaged frames are evaluated by means of the Dynamic Stiffness Method (DSM). This method employs precise member theory, utilizing frequency-dependent shape functions that are derived from solving the governing differential equations of motion for structural elements in free vibration. As a result, the DSM provides exact solutions for all natural frequencies and mode shapes, eliminating the need for approximations involved in FEM methods.

Once the dynamic stiffness matrix of the damaged frames is assembled, an arbitrary number of exact frequencies of vibration are obtained applying the Wittrick-Williams algorithm [6].

Several numerical applications have been developed varying the characteristics of the damaged element and comparisons with other frequently adopted damage models have been performed. In particular reference has been made to the models which assume that the stiffness reduction produced by damage can be represented by a decrease in the cross section dimensions for a certain extent related to damage intensity [1, 7, 8]. For concentrated damages a frequently adopted model consists in simply inserting a hinge with rotational spring whose stiffness is related to damage intensity [9, 10, 11, 12]. This simplified model has been adopted for damages of small extents in order to compare the results obtained by means of the FG beam element.

The proposed damaged element can represent a first step toward more accurate damage modelling strategies as it lends itself to possible extensions with fiber models of the beam in which the non-linear behavior of the material can also be taken into account. Its application could be therefore usefully adopted in accurate damage identification strategies coherently to the frame modelling approaches.

2 THE FUNCTIONALLY GRADED BEAM ELEMENT

The beam element considered in this study to model a damaged portion of a beam has length h_d, rectangular section of height h and width b with y as the beam axis (Figure 1). It is characterized by Young's modulus E and density ρ varying through the height of the section with a symmetric law with respect to the centroid. Without loss of generality a parabolic variation is considered in the following:

$$E(z) = E_2 + (E_1 - E_2)\left(\frac{z}{h/2}\right)^2 \qquad\qquad \rho = \rho_2 + (\rho_1 - \rho_2)\left(\frac{z}{h/2}\right)^2 \tag{1}$$

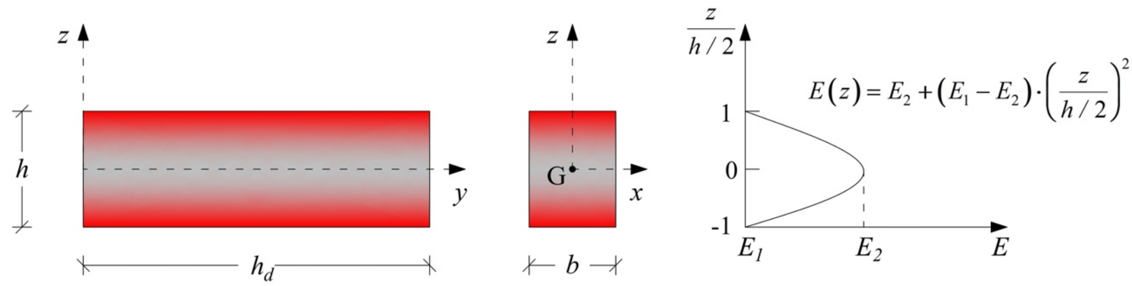


Figure 1: The functionally graded beam element.

where E_2 and ρ_2 are the properties of the beam at the centerline of the section, E_1 and ρ_1 are the properties at the top and bottom surfaces of the beam element, and ζ=$z/(h/2)$ is the dimensionless abscissa along the vertical axis represented in Figure 1. Furthermore, the material properties are assumed to be constant along the horizontal x and y axes.

To take into account the variation of the mechanical characteristics along the z axis, as already proposed in [13, 14] the following homogenization coefficients for the material properties have been considered:

$$A_i = \int_A z^i E(z)\, dA \qquad B_i = \int_A z^i \rho(z)\, dA \qquad i = 0,1,2 \tag{2}$$

The differential equations of dynamic equilibrium of FGM beams which take into account the material coefficients (2) are reported and solved in [5] and the free vibrations of symmetric functionally graded single beams have been studied in closed form.

In the present study the functionally graded beam element has been considered as a short Timoshenko beam while the undamaged structural members have been modelled as homogeneous Euler-Bernoulli beams.

3 THE DYNAMIC STIFFNESS MATRICES

For each beam element, both damaged or undamaged, the nodal displacements V, W, Φ and the nodal forces F, S, M are defined according to the convention reported in Figure 2.

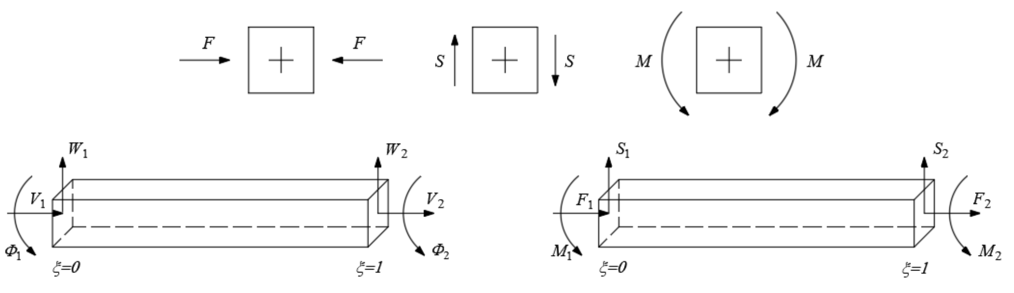


Figure 2: Displacement and nodal forces in each beam element.

The introduction of a damaged beam element in a framed structure requires the consideration of supplementary nodes at the ends of the elements and therefore clearly produces an increase of the degrees of freedom i.e of the order of the stiffness matrix.

For example, for a simple frame in which axial displacements are neglected, the three degrees of freedom of the undamaged structure (Figure 3a) increase by two for each insertion of a damaged element of length h_d placed at the bottom of the columns (Figure 3b_c)

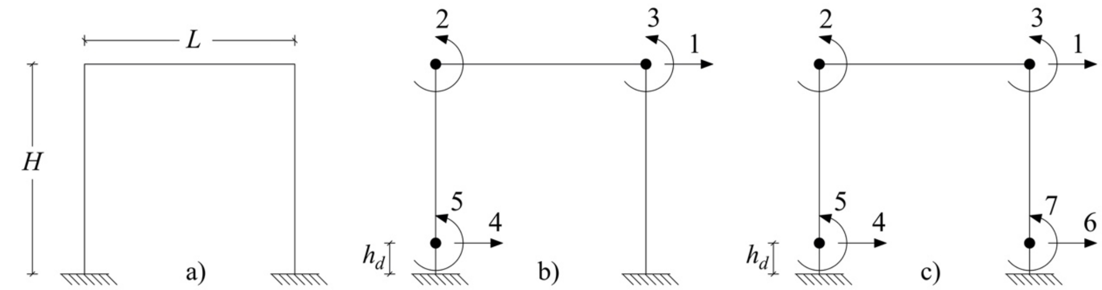


Figure 3: Degrees of freedom in an undamaged (a) and damaged frame with one damaged element at the bottom of the left column (b) and at the bottom of both columns (c).

For each frame the dynamic stiffness matrix $\mathbf{K}(\omega)$ relates the vector of the global nodal forces $\mathbf{P}$ to the corresponding displacement vector $\boldsymbol{\delta}$ as follows:

$$\mathbf{P} = \mathbf{K}(\omega)\boldsymbol{\delta} \tag{3}$$

The explicit expressions of the elementary matrices for Timoshenko and Euler-Bernoulli beams are reported in [5]. The exact frequencies of vibration of framed structures are obtained applying the Wittrick-Williams algorithm [6] to the assembled dynamic stiffness matrix of the structure. This algorithm allows the evaluation of the number J of vibration frequencies which are smaller than a trial value ω^* and, therefore, by means of an iterative procedure, to converge to any required accuracy. Consequently, this procedure allows to obtain exact results for all natural frequencies and mode shapes without the approximations involved in the finite element method. The free vibrations of damaged multistorey buildings can therefore be studied introducing any number of FG beam elements in frames with arbitrary size (Figure 4) and evaluating the correspondent stiffness matrices by means of assembling algorithms like those reported in [15].

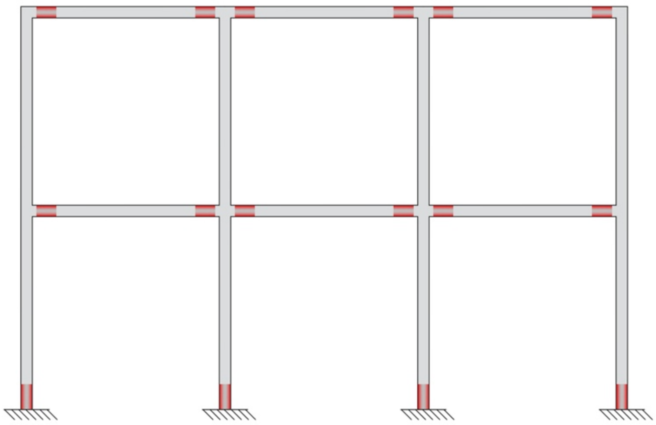


Figure 4: Example of multi-storey frame with different damages modelled with FGM elements.

4 NUMERICAL APPLICATIONS

The performed numerical applications concern the evaluation of the frequencies and modes of vibration of framed structures in presence of one or more damaged portions of beams introducing a localized stiffness reduction which affects the frame eigen-properties.

Namely, the here reported applications regard a sample frame, drawn in Figure 3, that is conceived to represent a simple prototype structure that could be the subject of future experimental investigations. The height H of the columns and the length L of the beam are assumed to be equal to 100 cm and a rectangular cross section of size 4x2 cm^2 is considered for the structural members.

This frame has been chosen in order to validate the obtained results by means of a comparison with those provided by Banerjee for homogeneous material [13].

The reference material is characterized by Young modulus E_2=200 GPa and mass density ρ_2=7500 kg/m^3. Aiming at modelling a reduction in the mechanical properties of the material due to damage, the reported values of E_2 and ρ_2 are considered representative of the inner material while the external ones in the FGM model are assumed by means of decreasing ratios with respect to E_2 and ρ_2. The frequencies of vibration evaluated considering the proposed FGM damaged element are compared to those obtained by means of classical damage model approaches reported in the literature. Reference will be made to the models that consider the effect of damage by introducing a stiffness reduction by means of the insertion of either a homogeneous element of smaller cross section or an equivalent hinge endowed with an elastic spring.

4.1 Frame with one damage

In the frame with one damage drawn (Figure 3b) the damaged zone is assumed to be located at the bottom of the left column. The FGM damaged element and the homogeneous one with reduced cross section are assumed to have variable length h_d. For small values of h_d the results have also been compared to those obtained modelling damage with a hinge and a rotational spring since this model has been commonly adopted in the scientific literature for concentrated damages. The stiffness of the elastic spring has been related to the length and the depth of the damaged portion of the beam as suggested in [11].

Table 1 shows the first three frequencies of vibration of the frame when the damage is modelled by means of a FGM beam element for decreasing ratios between the Young moduli of the outer and inner material E_1/E_2 and increasing values of the length h_d of the damaged element. It is worth to point out that, for the assumed numerical values, the flexural stiffness

EI of the cross section between two successive ratios E_1/E_2 has a constant reduction equal to 12%.

Since the variation between the mass densities of the outer and inner materials ρ_1/ρ_2 has negligible effect on the frequencies of vibration, it has not been considered in this application.

Table 2 shows the first three frequencies of vibration of the frame when the damage is modelled by means of a beam element of length h_d and rectangular cross section of width b and height h_r reduced with respect to h. In order to make a comparison with the results reported in Table 1, the same reduction of the flexural stiffness has been assumed in defining the values of h_r.

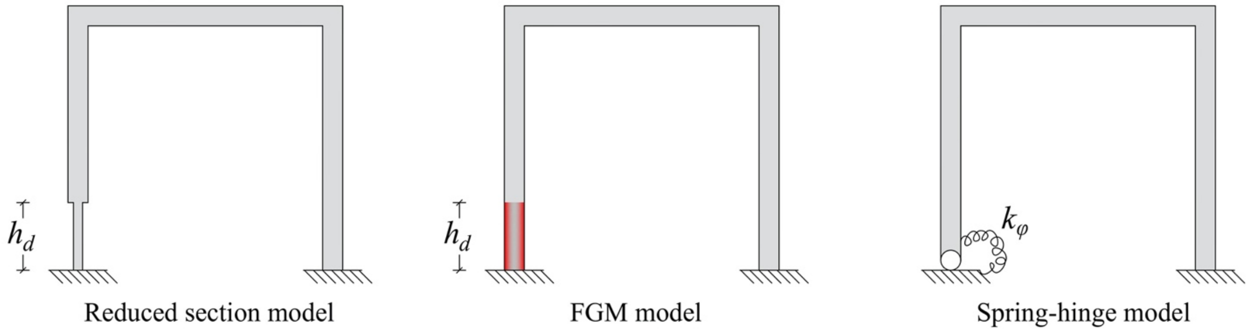


Figure 5: The considered damage models.

h_d	ω [rad/s]	$\frac{E_1}{E_2}=1$	$\frac{E_1}{E_2}=\frac{4}{5}$	$\frac{E_1}{E_2}=\frac{3}{5}$	$\frac{E_1}{E_2}=\frac{2}{5}$	$\frac{E_1}{E_2}=\frac{1}{5}$
1 mm	ω_1	95.5396	95.5293	95.5158	95.4972	95.4700
	ω_2	377.0245	377.0038	376.9766	376.9394	376.8849
	ω_3	614.8743	614.8020	614.7067	614.5756	614.3837
5 cm	ω_1	95.5384	95.0784	94.5003	93.7518	92.7446
	ω_2	377.0149	376.1707	375.1032	373.7111	371.8223
	ω_3	614.8232	611.9333	608.1426	603.0679	596.1347
10 cm	ω_1	95.5370	94.7188	93.7295	92.5092	90.9662
	ω_2	377.0051	375.6449	373.9901	371.9364	369.3249
	ω_3	614.7715	610.3017	604.6313	597.4888	588.5800
20 cm	ω_1	95.5344	94.2411	92.7684	91.0757	89.1088
	ω_2	376.9864	375.2393	373.2341	370.9032	368.1442
	ω_3	614.6763	609.3977	603.0233	595.4407	586.4786
30 cm	ω_1	95.5318	93.9983	92.3216	90.4785	88.4384
	ω_2	376.9702	375.1943	373.1710	370.8025	367.9036
	ω_3	614.6056	609.1342	602.3012	593.7088	582.6479

Table 1: Frequencies of vibration for a frame with one FGM element.

The results reported in Tables 1 and 2 for a damage of small extent have been also compared with the frequencies of vibration obtained adopting the damage model with hinge and spring. In particular only the values of h_d equal to 1 mm and 10 cm have been considered since they belong to the range $h_d < 10\%L$ as required in [11].
The stiffness of the spring has been equal to [11]:

$$k_\varphi = \frac{1-\beta}{b\beta} \tag{4}$$

where $\beta = (EI^U - EI^D)/EI^U$, $b = h_d/L$, EI^U and EI^D are the flexural stiffness of the undamaged and damaged element, respectively, and L is the length of the damaged structural element. Furthermore, EI^D has been chosen in order to obtain the same reductions of the flexural stiffness previously described for the others damage models.

h_d	ω [rad/s]	$h_r = h$	$h_r = \sqrt[3]{1-0.12} \cdot h$	$h_r = \sqrt[3]{1-0.24} \cdot h$	$h_r = \sqrt[3]{1-0.36} \cdot h$	$h_r = \sqrt[3]{1-0.48} \cdot h$
	ω_1	95.5396	95.5293	95.5158	95.4972	95.4700
1 mm	ω_2	377.0245	377.0038	376.9767	376.9394	376.8850
	ω_3	614.8743	614.8020	614.7067	614.5756	614.3838
	ω_1	95.5384	95.0785	94.5004	93.7519	92.7447
5 cm	ω_2	377.0149	376.1711	375.1039	373.7122	371.8238
	ω_3	614.8236	611.9355	608.1470	603.0751	596.1449
	ω_1	95.5371	94.7189	93.7297	92.5095	90.9666
10 cm	ω_2	377.0055	375.6463	373.9930	371.9413	369.3326
	ω_3	614.7744	610.3130	604.6545	597.5283	588.6411
	ω_1	95.5345	94.2418	92.7698	91.0781	89.1124
20 cm	ω_2	376.9882	375.2542	373.2673	370.9628	368.2435
	ω_3	614.6880	609.5217	603.3114	595.9578	587.3124
	ω_1	95.5321	94.0017	92.3289	90.4906	88.4566
30 cm	ω_2	376.9728	375.2661	373.3382	371.1068	368.4192
	ω_3	614.6131	609.7028	603.6926	596.2679	586.8644

Table 2: Frequencies of vibration for a frame with one element with reduced section.

h_d	ω [rad/s]	$h_r = h$	$h_r = \sqrt[3]{1-0.12} \cdot h$	$h_r = \sqrt[3]{1-0.24} \cdot h$	$h_r = \sqrt[3]{1-0.36} \cdot h$	$h_r = \sqrt[3]{1-0.48} \cdot h$
		$k_\varphi \to \infty$	$k_\varphi = 733.3$	$k_\varphi = 316.7$	$k_\varphi = 177.8$	$k_\varphi = 108.3$
	ω_1	95.5396	95.5293	95.5158	95.4972	95.4700
1 mm	ω_2	377.0245	377.0038	376.9767	376.9394	376.8850
	ω_3	614.8743	614.8020	614.7067	614.5756	614.3838
		$k_\varphi \to \infty$	$k_\varphi = 146.7$	$k_\varphi = 63.3$	$k_\varphi = 35.6$	$k_\varphi = 21.7$
	ω_1	95.5384	95.0785	94.5004	93.7519	92.7447
5 cm	ω_2	377.0149	376.1711	375.1039	373.7122	371.8238
	ω_3	614.8236	611.9355	608.1470	603.0751	596.1449
		$k_\varphi \to \infty$	$k_\varphi = 73.3$	$k_\varphi = 31.7$	$k_\varphi = 17.8$	$k_\varphi = 10.8$
	ω_1	95.5371	94.7189	93.7297	92.5095	90.9666
10 cm	ω_2	377.0055	375.6463	373.9930	371.9413	369.3326
	ω_3	614.7744	610.3130	604.6545	597.5283	588.6411

Table 3: Frequencies of vibration for a frame with spring and hinge.

As it can be seen from Tables and 2, an increasing intensity of damage, modelled by means of decreasing values of either E_1/E_2 in a FG beam element or h_r in the homogeneous model with reduced cross section, determines, as expected, a reduction of the frequencies of vibration. The same result is obtained when increasing h_d, since damage involves a larger portion of the structure reducing its global stiffness.

The values reported in Tables 1 and 2 show that the two damage models, opportunely calibrated as previously described, provide almost the same natural frequencies. However, an improvement of the FGM model without the homogenization approach reported in (2), could allow to consider more accurately the variation of the mechanical characteristics due to damage. With reference to the spring-hinge model it can be observed that the frequencies of vibration reported in Table 3 are slightly higher than the ones obtained with the other two damage models. This is due to the fact that the whole damage is concentrated in one section thus increasing the global stiffness of the structure. Furthermore, this damage model cannot be applied for damages of great extent.

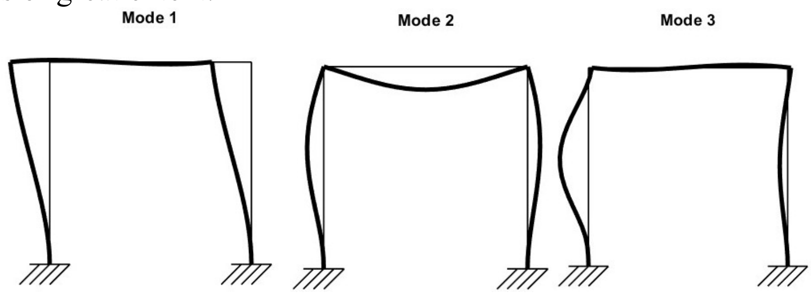
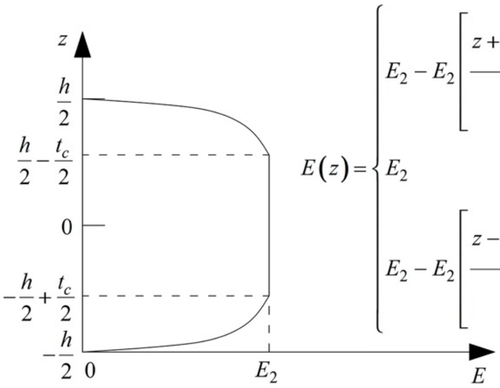


Figure 6: First three mode shapes of the frame with one damage with FGM model.

The modes of vibration of the frame with one damage are almost coincident with those of the undamaged frame. In Figure 6 the first three modes are plotted.

4.2 Frame with two damages

In this paragraph the frame plotted in Figure 3_c has been analyzed assuming the same sizes considered in the previous application. A degradation of the mechanical properties due to corrosion in the damaged portion at the bottom of the two columns has been assumed. In particular, it has been hypothesized the presence of an internal core of uniform material with Young's modulus E_2=200GPa and a corroded symmetric layer of depth t_c having decreasing mechanical properties as described in Figure 7. The results obtained with this model for the two damages of extension h_d, reported in Table 4, have been compared with those provided by the reduced section model, reported in Table 5, in which the corroded part has been completely eliminated.



$$E(z)=\begin{cases} E_2 - E_2\left[\dfrac{z+\left(\frac{h}{2}-\frac{t_c}{2}\right)}{\frac{t_c}{2}}\right]^2 & z<-\frac{h}{2}+\frac{t_c}{2} \\ E_2 & -\frac{h}{2}+\frac{t_c}{2}\le z\le \frac{h}{2}-\frac{t_c}{2} \\ E_2 - E_2\left[\dfrac{z-\left(\frac{h}{2}-\frac{t_c}{2}\right)}{\frac{t_c}{2}}\right]^2 & z>\frac{h}{2}-\frac{t_c}{2} \end{cases}$$

Figure 7: Variation of the mechanical characteristics along the height of the section with an internal uniform core and symmetric corroded layers.

The observation of the values reported in the last two tables shows that, for higher values of t_c the frequencies of vibration reduce due to an increase of the extension of the corroded area. The same results are obtained increasing h_d.

Comparing the results related to the two models it can be observed that the frequencies of vibration obtained with the reduced section model are much lower than the ones provided by the FGM model since the latter allows to consider the contribution of the degraded material.

h_d	ω [rad/s]	t_c = 0 cm	t_c = 0.2 cm	t_c = 0.4 cm	t_c = 0.6 cm	t_c = 0.8 cm	t_c = 1 cm
10 cm	ω_1	95.5344	95.4600	95.3925	95.3314	95.2765	95.2273
	ω_2	376.9856	376.8625	376.7506	376.6495	376.5586	376.4772
	ω_3	614.6679	614.2769	613.9224	613.6026	613.3155	613.0590
20 cm	ω_1	95.5292	95.4085	95.2992	95.2005	95.1120	95.0329
	ω_2	376.9482	376.7870	376.6409	376.5093	376.3913	376.2860
	ω_3	614.4785	614.0174	613.6013	613.2275	612.8936	612.5967
30 cm	ω_1	95.5240	95.3785	95.2469	95.1283	95.0221	94.9273
	ω_2	376.9158	376.7559	376.6117	376.4823	376.3669	376.2646
	ω_3	614.3380	613.9037	613.5151	613.1697	612.8650	612.5981

Table 4: Frequencies of vibration for a frame with two FGM elements.

h_d	ω [rad/s]	t_c = 0 cm	t_c = 0.2 cm	t_c = 0.4 cm	t_c = 0.6 cm	t_c = 0.8 cm	t_c = 1 cm
10 cm	ω_1	95.5344	91.3322	86.1121	79.8281	72.6426	65.0457
	ω_2	376.9863	370.0701	361.6586	351.9166	341.4038	331.0024
	ω_3	614.6738	593.9483	571.6547	548.9285	527.1574	507.4436
20 cm	ω_1	95.5293	89.1403	82.2384	75.1065	68.1448	61.8064
	ω_2	376.9517	368.4898	359.5241	350.3050	340.8121	330.1413
	ω_3	614.5018	591.6186	569.6528	548.3872	526.4735	500.1676
30 cm	ω_1	95.5245	88.1421	80.8187	73.8243	67.4034	61.6592
	ω_2	376.9209	368.6586	359.8656	349.9510	337.3501	318.4131
	ω_3	614.3529	592.2436	568.9684	542.2478	508.2357	461.3829

Table 5: Frequencies of vibration for a frame with two elements with reduced section.

5 CONCLUSIONS

The paper studies the possibility of modeling portions of damaged beams by introducing FGM elements with non-uniform mechanical properties. The possibility of describing continuous variations of the mechanical characteristics within the cross-section of the beam can provide more accurate damage models than those commonly used in scientific literature which refer to the introduction of elements with a reduced cross-section or hinges with rotational springs. The possibility of evaluating the dynamic stiffness matrix of a small beam element, modeled as a Timoshenko beam, made it possible to calculate the vibration frequencies in closed form for all three damage models analyzed.

The results reported in this study refer to a first comparison of damage models by maintaining elastic mechanical properties and appropriately calibrating the reduction in flexural stiffness due to the damage in order to obtain comparable results for the three models. The future development of the research foresees the possibility of modeling the beam following a fiber discretization of the damaged cross section by assigning to each fiber suitable mechanical properties, this latter approach will allow to consider the effect of damage in composite or layered beam-like structures.

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